Summary of scientific achievements

1. Name and surname:

Tomasz Grzywny

2. Scientific degrees:

2005 M.Sc. in Mathematics
Institute of Mathematics and Computer Sciences
Faculty of Fundamental Problems of Technology
Wroclaw University of Technology
Master's dissertation: *Potential theory of relativistic processes*
supervisor: Prof. Dr. Sc. Michał Ryznar

2008 Ph.D. in Mathematics
Institute of Mathematics and Computer Sciences
Faculty of Fundamental Problems of Technology
Wroclaw University of Technology
doctoral dissertation: *Potential theory for the relativistic α-stable process*
supervisor: Prof. Dr. Sc. Michał Ryznar

3. Information on previous employment in scientific institutions:

2008 – 2010	Assistant in the Institute of Mathematics and Computer Science Wroclaw University of Technology
2010 – 2014	Assistant Professor in the Institute of Mathematics and Computer Science, Wroclaw University of Technology
2012 – 2013	Post-doc position at Technische Universität Dresden, Germany
2013 – 2013	Post-doc position at Universität Bielefeld, Germany
2014* – 2015	Assistant Professor in Department of Mathematics, Wroclaw University of Technology
2015* – present	Assistant Professor in Faculty of Pure and Applied Mathematics, Wroclaw University of Science and Technology

* - these changes were caused by the organizational changes at the University, i.e. the transformation of the Institute of Mathematics and Computer Science into Department of Mathematics on Faculty of Fundamental Problem of Technology and then foundation of Faculty of Pure and Applied Mathematics.

4. The indication of the scientific achievement:

(a) The title of the scientific achievement:

Potential theory of isotropic unimodal Lévy processes

(b) The list of papers constituting the scientific achievement:

[H1] K. Bogdan, T. Grzywny, M. Ryznar, *Heat kernel estimates for the fractional Laplacian with Dirichlet conditions*, Annals of Probability 38(5), 1901–1923 (2010).

[H2] T. Grzywny, *On Harnack Inequality and Hölder Regularity for Isotropic Unimodal Lévy Processes*, Potential Analysis 41(1), 1–29 (2014).

[H3] K. Bogdan, T. Grzywny, M. Ryznar, *Density and tails of unimodal convolution semigroups*, Journal of Functional Analysis 266(6), 3543–3571 (2014).

[H4] K. Bogdan, T. Grzywny, M. Ryznar, *Dirichlet heat kernel for unimodal Lévy processes*, Stochastic Processes and their Applications 124(11), 3612–3650 (2014).

[H5] K. Bogdan, T. Grzywny, M. Ryznar, *Barriers, exit time and survival probability for unimodal Lévy processes*, Probability Theory and Related Fields 162, 155–198 (2015).

(c) A discussion of the above-mentioned papers and the obtained results, together with a discussion of their possible use

I. Introduction

Lévy processes form a wide and rich class of Markov processes. They are natural analogues of random walks in continuous time. The most prominent examples of the Lévy processes are the Poisson process, the Brownian motion, the Cauchy process and the isotropic stable Lévy processes. Lévy processes provides valuable guidance for general Markov processes. Very often a property is proved first for the Lévy processes and then it is being extended to Feller or Markov processes. Despite the relatively simple structure of Lévy processes (their transition probability is translation invariant), they have many specific features of interest and many prominent scientists study various aspects of these processes. In addition, more and more common, these processes are used to model many physical, chemical and economical phenomena ([2, 21, 38, 11, 47, 33, 36]).

The main aim of the papers included in the scientific achievement was to obtain sharp estimates of the transition densities of the processes killed after leaving an open set, that is estimates of the heat kernel of the generator with Dirichlet conditions. The transition probabilities form continuous operator semigroup and its generator acts on various function spaces. Explicit formulae for the transition densities are very rare therefore estimates and asymptotic properties are very essential. Even in the isotropic α-stable case the explicit formulae are not available (except for $\alpha \in \{1,2\}$) and the corresponding optimal estimates were first proved by Pólya [51] on the real-line and later generalized to the multidimensional setting by Blumenthal and Getoor [9]. In recent years there has been a remarkable progress in studying properties of transition semigroups (heat kernels) of

Markov, and especially of Lévy processes. Among many researchers who contributed to the subject we can mention Barlow, Bass, Bogdan, Burdzy, Chen, Song, Grigor'yan, Jakob, Kim, Knopova, Kulik, Kumagai, Saloff-Coste, Schilling, Sztonyk, Vondraček ([3, 29, 19, 13, 14, 41, 35, 43, 30, 31, 32, 5, 4, 10, 42, 37, 59, 58]).

The discussed scientific achievement concerns the isotropic unimodal Lévy processes. Their properties have been studied using probabilistic potential theory. In the paper [H1] we examined the isotropic α-stable processes on quite general sets, so-called κ-fat sets, obtaining sharp estimates for transition density of killed process. It should be added that estimates in such generality cannot be explicit. Explicit estimates were obtained for sets with $C^{1,1}$ boundary, both bounded and unbounded. Methods developed in the article [H1] where later used by others researchers to obtain estimates for the heat kernels of some subordinated Brownian motions, for instance for the relativistic stable processes and the sum of two independent isotropic stable processes with different indices [17, 16]. These studies became a motivation for us to unify and extend the obtained estimates to a wider class of processes. The appropriate class for such unification turned out to be the isotropic unimodal processes, for which we obtained estimates of the density of the potential measure and regularity results of harmonic functions with respect to examined processes ([H2]), estimates of the transition density of the free process ([H3]), and the estimates of the expected exit time from sets and survival probabilities ([H5]). The culmination of this study were the estimates of the Dirichlet heat kernels for a large class of bounded and unbounded sets in the paper [H4].

The following discussion of the papers series [H1] - [H5] is divided into sections according to ideas, and not according to the chronology of results.

- In Section I we set definitions and notions used in the rest of the thesis and we discuss the basic results concerning the Lévy processes.
- In Section II we present estimates of the heat kernel of the free processes (without killing).
- In Section III we describe results concerning harmonic functions and Green function on the whole space.
- Section IV focuses on the behaviour of the expected exit time and survival probabilities.
- Section V gives estimates of the Dirichlet heat kernels, which combine and use the results discussed in the preceding sections.

Lévy processes

Let $\{X_t\}_{t\geq 0}$ be a Lévy process in $\mathbb{R}^d$, $d \in \mathbb{N}$. This means that is a stochastic process with independent and stationary increments, càdlàg paths and starting from the origin. Lévy processes are characterized by the Lévy-Khinchine formula

$$\mathbb{E}e^{i\langle \xi, X_t\rangle} = \int_{\mathbb{R}^d} e^{i\langle \xi, z\rangle} P_t(dz) = e^{-t\psi(\xi)}, \quad \xi \in \mathbb{R}^d,$$

where ψ is the characteristic exponent (the symbol)

$$\psi(\xi) = \langle \xi, A\xi\rangle - i\langle \xi, \gamma\rangle - \int_{\mathbb{R}^d} \left(e^{i\langle \xi, z\rangle} - 1 - i\langle \xi, z\rangle \mathbf{1}_{|z|<1}\right) N(dz), \quad \xi \in \mathbb{R}^d, \tag{1}$$

$A = [A_{jk}]_{j,k=1,\dots,d}$ is a symmetric and non-negative definite matrix, $\gamma \in \mathbb{R}^d$ and N is a Lévy measure i.e. $N(\{0\}) = 0$ and $\int_{\mathbb{R}^d} (1 \wedge |z|^2)\, N(dz) < \infty$. The infinitesimal generator $\mathcal{L}$ of the transition semigroup of the process $\{X_t\}_{t\geq 0}$, to wit,

$$P_t f(x) = \mathbb{E} f(X_t + x), \quad x \in \mathbb{R}^d,$$

has the following form for $f \in C_b^2(\mathbb{R}^d)$,

$$\mathcal{L}f(x) = \sum_{j,k} A_{jk}\partial^2_{jk} f(x) + \langle \gamma, \nabla f(x)\rangle + \int_{\mathbb{R}^d} \left(f(x+z) - f(x) - \mathbf{1}_{|z|<1}\langle z, \nabla f(x)\rangle\right) N(dz). \qquad (2)$$

