Summary of scientific achievements

1. Name and surname:

Jacek Małecki

2. Scientific degrees:

2005 M. Sc. in Mathematics,
Institute of Mathematics and Computer Sciences, Faculty of Fundamental Problems of Technology, Wroclaw University of Technology,
Master's dissertation: *Potential theory on hyperbolic spaces.*
written under Prof. dr. hab. Tomasz Byczkowski direction.

2008 Ph.D. in Mathematics,
Institute of Mathematics and Computer Sciences, Faculty of Fundamental Problems of Technology, Wroclaw University of Technology,
doctoral dissertation: *Potential theory on hyperbolic spaces.*
written under Prof. dr. hab. Tomasz Byczkowski direction.

3. Information on previous employment in scientific institutions:

2008 – 2009	Assistant in the Institute of Mathematics and Computer Science Wroclaw University of Technology
2009 – 2014	Assistant Professor in the Institute of Mathematics and Computer Science, Wroclaw University of Technology
2010 – 2011	Post-doc position in LAREMA, Univeristé d'Angers, France
2014* – 2015	Assistant Professor in Department of Mathematics, Wroclaw University of Technology
2015* – present	Assistant Professor in Faculty of Pure and Applied Mathematics, Wroclaw University of Technology

* - these changes were caused by the organizational changes at the University, i.e. the transformation of the Institute of Mathematics and Computer Science to Department of Mathematics on Faculty of Fundamental Problem of Technology and then foundation of Faculty of Pure and Applied Mathematics.

4. An indication of the scientific achievement:

(a) The title of the scientific achievement:

Potential theory of Bessel processes in half-lines

(b) The list of papers constituting the scientific achievement:

[H1] T. Byczkowski, J. Małecki, M. Ryznar, *Hitting half-spaces by Bessel-Brownian diffusions*, Potential Anal. 33(1), 47–83 (2010).

[H2] T. Byczkowski, J. Małecki, M. Ryznar, *Hitting times of Bessel processes*, Potential Anal. 38(3), 753–786 (2013).

[H3] K. Bogus, J. Małecki, *Sharp estimates of transition probability density for Bessel process in half-line*, Potential Anal. 43(1), 1–22 (2015).

[H4] K. Bogus, J. Małecki, *Heat kernel estimates for the Bessel differential operator in half-line.*, Math. Nachr. doi: 10.1002/mana.201500163 (2016).

(c) A discussion of the above-mentioned papers and the obtained results, together with a discussion of their possible use

Introduction

Bessel processes appear naturally in the study of the norm of multidimensional Brownian motion. However, the spectrum of applications of Bessel processes and squared Bessel processes is much wider. A systematic study of Bessel processes was initiated by H.P. McKean in the paper [55], but even before, various so-called Bessel distributions occurred, for example in the study of the random walks (see [26], vol. II). The first surprising result in the theory of Bessel processes was the pioneer Ciesielski-Taylor theorem proved in 1962 in [14]. It states that the total time spend by $(d+2)$-dimensional Brownian motion in the unit ball is identical distributed as the first hitting time of a unit ball by d-dimensional Brownian motion. This theorem was generalized for Bessel processes of general (positive) index and in many other directions. There are also many interpretation of the result, for example, in terms of Brownian motion local times. Finally, the Ciesielski-Taylor theorem has been an inspiration for various research to many excellent mathematicians. Another deep and important result appeared in 1963 proved independently by D.B. Ray [62] and F.B. Knight [45]. The so-called Ray-Knight theorems identify Brownian motion local times, considered as functions of the starting points, stopped at appropriate times, as squared Bessel processes of dimension 2 or 0. These theorems are crucial for studying various delicate properties of local times. Moreover, there are deep relations between Bessel processes and the geometric Brownian motion and its integrals functionals, described by the Lamperti theorem. Note also that Bessel processes appear in a natural way in the study of the Brownian motion excursion theory, Brownian bridges and Brownian meanders as well as in the Williams decomposition of Brownian motion paths (see [67]), in the description of a Brownian motion reflected at its supremum and in potential theory of hyperbolic Brownian motion. Since the general inverse to the Bessel process local times of zero are α-stable subordinators, the Bessel processes can be applied to study jump processes. Finally, there are various practical applications, where Bessel process plays the crucial role. For example, recall the relations with Cox-Ingersoll-Ross processes being very common models in financial mathematics. It is impossible to describe all the important results in the theory of Bessel processes, so let u just mention that despite the above-mentioned researchers, such mathematicians as P. Biane, R.K. Getoor, J.T. Kent, J.W. Lamperti, H. Matsumoto, S.A. Molchanov, J. Pitman and M. Yor had their great contribution in the further development of the theory.

In the presented series of publications we continue research on Bessel processes, focusing on processes killed at the first exit time from the half-lines. More precisely, the aim of the habilitation thesis, consisting of the series of the four articles, was describing basic objects of potential theory of Bessel processes killed upon leaving the half-line (a, ∞) for fixed $a \geq 0$, such as first hitting times, hitting distributions and transition probability densities. By providing a description we mean here a derivation of explicit formulas or providing sharp two-sided estimates for the full range of considered parameters.

- In [H1] we examined the joint distributions of the first hitting time and hitting place of some sets of codimension 1 for the Bessel-Brown diffusions. We derived an explicit formula for the densities of these distributions in terms of the modified Bessel functions. The results were applied to describe hitting distributions related to some class of jumping processes.
- The paper [H2] relates to derive sharp two-sided estimates of the first hitting time of point $a > 0$, when the process starts from $x > a$.
- Finally, in the articles [H3] and [H4] we proved sharp two-sided estimates on transition probability densities of Bessel processes killed upon leaving the half-line (a, ∞). The estimates obtained in [H2], [H3] and [H4] explicitly describe the exponential behaviour of the objects.

Before we move on to detailed description of the results, we will present basic definitions, formulas and properties of Bessel processes that were used in the articles. We relate the Reader to [63] and [53], [54] for much more complete and detailed description. We also recommend very useful compendium of Bessel processes given in [6].

Bessel processes

There are several equivalent definitions of Bessel processes, which, depending on the nature of the studied problem and the required proof methods, turn out to be more or less useful. We will begin our discussion with the approach based on stochastic differential equations. It allows to pass naturally from norms of multidimensional Brownian motion to Bessel processes with general dimensions. Moreover, methods of stochastic analysis are crucial from the point of view of the results obtained in the first article [H1]. Note that defining the process

$$Z_t = B_1^2(t) + \ldots + B_n^2(t), \quad t \geq 0,$$

where $B^n = (B_1, \ldots, B_n)$ is Wiener process in $\mathbf{R}^n$, we obtain one-dimensional diffusion on a half-line $[0, \infty)$. Applying the Itô formula together with the Lévy characterisation of Wiener process we can show that Z_t is a solution to the following stochastic differential equation

$$dZ_t = 2\sqrt{|Z_t|}d\beta_t + ndt, \quad Z_0 = ||B(0)||^2,$$

where β_t is a Brownian motion in $\mathbf{R}$. However, there are no reason to be limited to the natural values of the constant appearing in the drift term. Consequently, we introduce the following definition.

DEFINITION 1. *For $x \geq 0$ the unique strong solution of the equation*

$$dZ_t = 2\sqrt{|Z_t|}d\beta_t + \delta dt, \quad Z_0 = x, \tag{1}$$

is called squared Bessel process of dimension δ starting from x and we denote it by $BESQ^\delta(x)$.

The strong uniqueness of solutions of (1) follows from the Yamada-Watanabe theorem originally proved in [68]. It is worth noting that classical uniqueness theorems for equations with Lipschitz coefficients can not be applied to (1), since a square root appears in the martingale part. It is easy to check that for $\delta = 0$ and $x = 0$ the process $Z_t \equiv 0$ is a solution and the comparison theorem (Chapter XI, Theorem 3.7 in [63]) implies that for $\delta \geq 0$ and $x \geq 0$ the process $BESQ^\delta(x)$ is non-negative. However, for $\delta < 0$ the trajectories of squared Bessel process become negative with probability 1 and after the first hitting time of zero the process behaves like minus squared Bessel process of positive dimension starting from zero. i.e. $-BESQ^{-\delta}(0)$ (see Section 3 in [31]). Although we still deal with diffusion processes, squared Bessel processes with negative dimensions are much more delicate and complex objects (for example formulas for the transition probability densities are much more complicated). Since the paths become negative, the definition of a Bessel process as a square root of $BESQ^\delta(x)$ requires killing of the process at the first hitting time of zero.

DEFINITION 2. *We define Bessel process $BES^\delta(x)$ of dimension $\delta \in \mathbf{R}$ starting from $x \geq 0$ as a squared root of $BESQ^\delta(x^2)$, where for $\delta < 0$ we kill the process $BESQ^\delta(x^2)$ at the first hitting time of zero. By $\mathbf{P}_x^\delta$ we denote the distribution of Bessel process (on $C(\mathbf{R}_+, \mathbf{R})$).*

The number δ is called the dimension of $BES^\delta(x)$. Moreover, we introduce the so-called **index** of Bessel process (squared Bessel process) by putting $\nu = \delta/2 - 1$. We will write $BESQ^{(\nu)}(x)$ and $BES^{(\nu)}(x)$, $\mathbf{P}_x^{(\nu)}$, when we will relate to index of the processes instead of their dimensions.

Distributions of Bessel processes are absolutely continuous with respect to Lebesgue measure and the corresponding densities are given in terms of the modified Bessel functions in the following way. For $\nu > -1$ we have

$$
\begin{aligned}
p^{(\nu)}(t,x,y) &= \frac{1}{t}\left(\frac{y}{x}\right)^\nu y \exp\left(-\frac{x^2+y^2}{2t}\right) I_\nu\left(\frac{xy}{t}\right), \quad x,y>0, \quad t>0,\\
p^{(\nu)}(t,0,y) &= \frac{y^{2\nu+1}}{2^\nu t^{\nu+1}\Gamma(\nu+1)} \exp\left(-\frac{y^2}{2t}\right), \quad y>0, \quad t>0
\end{aligned}
$$

and for $\nu \leq -1$

$$
p^{(\nu)}(t,x,y) = \frac{1}{t}\left(\frac{y}{x}\right)^\nu y \exp\left(-\frac{x^2+y^2}{2t}\right) I_{|\nu|}\left(\frac{xy}{t}\right), \quad x,y>0, \quad t>0.
$$

The above-given densities become symmetric functions of the space variables x and y if we consider the speed measure $m^{(\nu)}(dy) = 2y^{2\nu+1}dy$ as the reference measure. The absolute continuity of the distributions of Bessel processes with different indices is the key property, frequently used in our series of publications. Let us clarify that we work on the canonical space of continuous trajectories $C([0,\infty), \mathbf{R})$ and by $\mathcal{F}_t$ we denote the σ-algebras generated by the paths up to time t. Moreover, we define the first hitting time of a given level by

$$
T_a = \inf\{t > 0 : R(t) = a\}, \quad a \geq 0,
$$

defined for every $R \in \Omega = C([0,\infty), \mathbf{R})$. We will sometimes write $R^{(\nu)} = (R_t^{(\nu)})_{t\geq 0}$ to underline the index of the considered Bessel process. However, when we use different distributions on the canonical space, the notation of the corresponding indices will obviously appear only in the notation of the distributions and expected values, i.e. $\mathbf{P}_x^{(\nu)}$, $\mathbf{E}_x^{(\nu)}$. For every $\mu \geq 0$ and $\nu \in \mathbf{R}$ we have

$$
\left.\frac{d\mathbf{P}_x^{(\mu)}}{d\mathbf{P}_x^{(\nu)}}\right|_{\mathcal{F}_t} = \left(\frac{R(t)}{x}\right)^{\mu-\nu} \exp\left(-\frac{\mu^2-\nu^2}{2}\int_0^t \frac{ds}{R^2(s)}\right), \tag{2}
$$

where $x \geq 0$. The above-given relation holds $\mathbf{P}_x^{(\nu)}$-a.s. on the set $\{T_0^{(\nu)} > t\}$, i.e. up to the first hitting time of zero. This condition can be omitted in the case of non-negative indices since then $\mathbf{P}_x^{(\nu)}(T_0 = \infty) = 1$. For $\nu < 0$ ($\delta < 2$) the process hits zero almost surely. Additionally, for $0 < \delta < 2$ zero is instantaneously reflecting and for $\delta = 0$ the point 0 is absorbing (see Chapter IX, Theorem 1.5 in [63]). Moreover, Bessel process is transitive for $\nu > 0$ ($\delta > 2$) and recurrent in other cases. Note that these properties correspond to the well-known properties of multi-dimensional Brownian motion.

We can look at Bessel process from the point of view of one-dimensional diffusions. We define Bessel process with index $\nu \in \mathbf{R}$ as a diffusion on $[0, \infty)$ or $(0, \infty)$, with $\frac{1}{2}L^{(\nu)}$ as its infinitesimal generator, where

$$L^{(\nu)} = \frac{d^2}{dx^2} + \frac{2\nu + 1}{x}\frac{d}{dx}, \quad x > 0, \tag{3}$$

is the Bessel operator. Uniqueness of this definition requires indication of the domain of the generator or, equivalently, the boundary condition at zero for $\nu \in (-1, 0)$. Imposing reflecting condition at 0 coincides with the definition of Bessel processes based on SDE.

Distribution of the first hitting time and hitting place of Bessel-Brownian diffusions (paper [H1])

We begin investigations of Bessel processes killed upon reaching a level with the most natural situation, i.e. when a process hits zero. Taking into consideration the above-given properties of the Bessel paths, the problem is worth attention only in the case, where $\nu < 0$, since for $\nu \geq 0$ the first hitting time T_0 is infinite $\mathbf{P}_x^{(\nu)}$-a.s. However, the distribution of the first hitting time as well as the transition probability densities are very well-known. Indeed, for $\nu < 0$ and $x > 0$ we have [44, 30]

$$\mathbf{P}_x^{(\nu)}(T_0 \in dt) = \frac{2^\nu}{x^{2\nu}\Gamma(-\nu)t^{1-\nu}} \exp\left(-\frac{x^2}{2t}\right) dt, \quad t > 0. \tag{4}$$

From the other side, using the absolute continuity property (formula (2) for $\mu = -\nu$) we can easily express the transition probability density of Bessel process with negative index ν killed at T_0 in terms of function $p^{(-\nu)}(t, x, y)$. The situation becomes much more complicated if we attach an independent n-dimensional Wiener process $B^n = (B_1, \ldots, B_n)$ to the original Bessel process $R^{(\nu)}$. Then, we obtain $(n+1)$-dimensional process with independent coordinates $\mathbf{Y}(t) = (R^{(\nu)}(t), B^n(t))$, which is called **Bessel-Brownian diffusion**. Moreover, let us define, for an open set $\tilde{D} \subset \mathbf{R}^n$

$$\tau_D = \inf\{t > 0 : R^{(-\mu)}(t) = 0, \quad B^n(t) \notin \tilde{D}\}, \tag{5}$$

i.e. the first exit time of $\mathbf{Y}$ from $D = (\{0\} \times \tilde{D}^c)^c$ or equivalently, the first hitting time of $\{0\} \times \tilde{D}^c$ by $\mathbf{Y}$. In this case we will be interested in distributions of the first stopping time τ_D and $\mathbf{Y}(\tau_D)$. Due to the paths continuity and the definition of τ_D, the problem reduce to find the distribution of $(\tau_D, B^n(\tau_D))$. Note also that the problem is interesting (non-trivial) only when $\nu \in (-1, 0)$ and zero is reflecting for $R^{(\nu)}$. In the opposite case the question is trivial since the coordinates of $\mathbf{Y}$ are independent.