For a Lévy process $\{X_t\}_{t\geq 0}$ we define two functions, which turn out to be very useful for its description. The first one is a non-decreasing majorant of the real part of the symbol

$$\psi^*(r) = \sup_{|\xi|\leq r} \Re\psi(\xi), \quad r \geq 0.$$

The second function describes the combined intensity of the large and small jumps of the process and of the continuous part:

$$h(r) = \frac{||A||}{r^2} + \int_{\mathbb{R}^d} \min\{1, |z/r|^2\} N(dz), \quad r > 0.$$

The function h appeared for instance in the article of Pruitt [52], and earlier for the processes on the real line in the paper by Dupuis [24]. It turns out that the functions ψ^* and $r \mapsto h(1/r)$ are comparable and the comparability constant depends only on the dimension ([H2, Lemma 4])

$$\frac{1}{8(1+2d)} h(1/r) \leq \psi^*(r) \leq 2h(1/r), \quad r > 0. \qquad (3)$$

Most of the results of the scientific achievement concerns the isotropic unimodal Lévy processes, which we define now. A (Borel) measure $\mu(dx)$ on $\mathbb{R}^d$ is called *isotropic* if it is rotation invariant. If the Lévy measure is isotropic, $A = \sigma^2 \mathrm{I_d}$ and $\gamma = 0$, then the distribution $P_t(dx)$ of X_t is rotation invariant as well and the process is called *isotropic*. A measure is *isotropic unimodal* if on $\mathbb{R}^d \setminus \{0\}$ it is absolutely continuous with respect to the Lebesgue measure and has a finite *radial non-increasing* density function. Such measures may have an atom at the origin, they are of the form $a\delta_0(dx) + f(x)dx$, where $a \geq 0$, δ_0 is the Dirac measure,

$$f(x) = \int_0^\infty \mathbf{1}_{B_r}(x)\mu(dr) = \mu\big((|x|, \infty)\big) \qquad (a.e.),$$

and μ is a measure on $(0, \infty)$ such that $\mu\big((\varepsilon, \infty)\big) < \infty$ for all $\varepsilon > 0$. The Lévy process $\{X_t\}_{t\geq 0}$, is called isotropic unimodal if all its one-dimensional distributions $P_t(dx)$ are such. The processes were characterized by Watanabe in [62] as those isotropic Lévy processes that have the isotropic unimodal Lévy measure. A density of $P_t(dx)$ is denoted by $p_t(x)$. The characteristic exponent of the isotropic unimodal Lévy process is the following form

$$\psi(\xi) = \sigma^2|\xi|^2 + \int_{\mathbb{R}^d} (1 - \cos\langle \xi, z\rangle)\nu(z)dz,$$

for some radial function ν, with the profile $\nu(r) := \nu((r,0,\ldots))$, $r > 0$ being a non-increasing function ($N(dz) = \nu(z)dz$). The distribution of the isotropic unimodal process has an atom at the origin if and only if $\{X_t\}_{t\geq 0}$ is a compound Poisson process. Hence if the symbol ψ is unbounded ($\sigma > 0$ or $N(\mathbb{R}^d) = \infty$) then, for every $t > 0$, $P_t(dx)$ are absolutely continuous on $\mathbb{R}^d$.

The most important class of the isotropic unimodal Lévy processes are the subordinated Brownian motions. Let $\{B_t\}_{t\geq 0}$ be a Brownian motion on $\mathbb{R}^d$ and $\{T_t\}_{t\geq 0}$ be an independent subordinator, that is a Lévy process with non-decreasing paths. Then a process $\{B_{T_t}\}_{t\geq 0}$ we called *subordinated Brownian motion.* Since $T_t \geq 0$, the Laplace transform is useful to characterized these processes. Namely, we have

$$\mathbb{E}e^{-\lambda T_t} = e^{-t\varphi(\lambda)}, \quad \lambda \geq 0,$$

where φ is so-called the Laplace exponent, given by

$$\varphi(\lambda) = b\lambda + \int_0^\infty (1 - e^{-\lambda s})\mu(ds). \tag{4}$$

Here $b \geq 0$ and μ is a measure such that $\int_0^\infty \frac{s}{1+s}\mu(ds) < \infty$. Put differently, φ is a Bernstein function such that $\varphi(0) = 0$. We define two subclasses of Bernstein functions which are very important in the potential theory of Lévy processes. We say that a Bernstein function φ is *special* if the function $\lambda \mapsto \frac{\lambda}{\varphi(\lambda)}$ is also a Bernstein function. A subordinator with a special Bernstein function φ as the Laplace exponent is called a *special subordinator.* If φ' is a complete monotone function, then φ is called a *complete* Bernstein function and $\{T_t\}_{t\geq 0}$ is a complete subordinator. We should emphasized that complete Bernstein functions are special. The most important property of the special subordinators is the fact that the potential measure $U(A) = \int_0^\infty \mathbb{P}(T_t \in A)dt$ is absolutely continuous on $(0,\infty)$ and its density is non-increasing. The best reference to Bernstein functions and the subordination is [54].

The Lévy kernel of the Brownian motion, which in our setting has the transition density $g_t(x) = (4\pi t)^{-d/2}e^{-|x|^2/(4t)}$, after being subordinated by the subordinator $\{T_t\}_{t\geq 0}$ with the Laplace exponent (4) is given by

$$\nu(x) = \int_0^\infty (4\pi u)^{-d/2} e^{-\frac{|x|^2}{4u}}\mu(du).$$

Hence, ν is a radial and radially decreasing function. The Lévy-Khinchine exponent of the process $\{B_{T_t}\}_{t\geq 0}$ is equal to $\varphi(|\xi|^2)$. Hence $\{B_{T_t}\}_{t\geq 0}$ is an isotropic unimodal Lévy process. A prominent example of the subordinated Brownian motions is the isotropic α-stable process, where

$$\alpha \in (0,2], \quad \varphi(\lambda) = \lambda^{\alpha/2} \quad \text{and} \quad \psi(\xi) = |\xi|^\alpha.$$

A very important property of the isotropic unimodal Lévy processes is the fact that the characteristic exponent is almost increasing. Namely, we have (see [H3, Propositon 2] and [H2, Proposition 1])

$$\psi(\xi) \geq \pi^{-2}\psi^*(|\xi|), \quad \xi \in \mathbb{R}^d.$$

Combining this with (3) we obtain

$$\psi(\xi) \approx \psi^*(|\xi|) \approx h(1/|\xi|), \quad \xi \neq 0, \tag{5}$$

where $f(x) \approx g(x)$, $x \in A$ means that there is a (comparability) constant $C > 0$, such that

$$C^{-1}f(x) \leq g(x) \leq Cf(x), \quad x \in A.$$

Dirichlet condition (killed process)

In this section we define the Dirichlet heat kernel and the Green function. Let D be a non-empty open set in $\mathbb{R}^d$. By $\operatorname{diam}(D) = \sup\{|y-x| : x, y \in D\}$ we denote its diameter, and the distance to its complement is

$$\delta_D(x) = \operatorname{dist}(x, D^c), \qquad x \in \mathbb{R}^d.$$

We let $B(x,r) = \{y \in \mathbb{R}^d : |y-x| < r\}$, the open ball with center at $x \in \mathbb{R}^d$ and radius $r > 0$, and $B_r = B(0,r)$. We also let $\overline{B(x,r)}^c = \left(\overline{B(x,r)}\right)^c = \{y \in \mathbb{R}^d : |y-x| > r\}$ and $\overline{B}_r^c = \overline{B(0,r)}^c$. For $a \in \mathbb{R}$, we consider the upper half-space $\mathbb{H}_a = \{(x_1, \ldots, x_d) \in \mathbb{R}^d : x_d > a\}$. All other half-spaces are obtained by rotations. The ball, the complement of the ball and the half-space represent three distinctly different geometries at infinity which are in our focus. In papers [H1]-[H5] we considered the following classes of sets.