Although the above-given problem may seem to be apparently artificial, the motivation to take up this subject is explained by Lemma 1 (Proposition 3.1 in [H1]) establishing a relationship between joint distributions of first hitting time and place of Bessel-Brownian diffusions and harmonic measures of relativistic α-stable process with parameter $m \geq 0$, i.e. the Lévy process X^m on $\mathbf{R}^n$ with characteristic function given by

$$\mathbf{E}^0 e^{i\xi \cdot X^m(t)} = e^{mt} e^{-t(|\xi|^2 + m^{2/\alpha})^{\alpha/2}}, \quad \xi \in \mathbf{R}^n.$$

The infinitesimal generator of the relativistic process is then given by

$$H_\alpha = mI - (m^{2/\alpha}I - \Delta)^{\alpha/2}.$$

In particular, for $m = 0$, we get isotropic α-stable process. The basic object of our research in this context is a λ-harmonic measure of a set $\tilde{D} \subset \mathbf{R}^n$ defined by

$$P^{\lambda,m}_{\tilde{D}}(x, A) = \mathbf{E}^x\left[\tau_{\tilde{D}} < \infty; e^{-\lambda\tau_{\tilde{D}}}\mathbf{1}_A(X^m(\tau_{\tilde{D}}))\right], \quad x \in \tilde{D}, \quad A \in Borel(\mathbf{R}^n),$$

where $\tau_{\tilde{D}} = \inf\{t > 0 : X^m(t) \notin \tilde{D}\}$ is the first exit time of $X^m(t)$ from $\tilde{D}$. In the case $\lambda = m$ we will use a shorten notation $P^m_{\tilde{D}}(x, A)$ instead of $P^{m,m}_{\tilde{D}}(x, A)$. We have

Lemma 1 (Proposition 3.1 in [H1]). *Let $\tilde{D} \subset \mathbf{R}^n$ be open having exterior cone property at every point and let $x = (0, \tilde{x}) \in \{0\} \times \tilde{D}$. Moreover, let $\mathbf{P}^x(\tau_D < \infty)$, where τ_D is defined in* (5) *the first hitting time for $\mathbf{Y} = (R_t^{-\alpha/2}, B^n(t))$, $\alpha \in (0, 2)$, of $D = \{0\} \times \tilde{D}^c$. Then m-harmonic measure for relativistic α-stable process with parameter* $m \geq 0$ *is given by*

$$P^m_{\tilde{D}}(\tilde{x}, A) = \mathbf{E}^x[e^{-\frac{m^{2/\alpha}}{2}\tau_D}; B^n(\tau_D) \in A], \quad A \subset Borel(\mathbf{R}^n). \tag{6}$$

The conclusion is valid also for $m = 0$, that is for the harmonic measure for the standard isotropic α-stable process.

The exterior cone property is a technical assumption that guarantees regularity of the boundary points, i.e. $\mathbf{P}^y(\tau_D = 0) = 1$ for every $y \in \{0\} \times \tilde{D}$. The main idea of the proof is to show that the right-hand side of the desired equality satisfies the integral equation called the *sweeping out* formula

$$\int_{\tilde{D}^c} U^m_m(z - y)P^m_{\tilde{D}}(x, dz) = U^m_m(x - y), \quad x \in \tilde{D}, \quad y \in \tilde{D}^c,$$

that uniquely characterize the m-harmonic measure, i.e. the left-hand side of (6). Here $U^m_\lambda(x)$ is the λ-resolvent kernel, which in the special case $\lambda = m$ is given explicitly by the formula

$$U^m_m(x, y) = \frac{2^{1-(d+\alpha)/2}}{\Gamma(\alpha/2)\pi^{d/2}} \frac{K_{(d-\alpha)/2}(|x - y|)}{|x - y|^{(d-\alpha)/2}}, \tag{7}$$

where $K_\nu(z)$ is *the modified Bessel function of the second kind.* The general approach to look at jump Markov processes as traces of appropriate diffusions was introduced by Molchanov and Ostrowski in [56] and was used in different aspects among others by DeBlaisse, Bañuelos and Kulczycki, Kwaśnicki and Isozaki. Analogous techniques in analysis are widely known as Caffarelli–Silvestre extensions, since they were reinvented in paper [12]. Despite the fact that we do not use any known results of that kind, our research generally fits to this trend. Anyway, Lemma 1 implies that the knowledge of a distribution of $(\tau_D, B^n(\tau_D))$ (more precisely, the Laplace transform of time τ_D and distribution of hitting place $B^n(\tau_D)$) gives the formula for m-harmonic measure of relativistic α-stable process with parameter m. It is worth mentioning that explicit formulas for harmonic measures for jump processes are vary rare. From the other side, such knowledge usually helps in developing the potential theory related to the process. The interest in relativistic processes, which were intensively studied in resent years, comes from its applications in relativistic quantuum mechanics. More precisely, quasi-relativistic Hamiltonian (known also as Klein-Gordon square root operator)

$$\mathcal{H} = (-\hbar^2c^2\Delta + m^2c^4)^{1/2}, \tag{8}$$

describing the motion of a quasi-relativistic particle with mass m (here c is the speed of light, $\hbar$ is the Planck constant) can be easily described by the infinitesimal generator of the relativistic 1-stable process.

The other basic motivation comes from the very simple observation: a measure $P^m_{\tilde{D}}(x, A)$ is a harmonic measure of the set $\tilde{D}$ for the operator $-(m^{\alpha/2}I - \Delta)^{\alpha/2}$. This operator appears in the context of Bessel potentials $J_\alpha = (I - \Delta)^{-\alpha/2}$, the formal inverse to $(I - \Delta)^{\alpha/2}$. Such operators have convolution representation with the (Bessel) convolution kernel (7). The significance of Bessel potentials is that the Sobolev space $L^p_\alpha(\mathbf{R}^d)$ can be defined in terms of J_α as a subspace of $L^p(\mathbf{R}^d)$ consisting of all functions, which can be written in the form $J_\alpha g$, where $g \in L^p(\mathbf{R}^d)$ (see. [66] chap. V). The wide range of applications of Sobolev spaces in harmonic analysis and partial differential equations is very well-known. Surprisingly, the explicit formulas for harmonic measures and Green functions for half-lines (half-spaces) are known only recently. Such formulas for half-lines and half-spaces were provided in [D1] from my Ph.D. Thesis. Unfortunately, the proof was not constructive and consequently it can not be applied to find the formulas for different sets such as an interval. The results of [H1] are based on the different approach (rewriting the problem in the context of Bessel-Brownian diffusions), which in particular enable us to solve the problem for intervals and strips.

Half-line. We begin our considerations with the problem of two-dimensional Bessel-Brownian diffusion $\mathbf{Y} = (R_t^{-\alpha/2}, B(t))$ hitting one-dimensional half-line $\tilde{D} = (-\infty, 0) \subset \mathbf{R}$, i.e. we define

$$D_1 = \{(y_1, y_2) \in \mathbf{R}^2 : y_1 = 0, y_2 > 0\}^c = (\{0\} \times \tilde{D}^c)^c$$

and the first exit time of $\mathbf{Y}$ from the set D_1 by

$$\tau_{D_1} = \inf\{t > 0 : \mathbf{Y} \notin D_1\} = \inf\{t > 0 : R_t^{-\alpha/2} = 0 \wedge B(t) > 0\}.$$

We prove the following theorem.

Theorem 3 (Theorem 4.4 w [H1]). *For $(R_0^{(-\alpha/2)}, B(0)) = (z_1, z_2) \in D_1$, $z_1 > 0$, $\lambda > 0$ the measure*

$$\mathbf{E}^{(z_1,z_2)}\left[e^{-\frac{\lambda^2}{2}\tau_{D_1}}; B(\tau_{D_1}) \in A\right]$$

is absolutely continuous with respect to the Lebesgue measure on a half-line $r > 0$ with the density function given by

$$\frac{(|z| + z_2)^{\frac{\alpha}{4}}(|z| - z_2)^{\frac{\alpha}{2}}}{2^{\frac{3\alpha}{4}}\Gamma(\frac{\alpha}{2})r^{\alpha/4}} \int_\lambda^\infty e^{-(|z|+r)s}(s^2 - \lambda^2)^{\alpha/4} I_{-\frac{\alpha}{2}}\left(\sqrt{2r}\sqrt{|z| + z_2}\sqrt{s^2 - \lambda^2}\right) ds$$

For $z_1 = 0$ and $z_2 = u < 0$ we have

$$\mathbf{E}^{(0,u)}\left[e^{-\frac{\lambda^2}{2}\tau_{D_1}}; B(\tau_{D_1}) \in dr\right] = \frac{\sin(\pi\alpha/2)}{\pi}\left(\frac{-u}{r}\right)^{\alpha/2}\frac{e^{-\lambda(r-u)}}{r-u}. \tag{9}$$

The main idea of the proof is to adopt the very well-known conformal invariance property of Wiener process on a plane to our setting. More precisely, the image of complex Brownian motion $B_1 + iB_2$ by a holomorphic function $f : \mathbf{C} \to \mathbf{C}$ is a time-changed complex Brownian motion. The change of time is described by the integral functional $\int_0^t |f'(B(s))|^2 ds$. Surprisingly, a similar phenomenon appears also in the context of Bessel-Brownian diffusions. Since its proof depends on stochastic analysis methods, it is convenient to consider squared Bessel processes instead of Bessel processes. Let us denote by $Y = (Y_1, Y_2)$ a process described by

$$\begin{cases} dY_1 = 2\sqrt{Y_1}d\beta_1 + (2-\alpha)dt \\ dY_2 = d\beta_2 \end{cases}, \tag{10}$$

where β_1 and β_2 are independent Brownian motions.Consequently, Y is just Bessel-Brownian diffusion $\mathbf{Y}$ with the first coordinate squared (changing from Bessel process to corresponding squared Bessel process). Moreover, we consider two independent squared Bessel processes

$$\begin{cases} dX_1 = 2\sqrt{X_1}dB_1 + (2-\alpha)dt \\ dX_2 = 2\sqrt{X_2}dB_2 + (2-\alpha)dt \end{cases},$$

and define $Z = (Z_1, Z_2) = f(X_1, X_2)$, where $f(x,y) = (4xy, y^2 - x^2)$. We prove (see Section 2 in [H1]) that the function f transform two independent squared Bessel processes (X_1, X_2) into Y with time changed, i.e. $Y = Z \circ \sigma_1$, where $\sigma_1 = \inf\{t > 0 : A_1(t) > s\}$ is generalized inverse to the integral functional

$$A_1(t) = 4\int_0^t (X_1(s) + X_2(s))ds.$$

This fact follows from the Itô formula and the Lévy characterization. Choosing the form of the function f we took pattern from the holomorphic function $z \to -iz^2 = 2\Re z \Im z + i((\Im z)^2 - (\Re z)^2)$. The appearance of the coefficient $-i$ is technical. Since we work with squared Bessel processes we have to take square root of the coordinates and then we take the second power of the first coordinate, i.e. $f(x_1,x_2) = ((2\sqrt{x_1}\sqrt{x_2})^2, \sqrt{x_2}^2 - \sqrt{x_1}^2)$. Moreover, the function f transforms

$$H = \{(x_1,x_2) \in [0,\infty) \times [0,\infty) : x_1 > 0\}$$

into D_1 and the first exit time of (X_1, X_2) from H, i.e. $\tau_H = \inf\{t > 0 : X_1(t) = 0\}$, corresponds to the first hitting time of a half-line $\{0\} \times [0,\infty)$ by $Z = (Z_1, Z_2)$ (see Fig. 1). More precisely, we have

Lemma 2 (Lemma 4.1 w [H1]). *The distribution of* $(\tau_{D_1}, Y(\tau_{D_1}))$ *with respect to* $\mathbf{P}^{(y_1,y_2)}$ *is the same as the distribution of* $(A_1(\tau_H), f(X(\tau_H)))$ *with respect to* $\mathbf{P}^{(x_1,x_2)}$*, where* $f(x_1,x_2) = (y_1,y_2)$.

This result translates the problem of finding the distributions of hitting times and hitting places for Bessel-Brownian diffusion (here the hitting time depends on both coordinates) to the similar one, but for two independent squared Bessel processes (X_1, X_2), where the first hitting time depends only on the first of them. The cost of this transformation is an appearance of more complicated object $A_1(\tau_H)$ in a place of τ_{D_1}. However, using the fact that $A_1(t)$ is a sum of two integrals depending on X_1 and X_2 respectively, we can reduce the problem to one dimension. Indeed, independence of X_1 and X_2 implies that

$$\mathbf{E}^{(x_1,x_2)}\left[e^{-\frac{\lambda^2}{2}A_1(\tau_H)}; X_2(\tau_H) \in dr\right] = \int_0^\infty w(dt, x_1)\psi(t, x_2, dr),$$

where

$$w(t,x_2) = \mathbf{E}^{x_1}\left[\exp\left(-\frac{\lambda^2}{2}\int_0^{\tau_H} X_1(s)ds\right); \tau_H \in dt\right],$$

and

$$\psi(t,x_2,dr) = \mathbf{E}^{x_2}\left[\exp\left(-\frac{\lambda^2}{2}\int_0^t X_2(s)ds\right); X_2(t) \in dr\right].$$

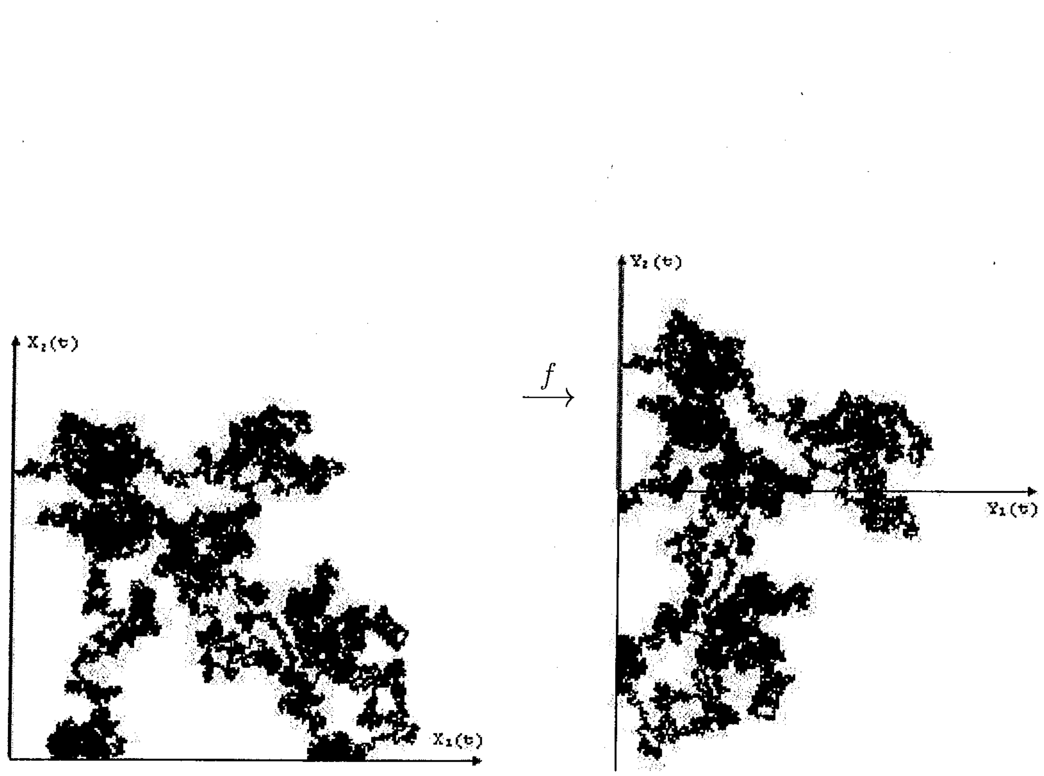

Figure 1: Trace of the trajectory of (X_1, X_2) stopped at the first exit time from H and its image by the function f.

The above-given expressions are indeed one-dimensional (τ_H depends only on X_1) and well-known in the theory of Bessel processes. The explicit formulas for these distributions in terms of the Whittaker functions and the modified Bessel functions are also obtained in Lemmas 4.2 and 4.3 in [H1] and they lead directly to the expressions given in Theorem 3.

Two half-lines. Next, we consider two-dimensional Bessel-Brownian diffusion $\mathbf{Y} = (R^{-\alpha/2}, B(t))$ hitting two distinct half-lines and this part relates to m-harmonic measure of an interval for relativistic process. Let us define

$$C_1 = \{(y_1, y_2) \in [0, \infty) \times \mathbf{R} : y_1 = 0, |y_2| > 1\}^c$$

and let τ_{C_1} time be the first exit of $\mathbf{Y}$ from C_1, i.e.

$$\tau_{C_1} = \inf\{t > 0 : R_t^{(-\alpha/2)} = 0 \wedge |B(t)| > 1\}.$$

We provide the following description of the Laplace transform of the first exit time and the distribution of the hitting place.

Theorem 4 (Theorem 4.6 in [H1]). *For $(R_0^{(-\alpha/2)}, B(0)) = (z_1, z_2) \in C_1$ and $r > x_2 \geq 1$ we have*

$$\mathbf{E}^{(z_1,z_2)}\left[e^{-\frac{\lambda^2}{2}\tau_{C_1}}; B(\tau_{C_1}) \in dr\right] == \frac{1}{(r^2-1)^{\frac{\alpha}{2}}} \frac{1}{2\pi i} \int_{\frac{\alpha}{4}-i\infty}^{\frac{\alpha}{4}+i\infty} m_{-\vartheta,\lambda}(x_1)\, w_\lambda(\vartheta) \phi^{\uparrow}_{\vartheta,\lambda}(x_2) \phi^{\downarrow}_{\vartheta,\lambda}(r) d\vartheta,$$

where

$$\begin{aligned} x_1 &= \frac{1}{2}\left(\sqrt{z_1^2 + (z_2+1)^2} + \sqrt{z_1^2 + (z_2-1)^2}\right), \\ x_2 &= \frac{1}{2}\left(\sqrt{z_1^2 + (z_2+1)^2} - \sqrt{z_1^2 + (z_2-1)^2}\right) \end{aligned}$$

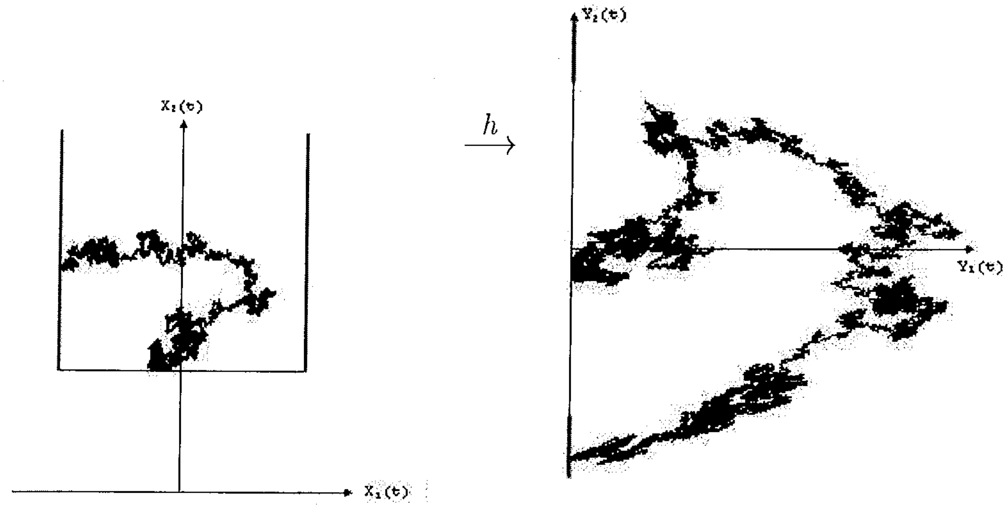

Figure 2: Trace of the trajectory of (X_1, X_2) stopped at the first exit time from H and its image by the function h.

and the function $m_{\vartheta,\lambda}(\cdot)$ is the solution of the following differential equation

$$(1-x^2)y''(x) - (2-\alpha)x\,y'(x) - (\lambda^2(1-x^2) + 2\vartheta)y(x) \quad = \quad 0, \quad |x| < 1, \tag{11}$$

with boundary conditions $m_{\vartheta,\lambda}(-1) = 0$, $m_{\vartheta,\lambda}(1) = 1$. The functions $\phi^{\uparrow}_{\vartheta,\lambda}(\cdot)$ i $\phi^{\downarrow}_{\vartheta,\lambda}(\cdot)$ are respectively increasing and decreasing independent positive solutions of the differential equation

$$(r^2-1)y''(r) + (2-\alpha)r\,y'(r) - (\lambda^2(r^2-1) + 2\vartheta)y(r) \quad = \quad 0, \quad r > 1. \tag{12}$$

satisfying $\lim_{x\to 1+} \phi^{\uparrow}_{\vartheta,\lambda}(x) = 0$, $\lim_{x\to\infty} \phi^{\downarrow}_{\vartheta,\lambda}(x) = 0$ *and*

$$w_\lambda(\vartheta) = \frac{2}{(1-r^2)^{\alpha/2-1} W\{\phi^{\uparrow}_{\vartheta,\lambda}, \phi^{\downarrow}_{\vartheta,\lambda}\}(r)},$$

where $W\{\phi^{\uparrow}_{\vartheta,\lambda}, \phi^{\downarrow}_{\vartheta,\lambda}\}$ is the Wronskian of the pair $\{\phi^{\uparrow}_{\vartheta,\lambda}, \phi^{\downarrow}_{\vartheta,\lambda}\}$.

The main idea of the proof is similar as in the previous case. We begin with the process Y described by the equations (10) and two independent one-dimensional diffusions

$$\begin{cases} dX_1 = \sqrt{|1-X_1^2|}dB_1 - \dfrac{2-\alpha}{2}X_1 dt \\ dX_2 = \sqrt{|X_2^2-1|}dB_2 + \dfrac{2-\alpha}{2}|X_2|dt \end{cases}, \tag{13}$$

where $|X_1(0)| < 1$ and $X_2(0) > 1$. The first one is a version of the Legendre process and the other one is the so-called hyperbolic Bessel process. We have $X_1(t) \in [-1,1]$ and $X_2(t) \geq 1$. We define $h : \mathbf{R}^2 \to \mathbf{R}^2$ by $h(x_1, x_2) = ((1-x_1^2)(x_2^2-1), x_1 x_2)$. The function h is a bijection between

$$G = \{(x_1, x_2) \in [-1,1] \times [1,\infty) : -1 < x_1 < 1\}$$

and C_1. As previously, we show that the process $Z = (Z_1, Z_2) = h(X_1, X_2)$ is just $Y = (Y_1, Y_2)$ with its time changed by σ_2, which is generalized inverse to the integral functional

$$A_2(t) = \int_0^t (X_2^2(s) - X_1^2(s))ds.$$

The form of h was postulated on basis of the formula $z \to \sin(z) = \sin(\Re z)\mathrm{ch}(\Im z) + i\mathrm{sh}(\Im z)\cos(\Re z)$ and a very simple observation: for $\alpha = 0$ the processes $X_1(t) = \sin B_1(t)$ and $X_2(t) = \cosh B_2(t)$ solve the above-given stochastic differential equations. Despite the fact that for $\alpha \in (0, 2)$ we do not have such nice representations, using the same function h leads to the relation between the distributions of the processes (X_1, X_2) and (Y_1, Y_2).

Lemma 3 (Lemat 4.5 w [H1]). *The distribution $(\tau_{C_1}, Y(\tau_{C_1}))$ with respect to $\mathbf{P}^{(y_1,y_2)}$ is the same as the distribution of $(A_2(\tau_G), h(X(\tau_G)))$ with respect to $\mathbf{P}^{(x_1,x_2)}$, where $h(x_1, x_2) = (y_1, y_2)$.*

Also in this case the functional $A_2(t)$ is a sum of two integrals depending only on X_1 and X_2 respectively, and the first exit time from G is independent from X_2. Consequently, we can translate the problem to one-dimensional ones. Unfortunately, the theory of the processes described by 13 is not well-developed. Thus, we use the general theory of Feynmann-Kac semi-groups to reduce our considerations to solving certain second-order differential equations. However, the equations, which are called *spheroidal wave equation*, are quite general and there is a lack of analytical tools to study and describe such equations and their solutions, which can lead to more transparent representation of the studied distributions. However, for $\lambda = 0$ the formulas can be used to provide the formula for the Poisson kernel of an interval for symmetric α-stable process (Corollary 4,7 in [H1]), which were originally obtained by M. Riesz using Kelvin transform.

Multi-dimensional case. We consider the following multi-dimensional generalizations of the sets C_1 and D_1

$$\begin{aligned} D_n &= \{y \in \mathbf{R}^{n+1} : y_1 = 0,\ y_2 > 0\}^c, \\ C_n &= \{y \in \mathbf{R}^{n+1} : y_1 = 0,\ |y_2| > 1\}^c. \end{aligned}$$

The first one is just the complement of n-dimensional half-space in $\mathbf{R}^{n+1}$ and the other is the complement of n-dimensional strip. We denote the first exit times of Bessel-Brownian diffusion $\mathbf{Y} = (R_t^{-\alpha/2}, B^n(t))$ from the sets by τ_{D_n} and τ_{C_n}. They are also the first hitting times of half-space and strip codimension 1. At the beginning we deal with set D_n assuming additionally that the diffusion starts from $y \in \mathbf{R}^{n+1}$ such that $y_1 = 0$. Note that this is the crucial case to determine m-harmonic measure for corresponding relativistic α-stable process.

Theorem 5 (Theorem 5.1 in [H1]). *For $y = (0, y_2, \ldots, y_{n+1})$ such that $y_2 < 0$ we have*

$$\mathbf{E}^y[\tau_{D_n} \in dt,\ \mathbf{Y}(\tau_{D_n}) \in d\tilde{\sigma}] = \frac{\sin(\pi\alpha/2)}{2^{n/2}\pi^{1+n/2}} \left(\frac{-y_2}{\sigma_2}\right)^{\frac{\alpha}{2}} \frac{1}{t^{1+n/2}} \exp\left(-\frac{|\tilde{\sigma} - y|^2}{2t}\right), \tag{14}$$

where $t > 0$ and $\tilde{\sigma} = (\sigma_2, \ldots, \sigma_{n+1}) \in \mathbf{R}^n$, $\sigma_2 > 0$.

This representation is proved by finding the distribution of $(\tau_{D_1}, B(\tau_{D_1}))$, which can be obtained by inverting the Laplace transform and the formula (9). Simplicity of the formula in (9) (compare with the case $y_1 > 0$), is crucial and enable us to compute the inverse Laplace transform effectively. Since the half-space D_n is a Cartesian product of D_1 and $\mathbf{R}^{n-1}$, the first exit time depends only on the first two coordinates and practically it coincides with τ_{D_1}. Thus we can find the distribution of $(\tau_{D_n}, B(\tau_{D_n}))$ using the one-dimensional result.

Since in the general case the formula in (9) is very complicated, inverting the Laplace transform and finding the distribution of $(\tau_{D_1}, B(\tau_{D_1}))$ for $y_1 > 0$ is very difficult. Thus, we limit ourselves to find the Laplace transform in time and distribution of place, i.e. the multi-dimensional generalization of Theorem 3. Here we use the Markov property, the formula (4) for the first hitting time of zero by Bessel process and the one-dimensional result.

Theorem 6 (Theorem 5.2 in [H1]). *For $y \in \mathbf{R}^{n+1}$ such that $(y_1, y_2, \dots, y_{n+1}) \in D_n$ we have*

$$\mathbf{E}^y[e^{-\frac{\lambda^2}{2}\tau_{D_n}}; \boldsymbol{Y}(\tau_{D_n}) \in d\tilde{\sigma}] = \frac{2y_1^\alpha \lambda^{\frac{n+\alpha}{2}}}{(2\pi)^{n/2}2^{\alpha/2}\Gamma(\alpha/2)} \frac{K_{\frac{n+\alpha}{2}}(\lambda|y-\tilde{\sigma}|)}{|y-\tilde{\sigma}|^{\frac{n+\alpha}{2}}} +$$

$$+c_{n,\alpha}\, y_1^\alpha \lambda^{n+\frac{\alpha}{2}} \int_{(-\infty,0)\times\mathbf{R}^{n-1}} \left(\frac{-z_2}{\tilde{\sigma}_2}\right)^{\frac{\alpha}{2}} \frac{K_{\frac{n+\alpha}{2}}(\lambda|y-\tilde{z}|)}{|y-\tilde{z}|^{\frac{n+\alpha}{2}}} \frac{K_{\frac{n}{2}}(\lambda|\tilde{\sigma}-\tilde{z}|)}{|\tilde{\sigma}-\tilde{z}|^{\frac{n}{2}}} d\tilde{z},$$

where $c_{n,\alpha} = \frac{4\sin(\pi\alpha/2)}{2^{n+\alpha/2}\pi^{n+1}\Gamma(\alpha/2)}$ and $\tilde{\sigma} = (\sigma_2, \dots, \sigma_{n+1}) \in \mathbf{R}^n$, $\sigma_2 > 0$.

For $y_1 = 0$ we get

$$\mathbf{E}^y[e^{-\frac{\lambda^2}{2}\tau_{D_n}}; \boldsymbol{Y}(\tau_{D_n}) \in d\tilde{\sigma}] = \frac{2\sin(\pi\alpha/2)\lambda^{n/2}}{2^{\frac{n}{2}}\pi^{\frac{n+2}{2}}} \left(\frac{-y_2}{\sigma_2}\right)^{\alpha/2} \frac{K_{n/2}(\lambda|y-\tilde{\sigma}|)}{|y-\tilde{\sigma}|^{n/2}}. \quad (15)$$

The formula for m-harmonic measure of half-space for relativistic α-stable process with parameter m, obtained previously in [D2] using different methods, follows immediately and we get

Corollary 7 (Corollary 5.3 in [H1]). *Let $\tilde{D}_n \subset \mathbf{R}^n$ be the half-space $\{\tilde{x} \in R^n;\ x_1 < 0\} \subset \mathbf{R}^n$. Then the m-Poisson kernel of $\tilde{D}_n$ for relativistic α-stable process with parameter $m > 0$ is given by*

$$P^m_{\tilde{D}_n}(\tilde{y}, \tilde{\sigma}) = \frac{2\sin(\pi\alpha/2)m^{\frac{n}{2\alpha}}}{2^{\frac{n}{2}}\pi^{\frac{n+2}{2}}} \left(\frac{-y_1}{\sigma_1}\right)^{\alpha/2} \frac{K_{n/2}(m^{\frac{1}{\alpha}}|\tilde{y}-\tilde{\sigma}|)}{|\tilde{y}-\tilde{\sigma}|^{n/2}},$$

where $\tilde{y} = (y_1, \dots, y_n) \in \tilde{D}_n$ i $\tilde{\sigma} = (\sigma_1, \dots, \sigma_n) \in \tilde{D}_n^c$. For $m = 0$ we obtain the Poisson kernel of $\tilde{D}_n$ for the standard isotropic α-stable process given by the formula

$$P_{\tilde{D}_n}(\tilde{y}, \tilde{\sigma}) = \frac{\sin(\pi\alpha/2)\Gamma(\frac{n}{2})}{\pi^{\frac{n+2}{2}}} \left(\frac{-y_1}{\sigma_1}\right)^{\alpha/2} \frac{1}{|\tilde{y}-\tilde{\sigma}|^n}.$$

Due to very complicated nature of the distributions given in Theorem 4, we introduce the following notation

$$H(z_1, z_2, \lambda, r) = \mathbf{E}^{(z_1,z_2)}\left[e^{-\frac{\lambda^2}{2}\tau_{C_1}}; B_2(\tau_{C_1}) \in dr\right], \ |r| > 1,$$

where $(z_1, z_2) \in C_1$. Moreover, let

$$h(\lambda, y, \tilde{\sigma}) = \mathbf{E}^y[e^{-\frac{\lambda^2}{2}\tau_{C_n}}; B^n(\tau_{C_n}) \in d\tilde{\sigma}],\ \tilde{\sigma} = (\sigma_2, \dots, \sigma_{n+1}) \in \mathbf{R}^n, |\sigma_2| > 1.$$

We characterise the Laplace transform of the hitting time and the distribution of the hitting place of a strip (i.e. the function h given above) by computing its Fourier transform.

Theorem 8 (Theorem 5.4 in [H1]). *Let $n \geq 2$. For $y = (0, y_2, \ldots, y_{n+1})$, such that $|y_2| < 1$ and $\bar{z} \in \mathbf{R}^{n-1}$ we have*

$$\int_{\mathbf{R}^{n-1}} h(\lambda, y, \tilde{\sigma}) e^{i(\bar{\sigma}, \bar{z})} d\bar{\sigma} = H(0, y_2, \sqrt{|\bar{y} - \bar{z}|^2 + \lambda^2}, \sigma_2). \tag{16}$$

Here $\tilde{\sigma} = (\sigma_2, \bar{\sigma})$.

An immediate consequence of the above-given result is the following

COROLLARY 9 (Corollary 5.5 in [H1]). *Assume that $n \geq 2$. Let $\tilde{C}_n \subset \mathbf{R}^n$ be the strip $\{\tilde{x} \in \mathbf{R}^n;\ |x_1| < 1\} \subset \mathbf{R}^n$. Then the $(n-1)$-dimensional Fourier transform of m-Poisson kernel of $\tilde{C}_n$ for the relativistic α-stable process with parameter $m \geq 0$ is given by*

$$\int_{\mathbf{R}^{n-1}} P^m_{\tilde{C}_n}(\tilde{y}, \tilde{\sigma}) e^{i(\bar{\sigma}, \bar{z})} d\bar{\sigma} = H(0, y_1, \sqrt{|\bar{y} - \bar{z}|^2 + m^{2/\alpha}}, \sigma_1)\ ,$$

where $\tilde{y} = (y_1, \ldots, y_n) \in \tilde{C}_n$ i $\tilde{\sigma} = (\sigma_1, \bar{\sigma}) \in \tilde{C}_n^c$.

In the case of interval (strip) the nature of the problem appeared to be very complicated. Nevertheless, we were able to provide closed description of the corresponding harmonic measures for the relativistic processes. However, these results show that it was impossible to "'guess"' the form of the distributions (as it was done in [D2]) without finding effective methods to determine them.

Hitting sets of codimension 2 and further applications.