DEFINITION **1.** *Let $D \subset \mathbb{R}^d$ be open.*

- *We say that D satisfies the* **inner ball condition** *at scale r if $r > 0$ and for every $Q \in \partial D$ there is ball $B(x',r) \subset D$ such that $Q \in \partial B(x',r)$.*
- *We say D satisfies the* **outer ball condition** *at scale r if $r > 0$ and for every $Q \in \partial D$ there is ball $B(x'',r) \subset D^c$ such that $Q \in \partial B(x'',r)$.*
- *We say that D is of* **class $C^{1,1}$ at scale** *r, if D satisfies the inner and outer ball conditions at the scale r.*
- *Let $\kappa \in (0,1/2]$ and $R \in (0,\infty]$. A set D is (κ, R)-***fat***, if for every $0 < r < R$ and $x \in D$ there exists $A_{x,r} \in D$ such that $B(A_{x,r}, \kappa r) \subset D \cap B(x,r)$.*

We call $B(x',r)$ and $B(x'',r)$ above the *inner* and *outer* balls for D at Q, respectively. Estimates of the potential-theoretic objects for $C^{1,1}$ sets D often rely on the inclusion $B(x',r) \subset D \subset \overline{B(x'',r)}^c$ and on explicit calculations for its extreme sides. If D is $C^{1,1}$ at some positive but unspecified scale (hence also at all smaller scales), then we simply say D is $C^{1,1}$. If D satisfies the inner ball condition at scale R, then D is $(1/2, R)$-fat. It is easy to see that half-spaces and the complements of balls are $(1/2, \infty)$-fat.

We are interested in describing the behaviour of the unimodal Lévy process $\{X_t\}_{t\geq 0}$ as it approaches the complement of the open set D. We shall employ the usual Markovian notation: for $x \in \mathbb{R}^d$ we write $\mathbb{E}^x$ and $\mathbb{P}^x$ for the expectation and distribution of $\{x + X_t\}_{t \geq 0}$, i.e. [53, Chapter 8],

$$\mathbb{E}^x f(\{X_t\}) = \mathbb{E} f(\{X_t + x\}).$$

We define the *time of the first exit* of $\{X_t\}_{t\geq 0}$ from open set $D \subset \mathbb{R}^d$:

$$\tau_D = \inf\{t > 0 : X_t \notin D\}.$$

Similarly, we define the *time of the first hitting* of closed set $F \subset \mathbb{R}^d$:

$$T_F = \inf\{t > 0 : X_t \in F\}.$$

Thus, $\tau_D = T_{D^c}$.

Let us consider a symmetric Lévy process $\{X_t\}_{t\geq 0}$ such that the one-dimensional distributions $P_t(dx) = \mathbb{P}(X_t \in dx)$ are absolutely continuous for every $t > 0$. We shall also write $p(t,x,y) := p_t(y-x)$, where p_t is a density of mention $P_t(dx)$. Then the transition density of the process $\{X_t\}_{t\geq 0}$ *killed* at the first exit from D is defined by the Hunt's formula, for $x, y \in \mathbb{R}^d$ and $t > 0$,

$$p_D(t,x,y) = p(t,x,y) - \mathbb{E}^x \left[p(t-\tau_D, X_{\tau_D}, y); \tau_D < t\right]. \tag{6}$$

For $A \subset D$, we have

$$\mathbb{P}^x(X_t \in A, \tau_D > t) = \int_A p_D(t,x,y)dy.$$

We call p_D the *Dirichlet heat kernel* of the process $\{X_t\}_{t\geq 0}$ (or of the generator $\mathcal{L}$ of its semigroup) on D, since it is a fundamental solution of the heat equation for the operator $\mathcal{L}$ with Dirichlet condition on the complement of D. Namely, for $s \in \mathbb{R}$, $x \in \mathbb{R}^d$, and $\phi \in C_c^\infty(\mathbb{R} \times D)$, we have

$$\int_s^\infty \int_D p_D(u-s,x,z) \left[\partial_u \phi(u,z) + \Delta_z^{\alpha/2}\phi(u,z)\right] dz du = -\phi(s,x).$$

The probability of process $\{X_t\}_{t\geq 0}$ not to leave D before time $t > 0$ may be expressed via p_D:

$$\mathbb{P}^x(\tau_D > t) = \int_{\mathbb{R}}^d p_D(t,x,y)dy, \qquad t > 0,\ x \in \mathbb{R}^d. \tag{7}$$

This is called the *survival probability*. The *Green function* of D for $\{X_t\}_{t\geq 0}$ is defined as

$$G_D(x,y) = \int_0^\infty p_D(t,x,y)dt, \qquad x, y \in \mathbb{R}^d. \tag{8}$$

Let us observe that $p_{\mathbb{R}^d}(t,x,y) = p_t(y-x)$. The Green function of the whole space (the kernel of the potential measure) we denote by $G(y-x) := G_{\mathbb{R}^d}(x,y)$.

The *expected exit time* from D is given by

$$\mathbb{E}^x\tau_D = \int_0^\infty \mathbb{P}^x(\tau_D > t)dt = \int_0^\infty \int_{\mathbb{R}^d} p_D(t,x,y)dydt = \int_{\mathbb{R}^d} G_D(x,y)dy, \qquad x \in \mathbb{R}^d.$$

If $x \in D$, then the $\mathbb{P}^x$-distribution of $(\tau_D, X_{\tau_D-}, X_{\tau_D})$ restricted to the event $\{\tau_D < \infty, X_{\tau_D-} \neq X_{\tau_D}\}$ is given by the following density function

$$(0,\infty) \times D \times D^c \ni (s,u,z) \mapsto \nu(z-u)p_D(s,x,u). \tag{9}$$

Integrating against ds, du and/or dz gives marginal distributions. For instance, if $x \in D$, then

$$\mathbb{P}^x(X_{\tau_D} \in A) = \left(\int_D G_D(x,u)\nu(A-u)du\right) dz, \tag{10}$$

for $A \subset (\overline{D})^c$ or even for $A \subset D^c$ if $\mathbb{P}^x(X_{\tau_D-} \in \partial D) = 0$. Such identities are called Ikeda-Watanabe formulae. They enjoy intuitive interpretations in terms of "the occupation time measures"

$p_D(s,x,u)duds$ and $G_D(x,u)du$ and "the intensity of jumps" $\nu(z-u)dz$. From the probabilistic point of view the Hunt's formula (6) is explicit, since it is expressed by $\mathbb{P}^x$ and explicit functionals of paths, i.e. by τ_D and X_{τ_D}. From the analytic point of view the definition (6) is implicit, since in the Ikeda-Watanabe formula the distribution of (τ_D, X_{τ_D}) is expressed by the heat kernel p_D. Nevertheless, it is tractable, for instance using the Hunt's formula one may check that $y \mapsto p_{B_r}(t,0,y)$ is a radial function for all $r,t>0$.

It is well known that p_D satisfies the Chapman-Kolmogorov equations, which yields the following simple connection of the heat kernel and the survival probability.