In the last part of the work [H1] we present an application of the obtained results to determine the distributions of the hitting time and the hitting place of $(n-1)$-dimensional sets by $(n+1)$-dimensional diffusions. Let $\mathbf{Y} = (Y_1, B_1, B_2, \ldots, B_n)$ be the Bessel-Brownian diffusion in $\mathbf{R}^{n+1}$, where Y_1 is Bessel process with index $-\alpha/2$ for $\alpha \in (1,2)$. Consider the complement in $\mathbf{R}^{n+1}$ of the $(n-1)$-dimensional half-space

$$H_n = \{y \in \mathbf{R}^{n+1} : y_1 = 0, y_2 = 0, y_3 > 0\}^c$$

and let us denote by τ_{H_n} the first exit time of $\mathbf{Y}$ from H_n, i.e. the first hitting time of the half-space of codimension 2. Note that the process $Z = \sqrt{Y_1^2 + B_1^2}$ is Bessel process with index $(1-\alpha)/2$, which follows from additive property of the squared Bessel processes. Consequently, the process $\mathbf{W} = (Z, B_3, \ldots, B_n)$ is n-dimensional Bessel-Brownian diffusion. Defining

$$\tilde{H}_n = \{\tilde{y} \in \mathbf{R}^n : y_1 = 0, y_2 > 0\}$$

observe that $\mathbf{Y}$ exits H_n when $\mathbf{W}$ leaves $\tilde{H}_n$ and consequently we can translate our question into the corresponding one for $\mathbf{W}$ and $\tilde{H}_n$. Thus, using the previous results with n replaced by $n-1$ and α changed into $\alpha - 1$, we get

Theorem 10 (Theorem 5.6 in [H1]). *For $y \in \mathbf{R}^{n+1}$, such that $y_1^2 + y_2^2 > 0$ i $1 < \alpha < 2$ we have*

$$\mathbf{E}^y[e^{-\frac{\lambda^2}{2}\tau_{H_n}}; \mathbf{Y}(\tau_{H_n}) \in d\bar{\sigma}] = \frac{2(y_1^2+y_2^2)^{\frac{\alpha-1}{2}} \lambda^{\frac{n-2+\alpha}{2}}}{(2\pi)^{\frac{n-1}{2}} 2^{\frac{\alpha-1}{2}} \Gamma(\frac{\alpha-1}{2})} \frac{K_{\frac{n-2+\alpha}{2}}(\lambda|y-\bar{\sigma}|)}{|y-\bar{\sigma}|^{\frac{n-2+\alpha}{2}}} +$$
$$+ G_{y_1^2+y_2^2} \int_{(-\infty,0)\times\mathbf{R}^{n-2}} \left(\frac{-z_3}{\sigma_3}\right)^{\frac{\alpha-1}{2}} \frac{K_{\frac{n-2+\alpha}{2}}(\lambda|y-\bar{z}|)}{|y-\bar{z}|^{\frac{n-2+\alpha}{2}}} \frac{K_{\frac{n-1}{2}}(\lambda|\bar{\sigma}-\bar{z}|)}{|\bar{\sigma}-\bar{z}|^{\frac{n-1}{2}}} d\bar{z},$$

where

$$G_{y_1^2+y_2^2} = \frac{4\sin(\pi\frac{\alpha-1}{2})(y_1^2+y_2^2)^{\frac{\alpha-1}{2}}\lambda^{\frac{2n-3+\alpha}{2}}}{2^{\frac{2n-3+\alpha}{2}}\pi^n\Gamma(\frac{\alpha-1}{2})}$$

and $\bar{\sigma} = (\sigma_3, \ldots, \sigma_{n+1}) \in \mathbf{R}^{n-1}$, $\sigma_3 > 0$. *For* $y_1 = y_2 = 0$ *we get*

$$\mathbf{E}^y[e^{-\frac{\lambda^2}{2}\tau_{H_n}}; \boldsymbol{Y}(\tau_{H_n}) \in d\bar{\sigma}] = \frac{2\sin(\pi\frac{\alpha-1}{2})\lambda^{\frac{n-1}{2}}}{2^{\frac{n-1}{2}}\pi^{\frac{n+1}{2}}}\left(\frac{-y_3}{\sigma_3}\right)^{\frac{\alpha-1}{2}}\frac{K_{\frac{n-1}{2}}(\lambda|y-\bar{\sigma}|)}{|y-\bar{\sigma}|^{(n-1)/2}}.$$

For $n = 2$ *the first formula can be simplified to*

$$\begin{aligned}\mathbf{E}^{(z_1,z_2,z_3)}\left[e^{-\frac{\lambda^2}{2}\tau_{H_2}}; B_3(\tau_{H_2}) \in dr\right] &= \frac{(|z|+z_3)^{\frac{\alpha-1}{4}}(|z|-z_3)^{\frac{\alpha-1}{2}}}{2^{\frac{3(\alpha-1)}{4}}\Gamma(\frac{\alpha-1}{2})r^{(\alpha-1)/4}}\times \\ &\times \int_\lambda^\infty e^{-(|z|+r)s}(s^2-\lambda^2)^{\frac{\alpha-1}{4}} I_{\frac{1-\alpha}{2}}\left(\sqrt{2r}\sqrt{|z|+z_3}\sqrt{s^2-\lambda^2}\right)ds, \qquad (17)\end{aligned}$$

where $|z| = \sqrt{z_1^2+z_2^2+z_3^2}$ *and* $z_1^2+z_2^2 > 0$.

Once again, we can translate the result to obtain the harmonic measures formulas for the relativistic processes

COROLLARY 11 (Corollary 5.7 in [H1]). *Let* $\tilde{H}_2 \subset \mathbf{R}^2$ *be the complement of the half-line* $\{\tilde{x} \in \mathbf{R}^2;\ x_1 = 0,\ x_2 > 0\} \subset \mathbf{R}^2$. *Then m-Poisson kernel* $\tilde{H}_2$ *for the relativistic α-stable process with parameter* $m > 0$ *and* $1 < \alpha < 2$ *is given by*

$$\begin{aligned}P^m_{\tilde{H}_2}(\tilde{y},r) &= \frac{(|\tilde{y}|+y_2)^{\frac{\alpha-1}{4}}(|\tilde{y}|-y_2)^{\frac{\alpha-1}{2}}}{2^{\frac{3(\alpha-1)}{4}}\Gamma(\frac{\alpha-1}{2})r^{(\alpha-1)/4}}\times \\ &\times \int_{m^{\frac{1}{\alpha}}}^\infty e^{-(|\tilde{y}|+r)s}(s^2-m^{2/\alpha})^{\frac{\alpha-1}{4}} I_{\frac{1-\alpha}{2}}\left(\sqrt{2r}\sqrt{|\tilde{y}|+y_2}\sqrt{s^2-m^{2/\alpha}}\right)ds,\end{aligned}$$

where $r > 0$ *and* $\tilde{y} = (y_1, y_2) \in \tilde{H}_2$, $y_1 \neq 0$. *If* $y_1 = 0$ *and* $y_2 < 0$ *we have*

$$P^m_{\tilde{H}_2}(\tilde{y},r) = \frac{\sin(\pi(\alpha-1)/2)}{\pi}\left(\frac{-y_2}{r}\right)^{(\alpha-1)/2}\frac{e^{-m^{1/\alpha}(r-u)}}{r-u}.$$

For $m = 0$ *we obtain the Poisson kernel of* $\tilde{H}_2$ *for the standard isotropic α-stable process given by the formula*

$$\begin{aligned}P_{\tilde{H}_2}(\tilde{y},r) &= \frac{(|\tilde{y}|+y_2)^{\frac{\alpha-1}{4}}(|\tilde{y}|-y_2)^{\frac{\alpha-1}{2}}}{2^{\frac{3(\alpha-1)}{4}}\Gamma(\frac{\alpha-1}{2})r^{(\alpha-1)/4}}\times \\ &\times \int_0^\infty e^{-(|\tilde{y}|+r)s}s^{\frac{\alpha-1}{2}} I_{\frac{1-\alpha}{2}}\left(s\sqrt{2r}\sqrt{|\tilde{y}|+y_2}\right)ds, \qquad (18)\end{aligned}$$

where $r > 0$ *and* $\tilde{y} = (y_1, y_2) \in \tilde{H}_2$, $y_1 \neq 0$. *If* $y_1 = 0$ *and* $y_2 < 0$ *we have*

$$P_{\tilde{H}_2}((0,y_2),r) = \frac{\sin(\pi(\alpha-1)/2)}{\pi}\left(\frac{-y_2}{r}\right)^{(\alpha-1)/2}\frac{1}{r-u}.$$

In the multidimensional case the result is as follows

COROLLARY 12 (Corollary 5.8 in [H1]). *Let $\tilde{H}_n \subset \mathbf{R}^n$ be the complement of the $(n-1)$-dimensional half-space $\{\tilde{x} \in R^n;\ x_1 = 0, x_2 > 0\} \subset \mathbf{R}^n$. Then m-Poisson kernel of $\tilde{H}_n$ for the relativistic α-stable process with parameter z parametrem $m > 0$ and $1 < \alpha < 2$ is given by*

$$P^m_{\tilde{H}_n}(\tilde{y}, \tilde{\sigma}) = \frac{2\sin(\pi(\alpha-1)/2) m^{\frac{n}{2\alpha}}}{2^{\frac{n-1}{2}} \pi^{\frac{n+1}{2}}} \left(\frac{-y_2}{\sigma_2}\right)^{(\alpha-1)/2} \frac{K_{(n-1)/2}(m^{\frac{1}{\alpha}}|\tilde{y}-\tilde{\sigma}|)}{|\tilde{y}-\tilde{\sigma}|^{(n-1)/2}},$$

where $\tilde{y} = (y_1, \dots, y_n); y_1 = 0, y_2 < 0$ and $\tilde{\sigma} = (\sigma_2, \dots, \sigma_n); \sigma_2 > 0$.

For $m = 0$ we obtain the Poisson kernel of the $\tilde{H}_n$ for the standard isotropic α-stable process given by the formula

$$P_{\tilde{H}_n}(\tilde{y}, \tilde{\sigma}) = \frac{\sin(\pi(\alpha-1)/2)\Gamma(\frac{(n-1)}{2})}{\pi^{\frac{n+1}{2}}} \left(\frac{-y_2}{\sigma_2}\right)^{(\alpha-1)/2} \frac{1}{|\tilde{y}-\tilde{\sigma}|^{n-1}}.$$

Note that these formulas were unknown even in the most classical case of the standard isotropic α-stable process (for $m = 0$)

Sharp two-sided estimates on first hitting time density (paper [H2])

The next natural question in this context relates to Bessel processes hitting a point $a > 0$. It turns out that this problem is much more complicated and significantly departs from the case $a = 0$. It should be emphasized that we consider situation when a process starts from $x > a$, i.e. leaving a half-space (a, ∞) (the opposite case when $x < a$ was studied in [P13]). Thus, we consider the whole range of the indices $\mu \in \mathbf{R}$, since in the obvious way we have $T_a < T_0$ $\mathbf{P}^{(\mu)}_x$-a.s. and we do not have to take care of the boundary conditions at zero (i.e. for every boundary condition the result is the same). From the other side, the absolute continuity property (2) allows to reduce the problem to the case $\mu \geq 0$ (or $\mu \leq 0$). It this place, it is worth mentioning that the first hitting times of a given level for Bessel process are related to the geometric Brownian motion. Using the Lamperti representation (see. [51]) we can show

$$T_a \stackrel{d}{=} A^{(\mu)}_x(\tau_a) := x^2 \int_0^{\tau_a} \exp(2\beta_s - 2\mu s)\, ds, \tag{19}$$

where the distribution of T_a is considered with respect to $\mathbf{P}^{(\mu)}_x$. Here β_t is Wiener process starting from 0 and $\tau_a = \inf\{t > 0 : x\exp(\beta_t - \mu t) = a\}$. The integral functional $A^{\mu}_x(t)$ has been a subject of study in substantial number of papers [1], [3], [4],[5], [25], [15], [16], [29], [33], [52], [53], [54], [69], [70] in connection with its wide range of applications in financial mathematics, diffusions theory or representation of hyperbolic Brownian motion. The hitting distributions T_a were studied in 1970s and 1980s by Kent [44], Getoor and Sharp[30], Pitman and Yor [60]. Kent [44] computed the Laplace transform of T_a, which is given by

$$\mathbf{E}^{(\mu)}_x \exp(-\frac{\lambda^2}{2}T_a) = \left(\frac{a}{x}\right)^{\mu} \frac{K_\mu(x\lambda)}{K_\mu(a\lambda)}, \quad \lambda \geq 0.$$

Various ratios of Bessel functions were proved to be completely monotonic functions hence they are Laplace transforms of probability distributions [37], [38], [39]. This topic became popular once again in the beginning of our century [3], [4], [10], since the development of the potential theory of hyperbolic Brownian motion required better description of the density function of $\mathbf{P}^{(\mu)}_x(T_a \in dt)$. The article [H2] fits to this trend and it is devoted to describe the behaviour of the density function of the first hitting time of $a > 0$

by Bessel process with index $\mu \in \mathbf{R}$. Let us denote by $g_{x,a}^{(\mu)}(t)$ the density of the distribution of T_a with respect to the Lebesgue measure, i.e.

$$q_{x,a}^{(\mu)}(t) = \frac{\mathbf{P}_x^{(\mu)}(T_a \in dt)}{dt}.$$

Using the scaling properties of Bessel processes we can write

$$q_{x,a}^{(\mu)}(t) = \frac{q_{x/a,1}^{(\mu)}(t/a^2)}{a^2}, \quad x > a > 0,$$

and it allow us to reduce our considerations to the case $a = 1$. Finding the estimates of $q_{x,a}^{(\mu)}(t)$ directly from the formula for the Laplace transform seems to be impossible. Thus, the key result here is the integral representation of $q_{x,1}^{(\mu)}(t)$ in terms of the modified Bessel functions provided in [11]

Theorem 1 (Byczkowski, Ryznar 2006). *For $\mu \geq 0$ there is a function w_λ such that*

$$q_{x,1}^{(\mu)}(t) = \lambda \frac{e^{-\lambda^2/2t}}{\sqrt{2\pi t}} \left(\frac{x^{\mu-1/2}}{t} + \int_0^\infty \left(e^{-\kappa/2t} - 1\right) w_\lambda(v) dv \right), \tag{20}$$

where $\kappa = \kappa(v) = (\lambda + v)^2 - \lambda^2 = v(2\lambda + v)$, and $\lambda = x - 1$.

The function w_λ appearing in the above-given representation is a sum of two components

$$w_\lambda(v) = w_{1,\lambda}(v) + w_{2,\lambda}(v), \tag{21}$$

where the first one is given in terms of the (complex) zeros $\{z_1, \ldots z_{k_\mu}\}$, $k_\mu \in \mathbf{N}$, of the analytical extension of $K_\vartheta(z)$

$$w_{1,\lambda}(v) = -\frac{x^\mu}{\lambda} \sum_{i=1}^{k_\mu} \frac{z_i e^{\lambda z_i} K_\mu(x z_i)}{K_{\mu-1}(z_i)} e^{z_i v}.$$

The other component has an integral representation

$$w_{2,\lambda}(v) = -\cos(\pi\mu) \frac{x^\mu}{\lambda} \int_0^\infty \frac{I_\mu(xu) K_\mu(u) - I_\mu(u) K_\mu(xu)}{\cos^2(\pi\mu) K_\mu^2(u) + (\pi I_\mu(u) + \sin(\pi\mu) K_\mu(u))^2} e^{-\lambda u} e^{-vu} u du. \tag{22}$$

This theorem is an excellent example showing how much the case $a > 0$ differs from the case $a = 0$. The complexity of these formulas contrasts with the very simple formula for $q_{x,0}^{(\mu)}(t)$ given in (4). From the other side it naturally raises the question of more accessible description of $q_{x,1}^{(\mu)}(t)$ by providing its estimates. Some results related to the asymptotics of $q_{x,1}^{(\mu)}(t)$ can be found in [11]. Note that some deeper analysis of the asymptotic behavior of $q_{x,a}^{(\mu)}(t)$ and the cumulative distributional function of T_a was made in later papers (the results of [36] were announced on arXiv.org 9 month later then [H2]) of Y. Hamana and H. Matsumoto [34], [35], [36]. The main aim of [H2] was to provided sharp two-sided estimates on $q_{x,1}^{(\mu)}(t)$ for every $x > 0$ and $t > 0$ and the result is given in the following theorem. Here $f(t,x) \stackrel{\mu}{\approx} g(t,x)$ means that there exist constants $c_1(\mu)$, $c_2(\mu)$ (depending only on μ), such that $c_1(\mu) g(t,x) \leq f(t,x) \leq c_2(\mu) g(t,x)$ for every x and t.

Theorem 2 (Theorem 2 in [H2]). *For every $\mu \neq 0$ we have*

$$q_{x,1}^{(\mu)}(t) \overset{\mu}{\approx} (x-1)\left(\frac{1}{1+x^{2\mu}}\right)\frac{e^{-(x-1)^2/2t}}{t^{3/2}}\frac{x^{2|\mu|-1}}{(t+x)^{|\mu|-1/2}} \tag{23}$$

for every $x > 1$ and $t > 0$. Moreover

$$q_{x,1}^{(0)}(t) \approx (x-1)\frac{e^{-(x-1)^2/2t}}{t^{3/2}}\frac{\sqrt{x+t}}{x}\frac{1+\ln x}{(1+\ln(1+t/x))(1+\ln(t+x))},$$

for every $x > 1$ and $t > 0$.

We have to indicate that the formulation (not the proof) of Theorem 2 in the published version of the article is incorrect. Instead of $(t+x)^{|\mu|-1/2}$ in the denominator of the right-hand side of (23) there is an expression $t^{|\mu|-1/2} + x^{|\mu|-1/2}$. For $|\mu| \geq 1/2$ both functions are comparable. However, for $|\mu| < 1/2$ there is a mistake, which was noticed and corrected in [H3]. The direct consequences of the result are the sharp two-sided estimates on the density of the first hitting time of unit sphere by the n-dimensional Wiener process

Theorem 3 (Theorem 3 in [H2]). *Let $\sigma^{(n)}$ e the first hitting time of a unit ball by n-dimensional Brownian motion $W^{(n)} = \{W_t^{(n)}, t \geq 0\}$, i.e.*

$$\sigma^{(n)} = \inf\{t > 0; |W_t^{(n)}| = 1\}.$$

Then, for $W_0^{(n)} = x \in \mathbf{R}^n$ such that $|x| > 1$ we have

$$\frac{P^x(\sigma^{(n)} \in dt)}{dt} \approx \frac{|x|-1}{|x|}\frac{e^{-(|x|-1)^2/2t}}{t^{3/2}}\frac{1}{t^{(n-3)/2} + |x|^{(n-3)/2}}, \quad n > 2,$$

for every $t > 0$. Moreover, we have

$$\frac{P^x(\sigma^{(2)} \in dt)}{dt} \approx \frac{|x|-1}{|x|}e^{-(|x|-1)^2/2t}\frac{(|x|+t)^{1/2}}{t^{3/2}}\frac{1+\log|x|}{(1+\log(1+\frac{t}{|x|}))(1+\log(t+|x|))}$$

Once again, the scaling properties implies the corresponding result for spheres with radius $r > 0$. Note that we described very complicated (in the analytical sense) function $q_{x,1}^{(\mu)}(t)$ by very simple elementary functions for the whole range of space and time parameters with constants depending only on the index of the process. On the other side, it seems to be very rare in the diffusion theory to find sharp two-sided estimates which precisely describe the exponential behaviour of the density. Moreover, even the result for the hitting time of a Brownian motion is new. Known results of this kind are usually only quantitatively sharp, i.e. the constants appearing in the exponential terms are different in the lower and upper bounds. Such estimates for Brownian motion in more general setting of Riemannian manifolds are given in [32], where we have the expressions $\exp(-c_i|x|^2/t)$ with different constants c_1 and c_2 in lower and upper bounds, respectively. It makes the estimates precise only for $|x|^2 < t$. Theorem 2 can be treated as an complement of [32], where we remove this defect and we generalize the result for Bessel processes of any dimension (index). Due to (19), we immediately obtain the estimates for the integral functionals of geometric Brownian motion.

Note also the different nature of the estimates for $\mu = 0$. The appearance of the logarithmic function is not surprising if we remember that the Bessel process with index 0 relates to the two-dimensional Brownian motion. Wiener process is quite exceptional just to mention its recurrent character or the logarithmic character of the compensated potentials in comparison with the power potentials in other dimensions. Technical reason for appearance of logarithmic functions for $\mu = 0$ is the fact that the modified Bessel function of the second kind $K_0(z)$ behaves as $-\ln z$ at zero, where for $\mu \neq 0$ we have $K_\mu(z) \sim z^{-\mu}$, as $z \to 0^+$. However, the logarithmic term in (23) matters only when t is large in comparison to x. In the other case, it is comparable with a constant and the estimates for $\mu = 0$ and $\mu \neq 0$ are of the same type.

Finally, in some cases we get more accurate results than it follows from Theorem 2. The following lemma gives the estimates when t/x is small, but in fact it is stronger result, which gives the second term of the asymptotic expansion for $t/x \to 0$.

Lemma 4 (Lemma 4 in [H2]). *For $\mu \geq 0$ we have*

$$q_{x,1}^{(-\mu)}(t) = \lambda \frac{e^{-\lambda^2/4t}}{(2\pi)^{1/2}t^{3/2}} x^{\mu-1/2}\left(1 + \frac{1-4\mu^2}{8}\frac{t}{x} + E(t,x)\right),$$

where the error term satisfies the following estimate

$$|E(t,x)| \leq C\frac{t}{x}(\sqrt{t} \wedge \frac{t}{\lambda}).$$

Moreover, for $0 \leq \mu < 1/2$ we get

$$\lambda \frac{e^{-\lambda^2/2t}}{(2\pi)^{1/2}t^{3/2}} x^{\mu-1/2} \leq q_{x,1}^{(-\mu)}(t) \leq \lambda \frac{e^{-\lambda^2/4t}}{(2\pi)^{1/2}t^{3/2}} x^{\mu-1/2}\left(1 + \frac{1-4\mu^2}{8}\frac{t}{x}\right)$$

for every $x > 1, t > 0$.

Proofs. In general, the proof of Theorem 2 relays on very delicate and often laborious estimates of the elements of the integral representation from [11], which makes the article very technical. However, it is important to emphasize that the main advantage of the paper is very precise and simple to formulate result with technical, laborious and arduous proof. Thus, we will discuss the main steps and ideas of the proof without going into complicated details. Since the absolute continuity property implies

$$q_{x,1}^{(-\mu)}(t) = x^{2\mu} q_{x,1}^{(\mu)}(t), \quad t > 0, x > 1, \tag{24}$$

we will consider only the case of non-positive indices $-\mu$ for $\mu \geq 0$. Moreover, we will use the following notation $\kappa = \kappa(v) = (\lambda + v)^2 - \lambda^2 = v(2\lambda + v)$ and $\lambda = x - 1$ introduced previously in the representation (20). In [H2], we dropped the index 1 in the notation of the density and wrote $q_x^{(\mu)}(t)$ instead of $q_{x,1}^{(-\mu)}(t)$, however we will write $q_{x,1}^{(-\mu)}(t)$ to keep the notation of the Summary consistent. The first step of the proof is to show the formula given in Lemma 4. Using the integral representation (20) together with the formula

$$\int_0^\infty w_\lambda(v)dv = \frac{x^{\mu-1/2}}{2x}(\mu^2 - 1/4),$$

proved in [11], we show that

$$E(x,t) = \frac{t}{x^{\mu-1/2}} \int_0^\infty e^{-\kappa/2t} w_\lambda(v)dv.$$

The estimates of $w_\lambda(v)$ given in the Appendix give then the desired upper-bounds for $E(x,t)$. Additional properties for $0<\mu\leq 1/2$ follows from the simple relations

$$0\leq \int_0^\infty (e^{-\kappa/2t}-1)w_\lambda(v)dv \leq -\int_0^\infty w_\lambda(v)dv = x^{\mu-1/2}(1/4-\mu^2)/2x.$$

The nest step is to extend the result for $x/t>C$ for arbitrary constant $C>0$. Recall that Lemma 4 provides the estimates for $x/t>C$ for C sufficiently large and consequently it remains to show them for $C<x/t<C'$, where C and C' are arbitrary constants.

Lemma 5 (Lemma 6 in [H2]). *For every $C>0$ there exists $c_1>0$ depending on C and $\mu>0$ such that*

$$\frac{1}{c_1}\lambda\frac{e^{-\lambda^2/2t}}{t^{3/2}}x^{\mu-1/2}\leq q_{x,1}^{(-\mu)}(t)\leq c_1\lambda\frac{e^{-\lambda^2/2t}}{t^{3/2}}x^{\mu-1/2},$$

whenever $x>Ct$ and $x>1$.

Unfortunately, the published version of the Lemma contains an error and instead of the correct inequality $x>Ct$ we have opposite one $x<Ct$. Moreover, the rôle of the constants C and C' was reversed in the proof. However, the key part of the proof for $C<x/t<C'$ is correct and it is about showing that the ratio of $q_{x,1}^{(\mu)}(t)$ and the expression on the right-hand side of (23) has a limit when x/t tends to a constant $c>0$ and at the same time $x\to\infty$ (Proposition 5 in [H2]). We explore here the explicit formula for $w_\lambda(v)$. Then, using the open cover method and the absolute continuity property (2) we provide the bounds. The result for $x/t>C'$ for C' sufficiently large (not for sufficiently small as it is written) follows immediately from Lemma 4.

Next two lemmas relates to the case, where t is relatively large in comparison to x and, as we have just mentioned, we have to distinguish $\mu\neq 0$ from $\mu=0$ in our considerations. This part of the work is the most technical and mostly relays on very careful estimates of the integrals. We begin with the special case, where $\mu-1/2\in\mathbf{N}$. The modified Bessel functions $K_{l+1/2}(z)$, $l\in\mathbf{N}$, are then elementary functions and consequently the function $w_\lambda(v)$ simplifies. In particular, $w_{2,\lambda}(v)\equiv 0$, since we have $\cos(\pi\mu)=0$ (see (22)). It leads to the following result.

Lemma 6 (Lemma 7 in [H2]). *Suppose that $\mu-1/2\in\mathbf{N}$. We have the following expansion*

$$q_{x,1}^{(-\mu)}(t)=\frac{(x^{2\mu}-1)}{\Gamma(\mu)2^\mu}e^{-\lambda^2/2t}\frac{1}{t^{\mu+1}}(1+E(x,t)).$$

There is a constant $c>0$ such that for $t>0$,

$$|E(x,t)|\leq c\frac{x}{t}.$$

The proof is based on the integral representation (24) from [11], which is some slight modification of the one given in (20) . It is about careful study of the proof of the asymptotic behaviour of $q_{x,1}^{\mu}(t)$, when $t\to\infty$ and $x>0$ is fixed, which was given in [11]. In our case, the dependence on x of the constant appearing as the limit is crucial and the proof concentrate on this relation. We use here the strong Markov property and the known results for the first hitting time of zero.

For $\mu-1/2\notin\mathbf{N}$ we split $q_{x,1}^{(\mu)}(t)$ into two parts relating to $w_{1,\lambda}(v)$, for which the proof is the same as in Lemma 6, and the one corresponding to $w_{2,\lambda}(v)$. We use the estimates of the integrand (22) to find the bounds of $w_{2,\lambda}(v)$ and we get

Lemma 7 (Lemma 8 in [H2]). *Let $\mu - 1/2 \notin \mathbf{N}$ and let $l = [\mu + 1/2]$. There are constants c_1, c_2, c_3 depending only on μ such that*

$$c_2 \frac{\lambda x^{2\mu-1}}{t^{\mu+1}} e^{-\lambda^2/2t} \left(1 - c_3 \left(\frac{x}{t}\right)^{l-\mu+1/2}\right) \le q_x^{(-\mu)}(t) \le c_1 \frac{\lambda x^{2\mu-1}}{t^{\mu+1}} e^{-\lambda^2/2t},$$

for $t > x > 1$. Note that $l - \mu + 1/2 > 0$.

The result of the last lemma covers the case $\mu = 0$ and the missing bounds for t/x large.

Lemma 8 (Lemma 9 in [H2]). *Let $\mu = 0$. For $t > 2x$,*

$$q_{x,1}^{(0)}(t) \approx \frac{\lambda}{x} \frac{e^{-\lambda^2/2t}}{t} \frac{1 + \log x}{(1 + \log \frac{t}{x})(1 + \log t)}.$$

Once again, the careful estimates of $w_\lambda(v)$ and the successive study of the behaviour of each integral give the result. The sharp estimates of $w_\lambda(v)$, which are crucial here as well as in many other parts of the proof, are given in Appendix (Lemma 14, Lemma 16 and Lemma 17) to the paper.

Applications. The first direct consequence of Theorem 2 are sharp estimates of the survival probabilities, i.e. the probability that the process starting from $x > 1$ does not hit 1 before time $t > 0$. We consider here the case of non-negative indices and using (24) we get

$$\mathbf{P}_x^{(-\mu)}(t < T_1) = x^{2\mu} \mathbf{P}_x^{(\mu)}(t < T_1 < \infty), \quad \mu > 0.$$

In particular, for $\mu > 0$ we have $\mathbf{P}_x^{(\mu)}(T_1 = \infty) = 1 - x^{-2\mu}$.

Theorem 4 (Theorem 4 in [H2]). *let $\mu > 0$. Then, for every $t \ge 0$ and $x > 1$, we have*

$$\mathbf{P}_x^{(\mu)}(t < T_1 < \infty) \approx \frac{x - 1}{\sqrt{x \wedge t} + x - 1} \frac{1}{t^\mu + x^{2\mu}}.$$

Moreover, for every $t \ge 0$ and $x > 1$, we have

$$\mathbf{P}_x^{(0)}(T_1 > t) \quad \approx \quad 1 \wedge \frac{\log x}{\log(1 + t^{1/2})}.$$

Another application of the results are the precise bounds for the Poisson kernel of the half-space for hyperbolic Brownian motion with drift. We consider here a half-space model of real hyperbolic space of dimension n

$$\mathbb{H}^n = \{y = (y_1, \ldots, y_{n-1}, y_n) : y_n > 0\}$$

and the Laplace-Beltrami operator associated with the hyperbolic metric on $\mathbb{H}^n$ given by

$$\Delta_{\mathbb{H}^n} = y_n^2 \sum_{i=1}^n \frac{\partial^2}{\partial y_i^2} - (n-2) y_n \frac{\partial}{\partial y_n}.$$

Hyperbolic Brownian motion with drift is a diffusion on $\mathbb{H}^n$, with the infinitesimal generator of the form $\frac{1}{2}\Delta_{\mathbf{H}^n}$. Replacing the term $(n-2)$ by $(2\mu - 1)$, $\mu > 0$, in the above-given formula we obtain an operator Δ_μ and the corresponding diffusion $Y_t^{(\mu)}$ is called hyperbolic Brownian motion with drift. The more detailed description of hyperbolic spaces and hyperbolic Brownian motion is given in Section 5.2, where we will discuss the potential theory on hyperbolic spaces. Let us consider the set $D = \{y \in \mathbb{H}^n : y_n > 1\}$ (the boundary of D is a horocycle in $\mathbb{H}^n$). We denote by $P^{(\mu)}(y, z)$, $y \in D$, $z \in \partial D$, its Poisson kernel, i.e. the density function (with respect to the surface measure) of a random variable $Y_{\tau_D}^{(\mu)}$, where $\tau_D = \inf\{t > 0 : Y_t^{(\mu)} \notin D\}$ is the first exit time from D.

Theorem 5 (Theorem 5 in [H2]). *For every $\mu > 0$ we have*

$$P^{(\mu)}(y,z) \approx \frac{y_n - 1}{|z-y|^n}\left(\frac{y_n}{\mathrm{cosh} d_{\mathbb{H}^n}(y,z)}\right)^{\mu-1/2}, \tag{25}$$

where $y = (\tilde{y}, y_n)$, $y_n > 1$ and $z = (\tilde{z}, 1)$, $\tilde{z} \in \mathbf{R}^{n-1}$.

The hyperbolic metric appearing above is characterized by the formula

$$\cosh d_{\mathbb{H}^n}(y,z) = 1 + \frac{|y-z|^2}{2y_n z_n}, \quad y, z \in \mathbb{H}^n.$$

The set D is unbounded in this metric and consequently we can not apply the general theory of comparability of potential theories (in particular Poisson kernels and Green functions) for strongly elliptic operators with their classical analogues. Moreover, such comparability does not hold for $\mu \neq 1/2$, and the difference between hyperbolic kernel and Euclidean kernel of D is described by $\left(\frac{y_n}{\mathrm{cosh} d_{\mathbb{H}^n}(y,z)}\right)^{\mu-1/2}$.

Sharp two-sided estimates on heat kernel on half-line (a,∞) (papers [H3] and [H4])

The last part of the habilitation thesis is devoted to provide sharp two-sided bounds on the transition probability density for Bessel process starting from $x > a$ and killed at the first hitting time of $a > 0$, i.e. the function

$$p_a^{(\mu)}(t,x,y) = \mathbf{E}_x^{(\mu)}[t < T_a; X_t \in dy]/m^\mu(dy), \quad x, y > a, \quad t > 0.$$

In other words, we consider Bessel process with index $\mu \in \mathbf{R}$ up to time T_a. The above-given density is described by the Hunt formula in the following way

$$p_a^{(\mu)}(t,x,y) = p^{(\mu)}(t,x,y) - \mathbf{E}_x^{(\mu)}[t > T_a, p^{(\mu)}(t-T_a, R(T_a), y)], \quad x, y > a, \quad t > 0.$$

Continuity of the paths implies that $R(T_a) = a$ a.s. and consequently we can write

$$r_a^{(\mu)}(t,x,y) := \mathbf{E}_x^{(\mu)}[t > T_a, p^{(\mu)}(t-T_a, R(T_a), y)] = \int_0^t p(t-s,a,y) q_{x,a}^{(\mu)}(s)ds. \tag{26}$$

The probabilistic approach is not the only one in this context. In the theory of partial differential equations the function $p_a^{(\mu)}(t,x,y)$ is a fundamental solution to the heat equation based on the Bessel operator, i.e. $\partial_t - \frac{1}{2}L^{(\mu)}$. In the most classical case, when the Bessel operator is replaced by Laplacian, the problem of estimating the Dirichlet heat kernels on subsets of $\mathbf{R}^n$ has a very long history and it is impossible to mention all the results and research on this topic. However, it goes back to 1980s and the works of E.B. Davies [22], [19], [20], [21], through the Q.S. Zhang work [71], where we can find the bounds for $C^{1,1}$ domains, up to the paper [64], where much more general operators were studied in this context. A quite comprehensive description of such results can be found in [65].