Lemma 2. *Let $\{X_t\}_{t\geq 0}$ be a symmetric Lévy process such that $e^{-t\psi} \in L_1(\mathbb{R}^d)$, $t>0$. Then, for all $t>0$ and $x,y\in\mathbb{R}^d$, we have $p_D(t,x,y) \leq p_{t/2}(0)\,\mathbb{P}^x(\tau_D > t/2)$ and*

$$p_D(t,x,y) \leq p_{t/2}(0)\,\mathbb{P}^x\left(\tau_D > \frac{t}{4}\right)\mathbb{P}^y\left(\tau_D > \frac{t}{4}\right).$$

The above lemma is very well known in a similar form for all ultracontractive semigroups. The structure of estimates i.e. the approximate factorization of the heat kernel is noteworthly and appears in many results of [H1]-[H5].

The following lemma is used to derive estimates of the Dirichlet heat kernel from above and below when the points x and y are relatively distant from each other. It is very important in estimating the Dirichlet heat kernel for jumping-type processes.

Lemma 3 (Lemma 1.10 in [H4]). *Let $\{X_t\}_{t\geq 0}$ be a symmetric Lévy process with Lévy kernel ν. Consider disjoint open sets $D_1, D_3 \subset D$. Let $D_2 = D\setminus(D_1\cup D_3)$. If $x\in D_1$, $y\in D_3$ and $t>0$, then*

$$\begin{aligned}
p_D(t,x,y) &\leq \mathbb{P}^x(X_{\tau_{D_1}}\in D_2)\sup_{s<t,\,z\in D_2} p(s,z,y) + (t\wedge \mathbb{E}^x\tau_{D_1})\sup_{u\in D_1,\,z\in D_3}\nu(z-u),\\
p_D(t,x,y) &\leq \mathbb{P}^x(X_{\tau_{D_1}}\in D_2)\sup_{s<t,\,z\in D_2} p_D(s,z,y) + \sup_{u\in D_1,\,z\in D_3}\nu(z-u)\times\\
&\quad\times\left(\mathbb{P}^x(\tau_{D_1}>t/2)\int_0^{t/2}\mathbb{P}^y(\tau_D>s)ds + \mathbb{P}^y(\tau_D>t/2)\int_0^{t/2}\mathbb{P}^x(\tau_{D_1}>s)ds\right),\\
p_D(t,x,y) &\geq t\,\mathbb{P}^x(\tau_{D_1}>t)\,\mathbb{P}^y(\tau_{D_3}>t)\inf_{u\in D_1,\,z\in D_3}\nu(z-u).
\end{aligned}$$

In order to use the above lemma we need estimates for: the transition density of the free process $p_t(x)$, the Lévy density ν, the distribution of the position of the process at the first exit from a set, the expected exit time, and the survival probability in the set. Estimates of these objects, given in the papers [H1] - [H5], are discussed in the following sections.

We note that a result similar to the first inequality in Lemma 3, appeared in the paper by Kulczycki and Siudeja for balls and the stable relativistic processes [45]. In [D3] we improved it and simplified the proof of this inequality for half-spaces. The next step was to obtain a similar inequality for the ball and the isotropic stable processes by Chen, Kim and Song [15]. The first and third inequalities in the form presented above for the isotropic stable processes and $t=1$ were proven in the paper [H1]. Using these inequalities and the boundary Harnack inequality we obtained sharp bounds for the Dirichlet heat kernels for κ-fat sets. Sharp estimate (bound) here means that the ratio of the upper bound and the lower bound is less than a constant. In the paper [H4] we proved the present statement of the lemma, which allowed to provide estimates of p_D for $C^{1,1}$ sets without using the boundary Harnack inequality.

A modification of this lemma was given and used for symmetric Markov processes with absolutely continuous jumping measure in the paper [Pre1].

Basics of fluctuation theory

In our study we used objects that are derived from the fluctuation theory. The most important of them is the harmonic function (with respect to the generator of the transition semigroup of a Lévy process) in the half-line, which is increasing and its derivative is also a harmonic function. Assume that $\{X_t\}_{t\geq 0}$ is a symmetric Lévy process on the real line which is not compound Poisson, i.e. there is the non-zero Gaussian part or the Lévy measure is infinite. Let $M_t = \sup_{s\leq t} X_s$ and let L_t be the local time at 0 for $M_t - X_t$, the process X_t reflected at the supremum ([25],[7]). We consider its right-continuous inverse, L_s^{-1}, called the ascending ladder time process for $\{X_t\}_{t\geq 0}$. We also define the ascending ladder-height process, $H_s = X_{L_s^{-1}} = M_{L_s^{-1}}$. The pair (L_t^{-1}, H_t) is a two-dimensional subordinator ([25],[7]). In fact, since $\{X_t\}_{t\geq 0}$ is symmetric and non Poisson, by [25, Corollary 9.7], the Laplace exponent of (L_t^{-1}, H_t) is

$$-\frac{1}{t}\log\left(\mathbb{E}\exp[-\tau L_t^{-1} - \lambda H_t]\right) = c_+ \exp\left\{\frac{1}{\pi}\int_0^\infty \frac{\log\left[\tau + \psi(\lambda\theta)\right]}{1+\theta^2} d\theta\right\}, \qquad \tau, \lambda \geq 0,$$

In what follows we let $c_+ = 1$, thus normalizing the local time L [25]. In particular, L_s^{-1} is then the standard 1/2-stable subordinator (see also [23, (4.4.1)]), and the Laplace exponent of H_t is

$$\kappa(\lambda) = -\frac{1}{t}\log\left(\mathbb{E}\exp[-\lambda H_t]\right) = \exp\left\{\frac{1}{\pi}\int_0^\infty \frac{\log\psi(\lambda\theta)}{1+\theta^2}\, d\theta\right\}, \quad \lambda \geq 0. \tag{11}$$

The renewal function V (the potential measure of intervals) of the ascending ladder-height process H is defined as

$$V(x) = \int_0^\infty \mathbb{P}(H_s \leq x)ds, \quad x \in \mathbb{R}. \tag{12}$$

Thus, $V(x) = 0$ if $x < 0$ and V is non-decreasing. It is also well known that V is subadditive,

$$V(x+y) \leq V(x) + V(y), \quad x, y \in \mathbb{R}, \tag{13}$$

and $\lim_{x\to\infty} V(x) = \infty$. Furthermore, from (11) we infer that the Laplace transform $\mathcal{L}V$ of V is

$$\mathcal{L}V(\lambda) = \frac{1}{\lambda\kappa(\lambda)}, \quad \lambda \geq 0. \tag{14}$$

Both V and its derivative V' were studied by Silverstein in [55]. If resolvent measures of $\{X_t\}_{t\geq 0}$ are absolutely continuous, then it follows from [55, Theorem 2] that $V(x)$ is absolutely continuous and harmonic on $(0, \infty)$ for the process $\{X_t\}_{t\geq 0}$, what means that $V(x) = \mathbb{E}^x V(X_{\tau_{(a,b)}})$, for every $0 < a < b < \infty$. Also, V' is a positive harmonic function for $\{X_t\}_{t\geq 0}$ on $(0,\infty)$, hence V is actually (strictly) increasing. Notably, the definition of V is rather implicit and the study of V poses problems. The explicit form of V for symmetric processes is known only for the α-stable Lévy processes, then $V(x) = (x_+)^{\alpha/2}$, $\alpha \in (0,2]$. Under structure assumptions satisfied for complete subordinate Brownian motions, V' is monotone, in fact completely monotone ([46, Proposition 4.5]). This circumstance stimulated much of the progress made in [17, 41].

We shall present sharp estimates of V by means of (simpler) functions ψ and h, but decay properties of V' are more delicate and they are not yet fully understood. It is known that V' is decreasing if $\{X_t\}_{t\geq 0}$ is subordinated Brownian motion governed by special subordinator. In addition, if the process has a non-zero Gaussian part, then V' is continuous and bounded on $(0,\infty)$.