The main results of the papers [H3] and [H4] are

Theorem 13 (Theorem 1 in [H3]). *Let $\mu \neq 0$ and $a > 0$. We have*

$$p_a^{(\mu)}(t,x,y) \overset{\mu}{\approx} \left[1 \wedge \frac{(x-a)(y-a)}{t}\right]\left(1 \wedge \frac{xy}{t}\right)^{|\mu|-1/2} \frac{1}{(xy)^{\mu+1/2}} \frac{1}{\sqrt{t}} \exp\left(-\frac{|x-y|^2}{2t}\right), \tag{27}$$

for every $x, y > a$ and $t > 0$.

Theorem 14 (Theorem 1 in [H4]). *For every $a > 0$ we have*

$$p_a^{(0)}(t,x,y) \approx \ln(x/a)\ln(y/a)\left(\ln\frac{3t}{ax+a\sqrt{t}}\ln\frac{3t}{ay+a\sqrt{t}}\right)^{-1}\frac{1}{t}\exp\left(-\frac{x^2+y^2}{2t}\right), \quad xy < t, \tag{28}$$

and

$$p_a^{(0)}(t,x,y) \approx \left(1\wedge\frac{(x-a)(y-a)}{t}\right)\frac{1}{\sqrt{xyt}}\exp\left(-\frac{(x-y)^2}{2t}\right), \quad xy \geq t,$$

where $x, y > a$ and $t > 0$.

Note the different nature of the estimates depending on whether μ is zero or not. This duality causes that the proof of the first part of Theorem 14 require different methods and consequently it has resulted in the separate publication. The both results give bounds for the whole range of space $x, y > a$ and time $t > 0$ parameters. Most importantly, the description of the exponential behaviour of $p_a^{(\mu)}(t,x,y)$ is very precise The known results for a Wiener process [71] (classical Laplacian with Dirichlet boundary conditions) are only quantitatively sharp, i.e. there are different constants in the exponential terms in the upper and lower bounds. Even more surprising, there are no sharp estimates for Laplacian on the two-dimensional ball as well as for heat kernels on manifolds [65]. The precise sharp result for the Fourier-Bessel kernels, i.e. the corresponding transition probability density of Bessel process killed upon leaving the interval $[0,1)$, were provided only recently in [P13]. Note that for $\mu \neq 0$ we can write

$$\frac{p_a^{(\mu)}(t,x,y)}{p^{(\mu)}(t,x,y)} \overset{\mu}{\approx} \left(1\wedge\frac{(x-a)(y-a)}{t}\right)\left(1\vee\frac{t}{xy}\right), \quad x,y>a, \quad t>0,$$

and

$$p_a^{(\mu)}(t,x,y) \overset{\mu}{\approx} \mathbf{P}_x^{(\mu)}(T_a^{(\mu)} > t)\mathbf{P}_y^{(\mu)}(T_a^{(\mu)} > t)p^{(\mu)}(t,x,y), \quad xy \leq t.$$

The last bound above is a very common way to estimate a heat kernel of a subset by a product of survival probabilities and a global heat kernel. Usually, the time variable t in $p^{(\mu)}(t,x,y)$ is multiplied by different constants in the lower and upper bounds. (see. Theorem 5.16 in [65]). In our result, the constants are the same. Moreover, we show that for $xy > t$ the kernel $p_a^{(\mu)}(t,x,y)$ does not behave like the right-hand side of the above-given estimates, i.e. the bounds for $p_a^{(\mu)}(t,x,y)/p^{(\mu)}(t,x,y)$, when $xy > t$, can not be factorize as a product of survival probabilities or any other product of the form $f(t,x)f(t,y)$ for some function f. It is caused by the accuracy of our estimates.

The absolute continuity property let us consider only the case, where μ has a constant sign. From technical reasons, we choose this time μ to be non-negative.

Proof of Theorem 13 is divided in the natural way into two parts, related to xy/t small and large respectively. The case $\mu = 1/2$ is the starting point for our considerations, since in this case the function $I_{1/2}(z)$ as well as $q_{x,1}^{(1/2)}(t)$ can be expressed in terms of elementary functions. Consequently, by the Hunt formula (26), it leads to

$$p_1^{(1/2)}(t,x,y) = \frac{1}{\sqrt{2\pi t}}\frac{y}{x}\left(\exp\left(-\frac{(x-y)^2}{2t}\right) - \exp\left(-\frac{(x+y-2)^2}{2t}\right)\right)$$

and immediately implies the estimates for $\mu = 1/2$. By the absolute continuity property, we also get

Proposition 15 (Proposition 1 in [H3]). *For $\mu \geq 1/2 \geq \nu \geq 0$ we have*

$$\left(\frac{y}{x}\right)^{\mu-1/2} p_1^{(\mu)}(t,x,y) \leq p_1^{(1/2)}(t,x,y) \leq \left(\frac{y}{x}\right)^{\nu-1/2} p_1^{(\nu)}(t,x,y)$$

for every $x, y > 1$ and $t > 0$.

The inequalities of the similar kinds hold also for the density of the first hitting times and are given in Lemma 1 in [H3]. The above-given bounds are optimal, i.e. they give the upper-bounds for $\mu \geq 1/2$ and the lower-bounds for $\nu \leq 1/2$ from (27), whenever $xy \geq t$. The proof of the remaining estimates (lower-bounds for $\mu \geq 1/2$ and upper-bounds for $\nu \leq 1/2$) is also divided into cases relating to the distance of the space variables x and y to the boundary of $(1, \infty)$. Assuming that x and y are bounded away from 1, the right-hand side of (27) is comparable with $p^{(\mu)}(t,x,y)$ and the estimates in this case are given in

Proposition 16 (Proposition 3 in [H3]). *Let $\mu \geq 1/2 \geq \nu > 0$. Then there exist constants $C_1^{(\nu)}, C_2^{(\mu)} > 0$ and $C_3^{(\mu)} > 1$ such that*

$$C_1^{(\nu)} \left(\frac{x}{y}\right)^{\nu+1/2} p_1^{(\nu)}(t,x,y) \leq \frac{1}{\sqrt{t}} \exp\left(-\frac{(x-y)^2}{t}\right) \leq C_2^{(\mu)} \left(\frac{x}{y}\right)^{\mu+1/2} p_1^{(\mu)}(t,x,y),$$

whenever $xy \geq t$ and the upper bounds are valid with additional assumption $x, y > C_3^{(\mu)}$.

The proof for $1/2 \geq \nu \geq 0$ follows immediately from the obvious relation $p^{(\nu)}(t,x,y) \leq p^{(\nu)}(t,x,y)$. For $\mu \geq 1/2$ we show that $r^{(\mu)}(t,x,y) \leq \frac{1}{2} p^{(\mu)}(t,x,y)$ and consequently $p_1^{(\mu)}(t,x,y) \geq \frac{1}{2} p^{(\mu)}(t,x,y)$. Here we use the results of Proposition 15, the mentioned inequalities for the first hitting times densities (Lemma 1 in [H3]) and the formula for $\mu = 1/2$. When x and y are bounded (i.e. close to boundary) the bounds are provided in

Proposition 17 (Proposition 4 in [H3]). *For fixed $m > 0$ and $\mu \geq 1/2 \geq \nu > 0$ there exist constants $C_4^{(\mu)}, C_4^{(\nu)} > 0$ such that*

$$C_4^{(\mu)} \left(\frac{x}{y}\right)^{\mu+1/2} p_1^{(\mu)}(t,x,y) \geq \left(1 \wedge \tfrac{(x-1)(y-1)}{t}\right) \tfrac{1}{\sqrt{t}} \exp\left(-\tfrac{(x-y)^2}{2t}\right)$$

and

$$\left(1 \wedge \frac{(x-1)(y-1)}{t}\right) \frac{1}{\sqrt{t}} \exp\left(-\frac{(x-y)^2}{2t}\right) \geq C_4^{(\nu)} \left(\frac{x}{y}\right)^{\nu+1/2} p_1^{(\nu)}(t,x,y)$$

whenever $(x \wedge y)^2 \geq mt$.

Once again we use (2) and we estimate the integral functional $\int_0^t R^{-2}(s)ds$ up to time T_b, where $b = ((x \wedge y) + 1)/2$, using the obvious relation $T_1 \geq T_b$. The scaling property allow us to change $p_b^{(\mu)}(t,x,y)$, appearing in the estimates as the result of replacing T_1 by T_b, back to $p_1^{(\mu)}(t,x,y)$. Moreover, for $xy \geq t$, $x, y < C$ for a fixed $C > 1$, we have $(x \wedge y)^2 \geq C^{-1}xy \geq C^{-1}$, and we can use the above-given result (for $m = 1/C$) to obtain the required bounds.

The last case, i.e. when one of the space variable is close to 1 and the other is large (unbounded), is studied in the following two propositions related to $\nu \in (0, 1/2)$ and $\mu \geq 1/2$ respectively.

Proposition 18 (Proposition 5 in [H3]). *For $\nu \in (0, 1/2)$ there exists constant $C_5^{(\nu)} > 0$ such that*

$$p_1^{(\nu)}(t,x,y) \le C_5^{(\nu)} \frac{1}{\sqrt{t}} \left(\frac{y}{x}\right)^{\nu+1/2} \exp\left(-\frac{(x-y)^2}{2t}\right) \left(1 \wedge \frac{(x-1)(y-1)}{t}\right)$$

for $1 < x \le 2 \le y$ and $xy \ge t$.

Proposition 19 (Proposition 6 in [H3]). *For every $\mu \ge 1/2$ and $c > 1$ there exists $C_6^{(\mu)}(c) > 0$ such that for every $1 < x \le c$ i $y \ge 5c(\mu+1)$ we have*

$$p_1^{(\mu)}(t,x,y) \ge C_6^{(\mu)}(c) \frac{1}{\sqrt{t}} \left(\frac{y}{x}\right)^{\mu+1/2} \exp\left(-\frac{(x-y)^2}{2t}\right) \left(1 \wedge \frac{(x-1)(y-1)}{t}\right),$$

whenever $xy \ge t$.

The proofs are in general analytical and based on a delicate analysis of the Hunt formula and an estimates of $r_1^{(\mu)}(t,x,y)$ taking into account the cancellations between $p^{(\mu)}(t,x,y)$ and $r_1^{(\mu)}(t,x,y)$, but at the same we can not lose control on the exponential behaviour of the density. We use the explicit formula for $p^{(\mu)}(t,x,y)$ and some properties of $I_\mu(z)$. In particular, we apply the estimates for the ration of the modified Bessel functions $I_\mu(x)/I_\mu(y)$ proved by A. Laforgia in [50].

We start the proof of (27) for xy/t small with providing the upper-bounds.

Proposition 20 (Proposition 7 in [H3]). *For every $\mu > 0$there exists constant $C_7^{(\mu)} > 0$ such that*

$$p_1^{(\mu)}(t,x,y) \le C_7^{(\mu)} \frac{x-1}{x} \frac{y-1}{y} \left(\frac{y^2}{t}\right)^{\mu+1/2} \frac{1}{\sqrt{t}} \exp\left(-\frac{x^2+y^2}{2t}\right),$$

whenever $xy \le t$.

In the easiest case, i.e. for $x, y > 2$ (both variables bounded away from 1), the inequality is just $p_1^{(\mu)}(t,x,y) \le p^{(\mu)}(t,x,y)$. Note that for $xy \le t$ assuming additionally $x, y < 2$, the expression $|x-y|^2/t$ is bounded and consequently the exponential terms in (27) vanish, which simplifies the consideration. We only have to take into account the decay of $p_1^{(\mu)}(t,x,y)$ near boundary. Using the simple inequality $p_1^{(\mu)}(t,x,y) \le p^{(\mu)}(t,x,y) \le c_1 \frac{y^{2\mu+1}}{t^{\mu+1}}$ together with the Chapmann-Kolmogorov equation we obtain the bounds in terms of the product of survival probabilities and $p^{(\mu)}(t,x,y)$. Then, the estimates from Theorem 4from [H2] give the result. Finally, for $1 < x < 2$ and $y \ge 2$ we apply the estimates for first hitting time density given in Theorem 2 and we make delicate estimates of the obtained expressions.

The case of the lower-bounds for $xy \le t$ is divided into two propositions separately covering the situation when $(y-1)^2/t$ is large and small.

Proposition 21 (Proposition 8 in [H3]). *For every $\mu > 0$ and $m \ge 1$ there exists constant $C_8^{(\mu)}(m) > 0$ such that*

$$\frac{p_1^{(\mu)}(t,x,y)}{p^{(\mu)}(t,x,y)} \ge C_8^{(\mu)}(m) \frac{x-1}{x}, \quad y > x > 1,$$

whenever $xy < mt$ and $\frac{(y-1)^2}{t} \ge 2(\mu+1)$.

Proposition 22 (Proposition 9 in [H3]). *For every $\mu > 0$ there exists $C_9^{(\mu)} > 0$ such that*

$$\frac{p_1^{(\mu)}(t,x,y)}{p^{(\mu)}(t,x,y)} \geq C_9^{(\mu)} \frac{x-1}{x}\frac{y-1}{y}, \qquad y > x > 1,$$

whenever $xy < t$ and $\frac{(y-1)^2}{t} \leq 2(\mu+1)$.

The proof of Proposition 21 is analytical and once again based on the properties of the modified Bessel functions and delicate estimates of the expressions appearing in the Hunt formula. The second part follows from the Chapmann-Kolmogorov equation where we change the range of integration in such way that we can use the previously proved estimates for $xy > t$. It leads to split the double integral into product of two depending only on x and y respectively. Moreover, these integrals can be estimated by using Proposition 21.

Applications. The bounds from Theorem 13 can be used to obtain sharp two-sided estimates of the λ-Green function of the half-space (the set bounded by a horocycle) for the hyperbolic Brownian motion with drift. This result was obtained in [P12] described below.

Proof of Theorem 14 relates only to the case $xy < t$. As we have mentioned, the estimates for xy/t are of the same form for every $\mu \in \mathbf{R}$ and the proofs given in [H3] can be directly applied for $\mu = 0$. In [H4] they are presented in the shorten version adapted to the case $\mu = 0$ (Propositions 4.1 and 4.2 in [H4]) for the completeness of the exposure and convenient of the Reader. Thus, we will not discuss them here.

We begin the description of the logarithmic behaviour for $xy < t$ with a simple observation that for $1 < x < y$ and $xy < t$ we have $x^2 < t$ and consequently, the estimates (28) can be rewritten in the simpler following forms

$$p_1^0(t,x,y) \approx \frac{\ln x \ln y}{\ln^2 3t} p^0(t,x,y), \qquad y^2 \leq t,$$

and

$$p_1(t,x,y) \approx \frac{\ln x}{\ln(3t/y)} p(t,x,y), \qquad y^2 > t.$$

We start with the first one, i.e. the case where y^2/t is bounded above by a constant.

Proposition 23 (Proposition 3.1 in [H4]). *For every $m \geq 2$ we have*

$$p_1^0(t,x,y) \overset{m}{\approx} \frac{\ln x \ln y}{\ln^2(3t)} p^0(t,x,y),$$

where $xy < t$ and $y^2 \leq mt$.

The upper-bounds can be obtained by applying (two times) the Chapmann-Kolmogorov equation, the estimate

$$p_1^{(0)}(t,x,y) \leq p^{(0)}(t,x,y) \leq 1/t$$

and the estimates of the survival probabilities (Theorem 4). The lower-bounds require changing the range of integration in the Chapmann-Kolmogorov equation to $[2\sqrt{mt}, 3\sqrt{mt}]$ and the following estimates

$$p_1^{(0)}(t,x,y) \geq c_m \frac{\ln x}{\ln(3t)} p^{(0)}(t,x,y), \quad 1 < x < y/2, \quad y > 1 + 2\sqrt{t}.$$

The above-given bound follows from the Hunt formula and the inequalities for the rations $I_0(z)/I_0(w)$ given in [50]. We also use here the monotonicity of $x \to p^{(0)}(t,x,y)$ for $2x < y$, $y^2 > 4t$ (see Remark 2.2 in [H4]), which follows from the estimates for the derivative of the function given in Lemma 2.1 in [H4]. The most difficult and the most important part of the proof relates to the case, where y^2/t is sufficiently large.