Lemma 4 (Proposition 2.4 in [H5]). *Let $\{X_t\}_{t\geq 0}$ be a symmetric Lévy process in $\mathbb{R}$ with the unbounded symbol ψ. Then*

$$V^2(r) \approx \frac{1}{h(r)} \approx \frac{1}{\psi^*(1/r)}, \qquad r > 0,$$

where the comparability constants are absolute (do not depend on anything).

We emphasized that in described papers we considered an isotropic process $\{X_t\}_{t\geq 0}$ in $\mathbb{R}^d$ and the renewal function V was defined for the ascending ladder-height process of a one-dimensional projection of the process $\{X_t\}_{t\geq 0}$ (for instance for one of coordinates). Since considered processes are rotation invariant, V does not depend on the choice of coordinates. In this sense, the claim of Lemma 4 holds also for $\mathbb{R}^d$ and in the estimates we used interchangeably V, h and ψ^*.

Weak scaling

Weak scaling is natural generalization of homogeneity and regular variation and nowadays is standard in the potential theory of Markov processes [39, 13]. We give a short survey of weakly scaling and almost monotone functions. The best referencestra on this topic are [8, 1].

DEFINITION **5.** *Let $\phi : I \to [0,\infty]$, for a connected set $I \subset [-\infty,\infty]$.*

- *We call ϕ* **almost increasing** *if there is (oscillation factor) $c \in (0,1]$ such that $c\phi(x) \leq \phi(y)$ for $x, y \in I$, $x \leq y$.*
- *If there is $C \in [1,\infty)$ such that $C\phi(x) \geq \phi(y)$ for $x, y \in I$, $x \leq y$, then we call ϕ* **almost decreasing** *(with oscillation factor C)*

Let

$$\phi^*(y) = \sup\{\phi(x) : \, x \in I, \, x \leq y\}, \qquad y \in I.$$

We easily check that ϕ^* is non-decreasing, $\phi \leq \phi^*$ and ϕ is almost increasing with oscillation factor c if and only if $c\phi^* \leq \phi$. Analogously, let

$$\phi_*(x) = \sup\{\phi(y) : \, y \in I, \, y \geq x\}, \qquad x \in I.$$

Then ϕ_* is non-increasing, $\phi \leq \phi_*$ and ϕ is almost decreasing with oscillation factor C if and only if $\phi_* \leq C\phi$. We also note that ϕ is almost increasing on I with factor c if and only if $1/\phi$ is almost decreasing on I with factor $1/c$.

Definition **6.** *Let* $\phi : (0,\infty) \mapsto \infty$.

- *We say that* ϕ *satisfies* **the weak lower scaling condition** *(at infinity) if there are numbers* $\underline{\alpha} > 0$, $\underline{\theta} \geq 0$, *and* $\underline{c} \in (0,1]$, *such that*

$$\phi(\lambda\theta) \geq \underline{c}\lambda^{\underline{\alpha}}\phi(\theta) \quad \text{for} \quad \lambda \geq 1, \quad \theta > \underline{\theta}. \tag{15}$$

In short we say that ϕ *satisfies* $WLSC(\underline{\alpha},\underline{\theta},\underline{c})$ *and write* $\phi \in WLSC(\underline{\alpha},\underline{\theta},\underline{c})$. *If* $\phi \in WLSC(\underline{\alpha},0,\underline{c})$, *then we say that* ϕ *satisfies the* **global** *weak lower scaling condition (for some* $\underline{\alpha} > 0$ *and* $\underline{c} \in (0,1]$*).*

- **The weak upper scaling condition** *holds if there are numbers* $\overline{\alpha} < 2$, $\overline{\theta} \geq 0$ *and* $\overline{C} \in [1,\infty)$ *such that*

$$\phi(\lambda\theta) \leq \overline{C}\lambda^{\overline{\alpha}}\phi(\theta) \quad \text{for} \quad \lambda \geq 1, \quad \theta > \overline{\theta}. \tag{16}$$

In short, $\phi \in WUSC(\overline{\alpha},\overline{\theta},\overline{C})$. *For* **global** *weak upper scaling we require* $\overline{\theta} = 0$ *in* (16).

It is possible to characterize of the scaling conditions in terms of almost monotone functions. We have $\phi \in \mathrm{WLSC}(\underline{\alpha},\underline{\theta},\underline{c})$ if and only if $\phi(\theta) = \kappa(\theta)\theta^{\underline{\alpha}}$ and κ is almost increasing on $(\underline{\theta},\infty)$ with oscillation factor $\underline{c}$. Similarly, $\phi \in \mathrm{WUSC}(\overline{\alpha},\overline{\theta},\overline{C})$ if and only if $\phi(\theta) = \kappa(\theta)\theta^{\overline{\alpha}}$ and κ is almost decreasing on $(\overline{\theta},\infty)$ with oscillation factor $\overline{C}$. Moreover if $\phi \geq 0$ is continuous and increases to infinity and $\phi \in \mathrm{WLSC}(\underline{\alpha},\underline{\theta},\underline{c})$ [or $\phi \in \mathrm{WUSC}(\overline{\alpha},\overline{\theta},\overline{C})$]. Then $\phi^{-1} \in \mathrm{WUSC}(1/\underline{\alpha},\phi(\underline{\theta}),\underline{c}^{-1/\underline{\alpha}})$ [$\mathrm{WLSC}(1/\overline{\alpha},\phi(\overline{\theta}),\overline{C}^{-1/\overline{\alpha}})$, respectively].

Weak scaling conditions may be considered for any exponents, but we will use them for the symbol ψ of a process $\{X_t\}_{t\geq 0}$, therefore we limit the ranges of exponents $\underline{\alpha}$ and $\overline{\alpha}$ in definitions. Since it is known from (5) that the characteristic exponent of the isotropic unimodal Lévy processes is an almost increasing function (hence (15) always holds for $\underline{\alpha} = 0$) and $\sqrt{\psi}$ is subadditive (thus (16) always holds for $\overline{\alpha} = 2$). Typical examples of the characteristic exponents satisfying weak scaling properties are $\psi(\xi) = |\xi|^\alpha + |\xi|^\beta$ and $\psi(\xi) = |\xi|^\alpha \ln^{1-\alpha/2}(1+|\xi|^\beta)$, for α, $\beta \in (0,2)$. More examples are presented in Section 3.4 in [H2], and in Section 4.1 in [H3].

Weak scaling conditions are also connected to O-regularly varying functions and Matuszewska indices. Let $\phi : [0,\infty) \to [0,\infty)$ and let $\beta(\phi) \leq \alpha(\phi)$ be the lower and upper Matuszewska indices [8, p. 68] of ϕ, respectively. By [8, Theorem 2.2.2], if $\phi \in \mathrm{WLSC}(\underline{\alpha},\underline{\theta},\underline{c})$ for some $\underline{\theta} \geq 0$ and $\underline{c} \in (0,1]$, then $\beta(\phi) \geq \underline{\alpha}$, and if $\phi \in \mathrm{WUSC}(\overline{\alpha},\overline{\theta},\overline{C})$ for some $\overline{\theta} \geq 0$ and $\overline{C} \in [1,\infty)$, then $\alpha(\phi) \leq \overline{\alpha}$. And a partial converse holds. Since we have that, if $\underline{\alpha} < \beta(\phi)$, then $\underline{\theta} \geq 0$ and $\underline{c} \in (0,1]$ exist such that $\phi \in \mathrm{WLSC}(\underline{\alpha},\underline{\theta},\underline{c})$, and if $\overline{\alpha} > \alpha(\phi)$, then $\overline{\theta} \geq 0$ and $\overline{C} \in [1,\infty)$ exist such that $\phi \in \mathrm{WUSC}(\overline{\alpha},\overline{\theta},\overline{C})$. We note that the scalings may, but need not hold for $\underline{\alpha} = \beta(\phi)$ and $\overline{\alpha} = \alpha(\phi)$. Furthermore, in what follows it is important to specify the ranges of θ for which the inequalities in (16) and (15) hold, in particular, the cases $\underline{\theta} = 0$ and $\overline{\theta} = 0$ are qualitatively different from the cases $\underline{\theta} > 0$ and $\overline{\theta} > 0$. These remarks explain why we need to state our assumptions in terms of weak scaling, rather than only use Matuszewska indices.