Proposition 24 (Proposition 3.2 in [H4]). *We have*

$$p_1^{(0)}(t,x,y) \approx \frac{\ln x}{\ln(3t/y)}\, p^{(0)}(t,x,y),$$

where $xy < t$ *and* $y^2 \geq 16t$.

We write

$$\frac{p_1^{(0)}(t,x,y)}{p^{(0)}(t,x,y)} \approx \frac{p^{(0)}(t,x,y) - p^{(0)}(t,1,y)}{p^{(0)}(t,1,y)} + \int_t^\infty q_{x,1}^{(0)}(s)ds + \int_0^t \frac{p^{(0)}(t,1,y) - p^{(0)}(t-s,1,y)}{p^{(0)}(t,1,y)} q_{x,1}^{(0)}(s)\, ds.$$

and reduce the problem to estimating the above-given three components. The first one, relating to the explicit formula for the probability transition density, can be simply estimated using the Lagrange theorem. The behaviour of the second one is known by the estimates of the survival probabilities. However, both of them are dominated by the third one. We provide the bounds for the integral component by splitting the range of integration into $[0,2x)$, $[2x, 4t^2/y^2)$, $[4t^2/y^2, t]$ and estimating the integrand separately on each of them. We show that for $0 < s < 4y^2/t^2$ we have

$$\frac{p^{(0)}(t,1,y) - p^{(0)}(t-s,1,y)}{p^{(0)}(t,1,y)} \approx s\frac{y^2}{t^2},$$

by delicate and quite technical estimates of the expressions involving the function $I_0(z)$. The bounds for $q_{x,1}^{(0)}(s)$ from Theorem 2 let us find the bounds for the integrals over $[0,2x)$ and $[2x, 4t^2/y^2)$. In the case $s > 4y^2/t^2$, once again analysing the formula for $p^{(0)}(t,x,y)$, we show that

$$\frac{p^{(0)}(t,1,y) - p^{(0)}(t-s,1,y)}{p^{(0)}(t,1,y)}$$

is comparable with a constant and consequently we get the estimates of the integral over $[4t^2/y^2, t]$ using the bounds for $q_{x,1}^{(0)}(s)$ from [H2]. This ends the proof.

5. Description of other scientific achievements

Besides the four papers, which constitute mono-thematic series of publications, after Ph.D., I published thirteen articles and additional one article was accepted and it has been waiting for publication since March 2015. Total number of my publications is 20, the number of citations, according to the Web of Science database ('Sum of the Times Cited' on 2016-04-13), is 98 (71 without auto-citations), and the h-index (Hirsh index) is 6. Total *impact factor* of the journals for four publications constituting the *scientific achievement*, according to the Journal Citation Reports, is 3,576; total *impact factor* of the journals for all 20 publications is 18,044, see Table 1.

Table 1: Impact factor of the journals according to Journal Citation Report from the publication year (or the year in parentheses for publications from 2016)

publication	journal	publication year	impact factor
[H1]	Potential Anal.	2010	0,853
[H2]	Potential Anal.	2013	1,048
[H3]	Potential Anal.	2015	0,992
[H4]	Math. Nachr.	2016	0.683 (2015)
[P1]	Colloq. Math.	2010	-
[P2]	Proc. London Math. Soc.	2010	1.243
[P3]	Demonstratio Math.	2012	–
[P4]	J. Differential Equations	2012	1.480
[P5]	Colloq. Math.	2012	0.403
[P6]	J. Math. Phys.	2013	1.176
[P7]	Stoch. Proc. Appl.	2013	1.046
[P8]	Annals Probab.	2013	1.431
[P9]	Rev. Math. Phys.	2013	1.448
[P10]	Electron. J. Probab.	2014	0.765
[P11]	Ann. Inst. Henri Poincaré (B)	2015*	1.059
[P12]	Studia Math.	2015	0.610
[P13]	J. Math. Anal. Appl.	2016	1.120 (2015)
[P14]	J. Math. Anal. Appl.	2016	1.120 (2015)
[D1]	Trans. Amer. Math. Soc.	2009	1,060
[D2]	Potential Anal.	2007	0.507
		Sum:	18,044

* - article [P11] has been waiting for publication since March 2015.

[P1] T. Byczkowski, J. Małecki, T. Żak, *Feynman-Kac formula, λ-Poisson kernels and λ-Green functions of half-spaces and balls in hyperbolic spaces*, Colloq. Math. 118, 201–222 (2010).

[P2] T. Kulczycki, M. Kwaśnicki, J. Małecki, A. Stós, *Spectral properties of the Cauchy process on half-line and interval*, Proc. London Math. Soc. 101(2), 589–622 (2010).

[P3] J. Małecki, G. Serafin, *Hitting hyperbolic half-space*, Demonstratio Math. 45(2), 337–360 (2012).

[P4] J. Lörinczi, J. Małecki, *Spectral properties of the massless relativistic harmonic oscillator*, J. Differential Equations 253, 2846-–2871 (2012).

[P5] T. Byczkowski, J. Chorowski, P. Graczyk, J. Małecki, *Hitting half-spaces or spheres by the Ornstein-Uhlenbeck type diffusions*, Colloq. Math. 129, 145–171 (2012).

[P6] P. Graczyk, J. Małecki, *Multidimensional Yamada-Watanabe theorem and its applications to particle systems*, J. Math. Phys. 54, 021503 (2013)

[P7] M. Kwaśnicki, J. Małecki, M. Ryznar, *First passage times for subordinate Brownian motions*, Stoch. Proc. Appl. 123(5), 1820—1850 (2013).

[P8] M. Kwaśnicki, J. Małecki, M. Ryznar, *Suprema of Lévy processes*, Annals Probab. 41(3B), 2047–2065 (2013).

[P9] K. Kaleta, M. Kwaśnicki, J. Małecki, *One-dimensional quasi-relativistic particle in the box*, Rev. Math. Phys. 25(8), 1350014 (2013).

[P10] P. Graczyk, J. Małecki, *Strong solutions of non-colliding particle systems*, Electron. J. Probab. 19(119), 1–21 (2014).

[P11] L. Chaumont, J. Małecki, *On the asymptotic behavior of the density of the supremum of Lévy processes*, Ann. Inst. Henri Poincaré (B), in print, http://imstat.org/aihp/accepted.html (2015).

[P12] K. Bogus, T. Byczkowski, J. Małecki, *Sharp estimates of Green function of hyperbolic Brownian motion*, Studia Math. 228(3), 197–222 (2015).

[P13] J. Małecki, G. Serafin, T. Żórawik *Fourier-Bessel heat kernel estimates*, J. Math. Anal. Appl. 439(1), 91-102 (2016).

[P14] K. Kaleta, M. Kwaśnicki, J. Małecki, *Asymptotic estimate of eigenvalues of pseudo-differential operators in an interval* J. Math. Anal. Appl. 439(2), 896–924 (2016).

Before Ph.D. I published the following two papers, which will not be discussed here.

[D1] T. Byczkowski, J. Małecki, M. Ryznar, *Bessel potentials, hitting distributions and Green functions*, Trans. Amer. Math. Soc. 361, 4871–4900 (2009).

[D2] T. Byczkowski, J. Małecki, *Poisson kernel and Green function of the ball in real hyperbolic spaces*, Potential Anal. 27(1), 1–26 (2007).

I will now discuss the results obtained in the papers [P1]-[P14]. For clarity, the description is divided into four sections relating to my main research interests.

5.1 Bessel process on an interval

The direct continuation of the papers [H1]-[H4] is [P13], where the estimates of the Fourier-Bessel heat kernels, i.e. the probability transition density for Bessel process killed upon leaving the interval $[0, 1)$, were studied. Since we now deal with the bounded set, the kernel has its representation in terms of eigenvalues and eigenfunctions, which in this case are given by the Bessel functions of the first kind $J_\nu(z)$ and its zeros $\lambda_{n,\nu}$.

$$G_t^\nu(x,y) = 2(xy)^{-\nu}\sum_{n=1}^{\infty}\exp\left(-\lambda_{n,\nu}^2 t\right)\frac{J_\nu(\lambda_{n,\nu}x)J_\nu(\lambda_{n,\nu}y)}{|J_{\nu+1}(\lambda_{n,\nu})|^2},\quad x,y\in(0,1),\quad t>0,$$

where $\nu > -1$. Fourier-Bessel expansions appeared in the famous book of J. Fouriera [27] in 1822 to study the heat propagation in cylinders. However, such expansions had been used even before, for example by D. Bernoulli (1732) to describe an oscillation of a hanging chain or L. Euler (1764) to examine vibrations of circular diaphragm. Despite the fact that it is an element of such classical theory the estimates of the Fourier-Bessel heat kernel for the whole range of parameters were proved only recently in [58], [59]. Once again the result is only quantitatively sharp, i.e. we have different constants in the exponential terms in upper and lower bounds. Note that the estimates for large ts can be quite easily obtained from the above-given spectral representation, since the first term decides about the behaviour of the whole series. However, for t small, this representation becomes of no use, because the terms of the series are highly oscillating and all of them matter. This is way we use the probabilistic approach. In particular, the Hunt formula for $G_t^\nu(x,y)$ enables us to find sharp two-sided estimates for the whole range of parameters of the form (Thm 1.1 in [P13])

$$G_t^\nu(x,y) \stackrel{\nu}{\approx} \frac{(1+t)^{\nu+2}}{(t+xy)^{\nu+1/2}}\left(1\wedge\frac{(1-x)(1-y)}{t}\right)\frac{1}{\sqrt{t}}\exp\left(-\frac{|x-y|^2}{4t}-\lambda_{1,\nu}^2 t\right),$$

for every $x,y\in(0,1)$ and $t>0$, where $\nu>-1$. We simultaneously provide sharp two-sided estimates for the transition probability density of Bessel process starting from $x<a$ killed at the first hitting time of a (Corollary 1.1 in [P13]) and analogous result for the first exit time from the interval $(0,1)$ (Corollary 1.2 in [P13]). In both results we precisely describe the exponential behaviour.

5.2 Potential theory on hyperbolic spaces

The publications [P1], [P3], [P5],[P12] are continuations of the research started in my Ph. D. thesis (see [D2]) and they are related to the potential theory of hyperbolic Brownian motion. In particular, we focus on finding descriptions of Poisson kernels and Green functions of balls, sets bounded by horocycles and half-spaces. The model of hyperbolic half-space is a set $\mathbb{H}^n = \{x\in\mathbf{R}^n : x_n>0\}$ equipped with the Riemann metric given by $ds^2 = dx^2/x_n^2$. The hyperbolic distance is then determined by the equation

$$\cosh(d_{\mathbb{H}^n}(x,y)) = 1+\frac{|x-y|^2}{2x_ny_n},\quad x,y\in\mathbb{H}^n,$$

and the Laplace-Beltrami operator (defined as the divergence of the gradient) is the second order differential operator

$$\Delta_{\mathbb{H}^n} = x_n^2\sum_{k=1}^{n}\frac{\partial^2}{\partial x_k^2}-(n-2)x_n\frac{\partial}{\partial x_n}.$$

In the disc model $\mathbb{D}^n = \{x\in\mathbf{R}^n : |x|<1\}$ with the metric $ds^2 = |dx|^2/(1-|x|^2)^n$ the hyperbolic distance is given by

$$\cosh(2d_{\mathbb{D}^n}(x,y)) = 1+\frac{2|x-y|^2}{(1-|x|^2)(1-|y|^2)},\quad x,y\in\mathbb{D}^n,$$

and the corresponding Laplace-Beltrami operator on $\mathbb{D}^n$ is

$$\Delta_{\mathbb{D}^n} = (1-|x|^2)^2 \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2} + 2(n-2)(1-|x|^2) \sum_{k=1}^{n} x_k \frac{\partial}{\partial x_k}.$$

Both models are isomorphic. However, it is more convenient to deal with balls in the disc model and with horocycles and half-spaces in the half-space model $\mathbb{H}^n$. Hyperbolic Brownian motion (HBM) $X = (X_t)_{t \geq 0}$ is the diffusion on $\mathbb{H}^n$ (on $\mathbb{D}^n$ respectively) with $\frac{1}{2}\Delta_{\mathbb{H}^n}$ (resp. $\frac{1}{2}\Delta_{\mathbb{D}^n}$) as its infinitesimal generator. If we replace the factor n by $2\mu+1$ in the formulas for Laplace-Beltrami operators, we will say about hyperbolic Brownian motion with drift. Recall that for $D \subset \mathbb{H}^n$ (resp. $\mathbb{D}^n$) and $\lambda \geq 0$ we define λ-harmonic measure as

$$P_D^\lambda(x, A) = \mathbf{E}^x[e^{-\lambda \tau_D}; X_{\tau_D} \in A], \quad x \in D, \quad A \in \partial D,$$

and λ-Green function of a set D is an integral kernel of the λ-Green operator

$$G_D^\lambda f(x) = \int_0^\infty e^{-\lambda t} \mathbf{E}[t < \tau_D; f(X_t)]\, dt, \quad x \in D.$$

Here τ_D is the first exit time of the process from D. In [P1] we unify, sort and expand the results of [D1], [10], [72] related to the harmonic measures and the Green functions of balls $D_r = \{x \in \mathbb{D}^n : |x| < r < 1\}$ and half-spaces $D_a = \{x \in \mathbb{H}^n : x_n > a > 0\}$. The balls with Euclidean radius $r < 1$ coincide with the centred hyperbolic balls (with hyperbolic radius $\frac{1}{2}\ln\frac{1+r}{1-r}$). The results for balls with any centre can be immediately obtained by using the group of hyperbolic isometrics and the invariance property of HBM. Using the Feynmann-Kac techniques we derive formulas for the Fourier transform λ-harmonic measure of D_a and the corresponding λ-Green function in terms of the modified Bessel functions $I_\mu(z)$ and $K_\mu(z)$. The results related to harmonic measures generalize those given in [10], where the formula for the Poisson kernel was provided (the case of $\lambda = 0$) and the characterization of λ-Green function had not been known before (even for $\lambda = 0$). The description of λ-harmonic measure and λ-Green function of a ball D_r is based on finding their Gegenbauer transform (i.e. the expansion of the functions in series of ultraspherical polynomials). Representing the process in terms of its radial and spherical parts we reduce the problem to solving the appropriate second order differential equation, which leads to the formulas in terms of the hypergeometric function ${}_2F_1$. These results generalize the results from [D2], where only the case $\lambda = 0$ was studied. We do a similar analysis in the complex case, which expands the results from [72] for any $\lambda > 0$.

The paper [P3] is devoted to study HBM with drift on $\mathbb{H}^n$ killed upon leaving the hyperbolic half-space, i.e. the set $D = \{x \in \mathbb{H}^n : x_1 > 0\}$. The name "hyperbolic half-space" is justified by the fact that there exists an isometry of $\mathbb{H}^n$ transforming D into the interior of its complement and D is in this sense a"'half"' of the space. We start with proving the reflection principle and we use it to find the transition probability density of HBM killed when leaving D. Then we provide formulas for the Green function and the Poisson kernel of D in terms of the Legendre functions $Q_a^b(z)$ and $P_a^b(z)$. These representations lead to sharp two-sided estimates and asymptotics behaviour descriptions of the objects. In the last part of the paper we apply the results to study λ-harmonic measure and λ-Green function of D.

The aim of the paper [P5] was to describe the general method leading to explicit formulas for the hitting distributions of a set $D \subset \mathbf{R}^n$ for Ornstein-Uhlenbec type processes, i.e. diffusions in $\mathbf{R}^n$ with $\Delta + F(x)\cdot\nabla$ as its infinitesimal generator. Here we assume additionally that the vector field F is orthogonal to the boundary of D. In particular, the method leads to new integral representations for the Poisson kernel of a half-space and a ball for HBM. Using new formulas we show some estimates and asymptotics of the kernel. The full description by providing sharp two-sided estimates are given in Theorem 5 from [H2]. Additionally, we apply the method to find the hitting distribution of a ball by classical Ornstein-Uhlenbeck process (i.e. for $F(x) = \lambda x$, $\lambda \in \mathbf{R}$).

The characterization of the λ-Green function of a half-space from [P1] is very complicated and it is hard to use it to provide estimates. Thus, the aim of the paper [P12] was to fulfil this gap and describe the Green function by providing sharp both-sided estimates for the whole range of parameters. Note once again that the considered set is unbounded and thus we can not apply the general theory of comparability of potential theory for strongly elliptic operators [17]. However, we use the representation of the HBM with drift on $\mathbb{H}^n$ in terms of geometric Brownian motion, subordinated Wiener process of codimension 1 and Lamperti theorem to reduce the problem to study the transition probability density for a Bessel process killed at point a. Thus, the main result of [H3] can be here applied to prove the desired bounds. Then, we aplly the estimates of the Green function for HBM with drift to find the estimates for the λ-Green function of the set.