We will present now the Abelian-type theorem. Namely, assuming weak scaling conditions for the Lévy kernel we proved that the symbol of the process also satisfies the weak scalings. The following proposition is useful to check whether the assumptions of the main theorems presented below are satisfied, when the starting point is the Lévy measure instead of the characteristic exponent.

Proposition 7 (Proposition 28 in [H3] and Proposition 8 in [H2]). *Let $\{X_t\}_{t\geq 0}$ be a pure-jump symmetric Lévy process in $\mathbb{R}^d$ with the Lévy measure $N(dx) = \nu(x)dx$ and the characteristic exponent ψ. Suppose that $\theta \in [0, \infty)$, constant $c \in (0, 1]$ and non-decreasing function $f : (0, \infty) \to (0, \infty)$ are such that*

$$c\frac{f(1/|x|)}{|x|^d} \leq \nu(x) \leq c^{-1}\frac{f(1/|x|)}{|x|^d}, \quad 0 < |x| < 1/\theta.$$

(i) If $f \in WLSC(\underline{\alpha}, \theta, \underline{c})$, then $\psi \in WLSC(\underline{\alpha}, \theta, C^*)$ for some constant C^*.

(ii) If $f \in WLSC(\underline{\alpha}, \theta, \underline{c}) \cap WUSC(\overline{\alpha}, \theta, \overline{C})$, then $f(|\xi|)$ and $\psi(\xi)$ are comparable for $|\xi| > \theta$.

II. Heat kernel of the whole $\mathbb{R}^d$

The study of the heat kernel is an important area of interactions between Probability, Analysis and Geometry. Transition density function provides direct access to path properties of a Markov process. It is also the fundamental solution of the heat equation with the infinitesimal generator of the corresponding process. Therefore the heat kernel gives also information about the spectral properties of the generator. Heat kernels for second order elliptic operators have been well studied and there are many beautiful results (i.e. [22, 28, 60, 27, 6, 49, 50, 63, 32, 31]). The heat kernels for symmetric non-local operators or equivalently transition densities of jump-type Markov processes are quite well understood when the jumping measure (the Lévy measure) is absolutely continuous and the Matuszewska indices at the origin of its density are strictly between $-2-d$ and $-d$. Estimates for rather general jump-type Markov processes were obtained for instance in papers [19, 14, 13, 3, 29]. Transition semigroups of Lévy processes allow for a deeper insight and direct approach from several directions thanks to their convolutional structure and the available Fourier techniques. For instance bounds for transition densities are obtained in [42, 58, 59, 37] by using Fourier inversion, complex integration, saddle point approximation or the Davies' method.

The main result of [H3] provides estimates for the tails of the one-dimensional distribution of the isotropic unimodal Lévy process $\{X_t\}_{t\geq 0}$ and its density function $p_t(x)$, expressed in terms of the Lévy-Khintchine exponent ψ. Since ψ is radially almost increasing, it is comparable with its radial non-decreasing majorant ψ^*, and we employ ψ^* in statements and proofs. The extensive usage of ψ (ψ^*) rather than ν is a characteristic feature of our development and may be considered natural from the point of view of pseudo-differential operators and spectral theory [34]. As usual for Fourier transform, the asymptotics of ψ at infinity translates into the asymptotics of p_t and ν at the origin. The following theorem is one of the main result of [H3].

Theorem 8 (Theorem 21 in [H3]). *Let $\{X_t\}_{t\geq 0}$ be an isotropic unimodal Lévy process in $\mathbb{R}^d$. If $\psi \in WLSC(\underline{\alpha}, \theta, \underline{c})$, then there is $C^* = C^*(d, \underline{\alpha}, \underline{c})$ such that*

$$p_t(x) \leq C^* \min\left\{ \left[\psi^-(1/t)\right]^d, \frac{t\psi^*(1/|x|)}{|x|^d} \right\} \quad \text{if } t>0 \text{ and } t\psi^*(\theta) < 1/\pi^2.$$

If $\psi \in WLSC(\underline{\alpha}, \theta, \underline{c}) \cap WUSC(\overline{\alpha}, \theta, \overline{C})$, then $c^* = c^*(d, \underline{\alpha}, \underline{c}, \overline{\alpha}, \overline{C})$, $r_0 = r_0(d, \underline{\alpha}, \underline{c}, \overline{\alpha}, \overline{C})$ exist such that

$$p_t(x) \geq c^* \min\left\{ \left[\psi^-\left(1/t\right)\right]^d, \frac{t\psi^*(1/|x|)}{|x|^d} \right\} \qquad \text{if} \quad t>0, \quad t\psi^*(\theta/r_0) < 1 \quad \text{and} \quad |x|\,\theta < r_0.$$

Hence under assumption that ψ satisfies weak upper and lower scalings with exponents strictly between 0 and 2 we obtained sharp estimates and they may be summarized as follows,

$$p_t(x) \approx \left[\psi^-\left(1/t\right)\right]^d \wedge \frac{t\psi^*(|x|^{-1})}{|x|^d} \approx p_t(0) \wedge [t\nu(x)], \tag{17}$$

We note that the corresponding estimates of ν, to wit,

$$\nu(x) \approx \frac{\psi^*(|x|^{-1})}{|x|^d}, \tag{18}$$

are simply obtained as a consequence of (17) and are not an element of the proof of estimates for the heat kernel. We also emphasized that comparability constants depend only on the dimension and the scaling characteristics.

The main idea used in the proof of Theorem 8 is unimodality of the density p_t. Since by radial monotonicity of $y \mapsto p_t(y)$ we have $p_t(x) \le p_t(0)$ and

$$p_t(x) \le \frac{\mathbb{P}\left(|x|/2 \le |X_t| < |x|\right)}{\left|B_{|x|} \setminus B_{|x|/2}\right|} \le \frac{1}{(1-2^{-d})|B_1|}\mathbb{P}\left(|X_t| \ge \frac{|x|}{2}\right)|x|^{-d}. \tag{19}$$

And, for any number $\lambda > 1$,

$$p_t(x) \ge \frac{\mathbb{P}\left(|x| \le |X_t| < \lambda|x|\right)}{\left|B_{\lambda|x|} \setminus B_{|x|}\right|} = \frac{d}{(\lambda^d - 1)\omega_d}\left(\mathbb{P}\left(|X_t| \ge |x|\right) - \mathbb{P}\left(|X_t| \ge \lambda|x|\right)\right)|x|^{-d}. \tag{20}$$

Hence, it was enough to proved appropriate estimates for the tail of one-dimensional distributions and $p_t(0)$ to get estimates for p_t. To do it we used Tauberian-like methods and the Fourier analysis.

At the beginning we obtained the following estimates for the Laplace transform of the tail function for $|X_t|^2$.

Lemma 9 (Lemma 4 in [H3]). *Let $f_t(\rho) = \mathbb{P}(|X_t|^2 > \rho)$. There is a constant $C_1 = C_1(d)$ such that*

$$C_1^{-1}\frac{1}{\lambda}\left(1 - e^{-t\psi^*(\sqrt{\lambda})}\right) \le \mathcal{L}f_t(\lambda) \le C_1\frac{1}{\lambda}\left(1 - e^{-t\psi^*(\sqrt{\lambda})}\right), \qquad \lambda > 0.$$

A consequence of the above lemma and monotonicity of the tail function is the upper bound for this function. In fact this estimate holds for every symmetric Lévy processes.