5.3 Spectral analysis of jump processes

The papers [P2], [P4], [P9] and [P14] are devoted to spectral analysis of operators related to one-dimensional jump processes. On the one side, the research contribute in the theory of non-local operators. On the other side, the knowledge of the semigroup of a killed process provides deeper understanding of its behaviour and properties. This subject was intensively developed in the last years by, among others, Bañuelos, Chen, DeBlassie, Kulczycki, Song in the context of α-stable processes and related processes.

In the paper [P2] we study the Cauchy process on a half-space using the relations of the generator of the process, i.e. the fractional Laplacian ($\alpha = 1/2$) and the Dirichlet-Neumann operator for two-dimensional Laplacian in the upper half-space. Translating the one-dimensional problem to the corresponding two-dimensional one for a Wiener process (with appropriate boundary conditions) we provide explicit formula for generalized eigenfunctions for square root of Laplacian on a half-line. Then we use the result to approximate the eigenvalues and eigenfunctions for the operator on the interval $(-1, 1)$. I particular, we show that all the eigenvalues are simple and we derive their asymptotics. We also prove the upper and lower bounds for the eigenvalues using the Rayleigh-Ritz method and the approximation based on Legendre polynomials, which then are applied to numerical computations.

We continue the research on $-\sqrt{-\frac{d^2}{dx^2}}$ in [P4], however this time we do not kill the process upon leaving a set, but we consider killing with time by adding the Schrödinger potential of the form x^2. Then the killed semigroup generated by the operator is compact and it has a discreet spectrum. Using the Fourier transform we reduce the problem of finding the eigenvalues and eigenfunctions to solving the Airy equation and consequently we identify the eigenvalues as the appropriate zeros of Airy functions and the eigenfunctions as integrals of Airy functions $Ai(z)$. In the paper [P4] we provide numerical computations and the asymptotic expansions for the eigenvalues as well as the asymptotics of their trace and the estimates of the spectral gap, i.e. the difference between the first two eigenvalues. We derive very accurate asymptotic expansions for the eigenfunctions at infinity, their Maclaurin series expansions and we prove the uniform boundedness of the eigenfunctions. Moreover, we show that every eigenfunction has finite number of zeros, the first eigenfunction (the ground state) decreases on $[0, \infty)$ and it is concave on some neighbourhood of zero and convex for sufficiently large arguments. We finish the article with some analysis of the transition probability density $p(t, x, y)$ of the process showing its continuity on $[0, \infty) \times \mathbf{R}^2$ and providing sharp two-sided estimates for $t > 1$.

The articles [P9] and [P14] continue the research initiated in [P2] and further developed in [49] related to approximation of the eigenfunctions on intervals λ_n by the generalized eigenfunctions on a half-line derived in [48]. This method gives asymptotics of the eigenvalues λ_n, when $n \to \infty$. In the paper [P9] we consider the special case of the Klein-Gordon square-root operator (8) (i.e. quasi-relativistic Hamiltonian), which has very important applications in Physics. In particular, the eigenvalues for the Klein-Gordon operator on intervals $(-a, a)$ relate to the energy levels of quasi-relativistic particle in the infinite square potential well. We show that all the energy levels E_n are non-generated (i.e. the eigenvalues are simple) and E_n behaves at infinity as $(\frac{n\pi}{2} - \frac{\pi}{8})\frac{\hbar c}{a} + O(1/n)$ (we assume here that $m = 1$). Moreover, we provide the estimates, which give control on the lower order component at the expense of transparency of the main term or its accuracy. Finally, we show the uniform boundedness of the eigenfunctions. In [P14] we expand the results for the operators of the form $\psi(-\Delta)$, where ψ is a complete Bernstein function such that $\xi\psi(\xi)$ tends to infinity when $\xi \to \infty$. From probabilistic point of view, we consider subordinated Brownian motions, where the Laplace exponent of the subordinator is $\psi(\xi)$. We prove that the eigenvalues for $\psi(-\Delta)$ on interval $(-1, 1)$ fulfils $\lambda_n = \psi(\mu_n) + O(1/n)$, where the sequence μ_n is defined by a functional equation, which enable us to find the asymptotics of μ_n at infinity. Since the considered problem is much more general, the description here is less precise. However, such accurate control on the constants appearing in the estimates as in [P9] is not available in full generality.

5.4 Supremum functional of Lévy processes

In the papers [P7], [P8], [P11] we consider a one-dimensional Lévy motion $X = (X_t)_{t\geq 0}$, i.e. a stochastic process starting from zero and having independent, stationary increments and càdlàg trajectories. We denote by $\Psi(\xi)$ its characteristic Lévy-Khintchine exponent, i.e.

$$\mathbf{E}[\exp(i\xi X_t)] = e^{-t\Psi(\xi)}, \quad \xi \in \mathbf{R}.$$

One of the most important object of the fluctuation theory of Lévy process is a supremum functional $\overline{X_t} = \sup\{X_s : 0 \leq s \leq t\}$. The supremum functional increases only at first passage times $\tau_a = \inf\{t \geq 0 : X_t \geq a\}$, and the set $(\tau_a, X_{\tau_a}) : a \geq 0$ has the regenerative property. Therefore, it coincides with the range of a two-dimensional Lévy process with increasing coordinates (a bi-variate subordinator), which is called the ascending ladder process. Its Laplace exponent is denoted by $\kappa(t, z)$, where the first coordinate relates to the co-called ladder-time process and the second one to the ladder-height process. We also denote by $h(x)$ the increasing co-harmonic function in $(0, \infty)$, which from the other-side is the renewal function for the ladder-height process. Despite the great theoretical significance, numerous practical applications (including risk theory or queuing theories) and the enormous literature raised on the subject, the explicit formulas for the distribution of $\overline{X_t}$ or the description in terms of the estimates were known until recently only in very few cases. Baxter and Donsker [2] derived double Laplace transform of $\overline{X_t}$ (with respect to the time and space variables), however, inverting the formula is a very difficult task. Consequently, the explicit formulas for the supremum distribution are known only for a Wiener process (due to the refection principle), a symmetric Cauchy process [18], a compound Poisson process with $\Psi(\xi) = 1 - \cos\xi$ [2] and a Poisson process with drift [61]. The next results of this kind were published fifty years later by Kuznetsov [47] and are related to series representations of the supremum distribution for stable processes. The results of the papers [P7], [P8], [P11] can be treated as a continuation of the classical research from 1950s and the motivation was to describe the supremum functional for the broadest possible class of Lévy processes.

In the article [P7], we study some properties of the cumulative distribution function of $\overline{X_t}$, i.e. the function $\mathbf{P}(\overline{X_t} < x)$. Under mild assumptions, we show the following bounds

$$\mathbf{P}(\overline{X_t} < x) \approx \min\{1, h(x)\kappa(1/t, 0)\}, \quad x, t > 0.$$

In the symmetric case, under some not very demanding assumptions on regularity of $\Psi(\xi)$, we show that $h(x) \approx 1/\sqrt{\Psi(1/x)}$. Since for symmetric processes we always have $\kappa(z, 0) = \sqrt{z}$, it gives the estimates of the cumulative distribution function for every $t, x > 0$ in terms of the Lévy-Khintchine exponent. Moreover, the constants are absolute and they do not depend on the process. Furthermore, for $\Psi(\xi)$ being the increasing characteristic exponent of symmetric process, we provide the integral representation of the Laplace transform of $\overline{X_t}$, i.e. we invert the Baxter-Donsker formula with respect to the time variable. Finally, we derive the integral formulas for $h(x)$ and its derivative $h'(x)$ in the special case, when $\Psi(\xi)$ is given by a complete Bernstein function.

The paper [P8] is devoted to deeper analysis of the symmetric case under additional assumption that $\Psi(\xi) = \psi(\xi^2)$, where $\psi(\xi)$ is a complete Bernstein function. It means that the original Lévy process is just a subordinated Brownian motion, where the Lévy measure of the subordinator has a complete monotone density. It is easy to see that $\mathbf{P}(\tau_x > t) = \mathbf{P}(\overline{X}_t < x)$, where τ_x is the previously defined first passage time of a given level $x > 0$. Thus, the results of [P7] can be directly translated into the corresponding ones for $\mathbf{P}(\tau_x > t)$. In particular, we use the integral representation of the Laplace transform of $\overline{X_t}$ from [P7] to derive the integral representation of $\mathbf{P}(\tau_x > t)$ (equivalently for $\mathbf{P}(\overline{X}_t < x)$) in terms of the generalized eigenfunctions $F_\lambda(x)$ studied by Kwaśnicki [48]. Moreover, we provide the analogous representations for the derivatives $d^n/dt^n \mathbf{P}(\tau_x > t)$. We also generalize the results of [48] by proving the so-called $\pi/2$-conjecture (Lemma 3.1 in [P8]). Careful analysis of the behaviour of $F_\lambda(x)$ let us derive sharp two-sided bounds for $d^n/dt^n \mathbf{P}(\tau_x > t)$ for large times t and small x and the corresponding asymptotics for $t \to \infty$ and $x \to 0^+$.

The last publication [P11] is devoted to study the properties of the supremum density function $f_t(x) = \mathbf{P}(\overline{X}_t \in dx)/dx$, $x > 0$. It is a natural continuation of [P7] and the research of Chaumont from [13], where the absolute continuity of different distributions related to supremum function was studied. In particular, the very useful representation of $f_t(x)$ in terms of the entrance law of the reflected excursions $q_t(dx)$ was proved. Under the mild assumption that there exists bounded probability density function of the original process, excluding compound Poisson processes and subordinators, we show that $f_t(x)/h'(x)$ tends to $n(t < \zeta) = \int_0^\infty q_t(dx)$ uniformly on $[t_0, \infty)$ for every given $t_0 > 0$. Under additional assumption that the Laplace exponent of the ladder time process is regularly varying, we establish the asymptotic of $f_t(x)$ for $t \to \infty$. We also prove the bounds for $f_t(x)$ for small x and large t, which become sharp under regularity assumption for $\kappa(\alpha, 0)$. Moreover, we show that the continuity of $f_t(x)$ at some $x > 0$ is equivalent to continuity at every $x > 0$ and also to continuity of $h'(x)$. Then we use the results to study the properties of the Lévy processes conditioned to stay positive and its meanders.

5.4 Non-colliding particle systems

We will discuss the results from [P6] and [P10], which are devoted to study properties of solutions to stochastic differential equations describing the so-called non-colliding particle systems and the corresponding matrix stochastic differential equations. Natural step in developing the Bessel processes theory is finding its matrix analogue. Thus, let us denote by N_t a Brownian $p \times n$ matrix for $n = 1, 2, \ldots$ (i.e. a matrix process, where the entries are independent one-dimensional Brownian motions). Put $X_t = N_t^T N_t$ and it is an analogue and generalization of the construction of the squared Bessel process we begin this Summary with. Similarly as in the one-dimensional case, we can show that such process with values in S_p (symmetric $p \times p$ matrices) is a solution to the following stochastic differential equation

$$dX_t = \sqrt{|X_t|}dW_t + dW_t^T\sqrt{|X_t|} + \alpha I dt, \quad X_0 = x_0, \tag{29}$$

with $\alpha = n$, where $W = (W_t)$ is a Brownian $p \times p$ matrix. The matrix equation (29) was studied by M.F. Bru [7, 8, 9], who showed in particular that for $x_0 \in \overline{S_p^+}$ (positive-definite matrices) and $\alpha > p - 1$ there exists a unique weak solution of (29). Moreover, for $x_0 \in S_p^+$ and $\alpha \geq p + 1$ there exists unique strong solution and any solution starting from $x_0 \in \overline{S_p^+}$ stays in $\overline{S_p^+}$, whenever $\alpha > p - 1$. Such solutions are called *matrix squared Bessel processes* or *Wishart processes*. We do not know if the pathwise uniqueness holds for (29) in the case $\alpha \in (p-1, p+1)$, since there is no multidimensional (matrix) version of the one-dimensional Yamada-Watanabe theorem. Consequently, there are no tools to show the uniqueness of a strong solution in the case, where solutions hit the boundary of S_p^+ (i.e. for $\alpha \in (p-1, p+1)$). Bridging the gap in the theory was the motivation for the research done in [P6] and [P10].

It turns out that the process $\Lambda(t) = (\lambda_1(t), \ldots, \lambda_p(t))$ of increasingly ordered eigenvalues of X_t is very important in studying the properties of X_t. For $x_0 \in \tilde{S}_p$ (the set of symmetric matrices with distinct eigenvalues) it is described by the following system of SDEs

$$d\lambda_i = 2\sqrt{|\lambda_i|}dB_i + \left(\alpha + \sum_{j=\neq i} \frac{|\lambda_i| + |\lambda_j|}{\lambda_i - \lambda_j}\right) dt, \quad i = 1, \ldots, p.$$

Note the similarities between these equations and (1) and the additional expression appearing in the drift parts related to the repulsive forces between eigenvalues. The interest in such systes of SDEs comes not only from the above given relation to matrix SDEs [23], [24], [46], but must of all with their wide range of applications in various models of mathematical physics and physical statistics [40], [41], [42], [43].

In the paper [P6] we consider the generalization of (29) of the form

$$dX_t = g(X_t)dW_t h(X_t) + h(X_t)dW_t^T g(X_t) + b(X_t)dt, \quad X_0 = x_0,$$

where functions $g, h, b : \mathbf{R} \to \mathbf{R}$ appearing above act spectrally on S_p. First of all, we derive the systems of SDEs describing the eigenfunctions and eigenvalues of a solution in both real and complex cases. In particular, assuming that $x_0 \in \tilde{S}_p$, the SDEs for the eigenvaleus are goven by

$$d\lambda_i = 2g(\lambda_i)h(\lambda_i)dv_i + \beta\left(b(\lambda_i) + \sum_{j\neq i} \frac{G(\lambda_i, \lambda_j)}{\lambda_i - \lambda_j}\right) dt, \quad i = 1, \ldots, p. \tag{30}$$

for $\beta = 1$ (real case) and $\beta = 2$ (complex case). Here $G(x,y) = g^2(x)h^2(y) + g^2(y)h^2(x)$. However, applications in physical statistics statistic mechanical models for *log-gases*, see [28]) require study of the whole range of $\beta > 0$. We prove some special multidimensional version of the Yamada-Watanabe theorem and we apply it to show existence and uniqueness of solutions to the systems describing eigenvalues and eigenfunctions up to the first collision time. Moreover, we provide some sufficient conditions on the coefficients of the SDEs which prevent the eigenvalues λ_i from collisions in finite time. Uniqueness of the solutions to the SDEs for eigenvalues and eigenfunctions does not imply the corresponding result for the matrix SDE, but it emphasizes the advantages of the description of the solutions to (30) in terms of its eigenvalues and eigenfunctions.

In all the results from [P6] we assume that the eigenvalues of the starting point are all distinct. Unfortunately, the start from collision points (all the eigenvalues equal for $t = 0$) is quite often crucial in applications. For example, if we consider the simplest case of Brownian motion in S_p, it is the most natural to assume that it starts from 0, i.e. $\lambda_i(0) = 0$ for every $i = 1, \ldots, p$. Analogously, applications of the square Bessel particle systems in some models of mathematical physics require studying the case, where $\lambda_i(0) = 0$ for every i. The paper [P10] is devoted to this issue, i.e. we study the existence, uniqueness and behaviour of solutions to the following SDEs system

$$dx_i = \sigma_i(x_i)dB_i + \left(b_i(x_i) + \sum_{j\neq i} \frac{H_{ij}(x_i, x_j)}{x_i - x_j}\right) dt, \quad i = 1, \ldots, p, \tag{31}$$

$$x_1(t) \leq x_2(t) \leq \ldots \leq x_p(t), \quad t \geq 0. \tag{32}$$

Note that such systems are more general than previously, the coefficients in the martingale parts and drift parts can change (they depend on i and j). Additionally, now we do not require that the particle are distinct for $t = 0$, i.e. we allow that $x_i(0) = x_j(0)$ for some $i \neq j$. However, we assume that $H_{ij}(x, y) = H_{ji}(y, x)$, i.e. the particles push each other with the same force. The key idea to deal with such systems of SDEs is to look at symmetric polynomials of $(x_1, \ldots, x_p)$. The symmetry of H described above implies that in the SDEs for symmetric polynomials the problematic expressions $(x_i - x_j)^{-1}$ vanish and the collisions between particles does not cause any problem for the symmetric polynomials. Thus, we find the SDEs for the symmetric polynomials and use them to show that, under appropriate assumptions on the coefficients, the eigenvalues become immediately distinct (if starting from collisions) and never collide again. It enable us to construct a solution of (31) and show that pathwise uniqueness holds (by analogue of Yamada-Watanabe). Our assumptions and requirements on the coefficients of the equations have their heuristic (physical) explanations or motivated by some examples. Finally, we apply our theory to many important and classical examples of particle systems having their theoretical and practical applications.

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