Proposition 10 (Corollary 6 in [H3]). *For $r > 0$ we have $\mathbb{P}(|X_t| \ge r) \le \frac{2e}{e-1}(2d+1)\left(1 - e^{-t\psi^*(1/r)}\right)$.*

As a consequence of the above proposition and (19) we obtained a general upper bound for transition density of unimodal Lévy processes.

Proposition 11 (Corollary 7 in [H3]). *There is $C = C(d)$ such that*

$$p_t(x) \le Ct\psi^*(1/|x|)/|x|^d, \qquad x \ne 0.$$

To obtain estimates for $p_t(0)$ we used the inverse Fourier transform, since the weak lower condition ensure that $e^{-t\psi}$ is integrable. Hence, we have

$$p_t(0) = (2\pi)^{-d}\int_{\mathbb{R}^d} e^{-t\psi(\xi)}d\xi$$

and under WLSC for ψ we proved that $p_t(0) \le C[\psi^-(1/t)]^d$. We should note that the lower bound in such form always holds, since

$$p_t(0) \ge (2\pi)^{-d}\int_{\psi^*(|\xi|)\le 1} e^{-t\psi^*(|\xi|)}d\xi \ge c(d)[\psi^-(1/t)]^d.$$

These imply that under WLSC for the symbol $p_t(0) \approx [\psi^-(1/t)]^d$.

Using WUSC condition for the symbol ψ and Lemma 9 we obtained the lower bound for the tail function.

Lemma 12 (Lemma 14 in [H3]). *$C = C(d)$ exists such that if $\psi \in WUSC(\overline{\alpha}, \overline{\theta}, \overline{C})$ and $a = [(2-\overline{\alpha})C]^{\frac{2}{2-\overline{\alpha}}} \overline{C}^{\frac{\overline{\alpha}-2}{2}}$, then*

$$\mathbb{P}(|X_t| \geq r) \geq a\left(1 - e^{-t\psi^*(1/r)}\right), \qquad 0 < r\overline{\theta} < \sqrt{a}.$$

Thus, under WUSC assumption for the Lévy-Khinchine exponent, we obtained a sharp estimate of the tail of distribution $|X_t|$. Using weak convergence of appropriate normed distribution of X_t to the Lévy measure and [H5, Proposition 5.2 (i)] we provided that the lower bound in Lemma 12 is equivalent to the condition WUSC. A consequence of the above lemma, the weak lower scaling condition for ψ and (20) is the lower bound of Theorem 8.

We emphasized the lower bound of the density p_t in Theorem 8 holds if and only if ψ has lower and upper scalings of order strictly between 0 and 2, equivalently if the lower and upper Matuszewska indices are strictly between 0 and 2. Namely, we showed that for the unimodal Lévy processes, the scaling of ψ (at infinity) is *equivalent* to the bounds (17) for the transition density and the Lévy measure (at the origin). In fact, already the lower bound $\nu(x) \geq c\psi^*(|x|^{-1})/|x|^d$ implies such scalings of ψ.

Theorem 13 (Theorem 26 in [H3]). *Let $\{X_t\}_{t\geq 0}$ be an isotropic unimodal Lévy process in $\mathbb{R}^d$ with transition density p, Lévy-Khintchine exponent ψ and Lévy measure density ν. The following are equivalent:*

(i) WLSC and WUSC hold for ψ.

(ii) There are $r, c > 0$, such that

$$p_t(x) \geq c\frac{t\psi^*(|x|^{-1})}{|x|^d}, \qquad 0 < |x| < r,\ 0 < t\psi^*(|x|^{-1}) < 1.$$

(iii) There are $r, c > 0$, such that

$$\nu(x) \geq c\frac{\psi^*(|x|^{-1})}{|x|^d}, \qquad 0 < |x| < r.$$

If we instead assume global WLSC and WUSC in (i), and let $r = \infty$ in (ii) and (iii), then the three conditions are equivalent, too.

Theorem 8 yields the implication $(i) \Rightarrow (ii)$. The implication $(ii) \Rightarrow (iii)$ follows because $\lim_{t\to 0^+} p(t,x)/t = \nu(x)$ vaguely on $\mathbb{R}^d \setminus \{0\}$. To prove that (iii) implies (i), we constructed a complete Bernstein function φ with the Lévy kernel μ such that $\varphi(|x|^2) \approx \psi(x)$ and $|x|^{d-2}\nu(x) \leq c(d)\mu(|x|^2)$, for some constant $c(d)$. Next combining a fact that μ is completely monotone with Tauberian-like theorems we proved that $\varphi(r) \approx r\varphi'(r)$, what implies φ satisfies weak scaling conditions and therefore ψ as well. To infer from (iii) WUSC condition for ψ, we used the conjugate Bernstein function $\varphi_1(r) = r/\varphi(r)$. Using Tauberian-type theorem on the density of the potential measure of a subordinator associated with the function φ_1 we showed that $\varphi_1(r) \approx r\varphi_1'(r)$. This comparability implies the lower scaling condition for φ_1, which is equivalent to the upper scaling condition for φ with exponent less than 1 and in consequence WUSC for ψ with exponent strictly less than 2.

In the paper [H4] using Lemma 3 we proved the general lower bound.

Proposition 14 (Lemma 1.11 in [H4]). *For all $t > 0$ and $x \neq 0$,*

$$p_t(x) \geq 4^{-d} t\, \nu(x) \left[\mathbb{P}^0(\tau_{B_{|x|/2}} > t)\right]^2 .$$

Combining the above proposition with estimates for the survival probability due to Pruitt [52, page 954] we obtain the following bound for the heat kernel.

Corollary 15 (Corollary 1.13 in [H4]). *There is $c = c(d) > 0$ such that if $x \neq 0$ and $0 < t < c/\psi(1/|x|)$, then $p_t(x) \geq 4^{-d-1} t\nu(x)$.*

We obtained the following bounds for any isotropic unimodal Lévy process, for $0 < t < c/\psi^*(1/|x|)$,

$$4^{-d-1}\, t\, \nu(x) \leq p_t(x) \leq c(d)\, t\, \psi^*(1/|x|)|x|^{-d}.$$

The above bounds were extended in the paper [Pre2]. We provided estimates of the survival probability for process in a ball when it starts from the center for every $t > 0$. In addition, we improved the upper bound for the heat kernel p_t. Finally we obtained existing constants $c = c(d)$ and $C = C(d)$ such that

$$C^{-1} t\nu(x) e^{-c\, h(|x|)t} \leq p_t(x) \leq Ct(-h'(|x|))|x|^{1-d}, \qquad t > 0,\ x \neq 0. \tag{21}$$

We emphasized that $0 < -r\, h'(r) \leq 2h(r)$. The above estimate is sharp for $0 < t < c/\psi^*(1/|x|)$, not only for processes, whose the characteristic exponents satisfy WLSC and WUSC conditions, but also, when ψ is slowly varying function.

III. Regularity of harmonic functions and estimates for Green function

We introduce several definitions. It is known that the classical harmonic functions have the mean value property. Similarly, we define harmonic functions with respect to a stochastic process. A function $f : \mathbb{R}^d \to [0, \infty)$ is said that is *harmonic* with respect to $\{X_t\}_{t\geq 0}$ in an open set D if for any bounded open set B such that $\bar{B} \subset D$

$$f(x) = \mathbb{E}^x f(X_{\tau_B}), \qquad x \in B.$$

If the mean value property holds also for $B = D$ we say that a function is *regular harmonic.*

We say that the *scale invariant Harnack inequality* holds for a process $\{X_t\}_{t\geq 0}$ if for any $R > 0$ there exists a constant $C = C(R)$ such that for any function g non-negative on $\mathbb{R}^d$ and harmonic in a ball B_r, $r \leq R$,

$$\sup_{x\in B_{r/2}} g(x) \leq C \inf_{x\in B_{r/2}} g(x).$$

We say that the *global* scale invariant Harnack inequality holds if the constant in the above inequality does not depend on R that means $\sup_{R>0} C(R) < \infty$.

The main purpose of [H2] was proving the scale invariant Harnack inequality and regularity properties for harmonic functions with respect to the isotropic unimodal Lévy process with the characteristic exponent satisfying the weak lower scaling condition. Our main contribution is the fact that we assumed only a mild condition for the characteristic exponent but we do not use in our proofs any properties of the Lévy measure except its isotropy and unimodality. Usually in the existing literature on the Harnack inequality for Lévy processes the assumptions are given in terms of the behaviour of the Lévy measure (see [57, Section 3]) or the initial step relies on describing its behaviour ([41]). Our result seems to be important for application to subordinated Brownian motions because there are examples when the characteristic exponent is known, while estimates for the Lévy measure are not. We should also notice that our approach allows to deal with isotropic unimodal processes with the Lévy-Khinchine exponent behaving at infinity almost like the exponent for the Brownian motion, which were not treated in the literature, except a few particular cases. Namely, we can take $\psi(x) = |x|^2\ell(|x|)$, where ℓ is slowly varying and goes to 0 at infinity. An example of such process is for instance a process with density of its Lévy measure equal to $|x|^{-d-2}\log^{-2}(2 + |x|^{-1})$. Moreover, our result allows to extend the scale invariant Harnack inequality to its global version for many processes. For instance we get the global scale invariant Harnack inequality for the α-stable relativistic processes.

Theorem 16 (Theorem 1 in [H2]). *Let $d \geq 3$. Suppose that $\{X_t\}_{t\geq 0}$ is an isotropic unimodal Lévy process in $\mathbb{R}^d$. If $\psi \in WLSC(\underline{\alpha}, \underline{\theta}, \underline{c})$, then the scale invariant Harnack inequality holds. Moreover, if ψ satisfies the global weak lower scaling condition, then the global scale invariant Harnack inequality holds.*

The next theorem deals with regularity of harmonic functions.

Theorem 17 (Theorem 2 in [H2]). *Let $d \geq 3$. Suppose that $\{X_t\}_{t\geq 0}$ is an isotropic unimodal Lévy process in $\mathbb{R}^d$ and $\psi \in WLSC(\underline{\alpha}, \underline{\theta}, \underline{c})$. For any $R > 0$ there exist constants $c = c(R)$ and $\delta > 0$ such that, for any $0 < r \leq R$, and any bounded function g, which is harmonic in B_r,*

$$|g(x) - g(y)| \leq c||g||_\infty \left(\frac{|x-y|}{r}\right)^\delta, \qquad x, y \in B_{r/2}.$$

The assumption $d \geq 3$ in the two above theorems can be removed in the case of subordinated Brownian motions (see Theorem 7 in [H2]). For a general isotropic unimodal Lévy process $\{X_t\}_{t\geq 0}$ the assumption $d \geq 3$ ensures not only that $\{X_t\}_{t\geq 0}$ is transient but also that for some $\varepsilon \in (0,1)$, the function $r \to r^{d-\varepsilon}\psi^*(1/r)$ is almost increasing. This property was necessary in our approach.

We follow in the approach of Bass and Levin ([5]) to prove the above theorems. The key ingredient of the proofs of the Harnack inequality and local Hölder continuity of harmonic functions is the Krylov-Safonov type estimate, which says that there are c, $R > 0$ and $\lambda < 1$ such that for $r < R$ and a closed set $A \subset B_{\lambda r}$,

$$\mathbb{P}^x(T_A < \tau_{B_r}) \geq c\frac{|A|}{|B_r|}, \qquad x \in B_{\lambda r}. \tag{22}$$

We proved this estimate for the isotropic unimodal Lévy processes with the symbol satisfying WLSC.

Proposition 18 (Proposition 7 in [H2]). *Let $d \geq 3$ and $\psi \in WLSC(\underline{\alpha}, \underline{\theta}, \underline{c})$. There exist $c = c(d, \underline{\alpha}, \underline{c})$ and $\lambda = \lambda(d, \underline{\alpha}, \underline{c}) \in (0,1)$ such that (22) holds for $r\underline{\theta} < 1$.*

We notice if ψ satisfies global WLSC, then Krylov-Safonov type estimate holds for every $r > 0$.

In addition to the Krylov-Safonov type estimate there are required two another estimates to use the approach of Bass and Levin. The first one holds for every symmetric Lévy processes whose the characteristic exponent of every one-dimensional projection satisfies WLSC. The following lemma is a consequence of the Dynkin formula

$$\mathbb{E}^x f(X_{\tau_D}) - f(x) = G_D\mathcal{L}f(x), \quad f \in Dom(\mathcal{L}) \tag{23}$$

estimates for the expected exit time from the ball when process starting from its center ([52]) and comparability of Pruitt function h and ψ^*, see (3).

Lemma 19 (Part of Corollary 2 in [H2]). *Let $\{X_t\}_{t\geq 0}$ be a symmetric Lévy process. Assume that for every $\Theta \in \mathbf{S}^{d-1}$ a function $f(t) = \psi(t\Theta)$ belongs to $WLSC(\underline{\alpha}, \underline{\theta}, \underline{c})$. Then there is $C = C(d, \underline{\alpha}, \underline{c}, \sup_{|y|=\underline{\theta}} \psi(y)/\psi^*(\underline{\theta}))$ such that, for $0 < r\,\underline{\theta} < 1$ and $s \leq r/2$,*

$$\mathbb{P}^x(|X_{\tau_{B_s}}| \geq r) \leq C\left(\frac{s}{r}\right)^{\underline{\alpha}}, \qquad |x| \leq s.$$

In particular the constant C in the above lemma depends only on the dimension and the scaling characteristics in the isotropic unimodal setting. The second result that we need in our discussion is the following lemma.

Lemma 20 (Proposition 6 in [H2]). *Let $d \geq 3$ and $\psi \in WLSC(\underline{\alpha}, \underline{\theta}, \underline{c})$. There exist $c = c(d, \underline{\alpha}, \underline{c})$ and $\lambda = \lambda(d, \underline{\alpha}, \underline{c}) \in (0,1]$ such that for any $r\,\underline{\theta} < 1$, and any non-negative function H such that* $\mathrm{supp} H \subset \overline{B_r}^c$,

$$\mathbb{E}^x H(X_{\tau_{B_{\lambda r}}}) \leq C\mathbb{E}^y H(X_{\tau_{B_r}}), \qquad x, y \in B_{\frac{\lambda r}{2}}.$$

Using the Ikeda-Watanabe formula, the equilibrium measure and isoperimetric inequality proved by Watanabe [62] we reduced proofs of the above lemma and the Krylov-Safonov type estimate to sharp estimates of the capacity and the potential measure of balls and the Green function of $\mathbb{R}^d$ of the underlying process. In [H2, Proposition 2] we obtained for $d \geq 3$ that the potential measure of balls satisfies $U(B_r) = \int_0^\infty \mathbb{P}(X_t \in B_r)dt \approx 1/\psi^*(1/r)$, where comparability constant depends only on the dimension. To prove it we used a Tauberian-like theorem. Next, we used estimates for $U(B_r)$ to obtain the estimates for 0-order capacity of the ball. Namely, we used a formula

$$\mathbb{P}^x(T_{B_r} < \infty) = \int_{\mathbb{R}^d} G(y-x)\rho_B(dy),$$

where ρ_{B_r} is the equilibrium measure for the closed ball $\overline{B}_r$, to prove that

$$Cap(\overline{B_r}) = \rho_{B_r}(\overline{B_r}) \approx r^d/U(B_r) \approx r^d\psi^*(1/r).$$

The comparability constants again depend only on the dimension. Another consequence of estimates for the potential measure of balls is an upper bound for the potential kernel. By unimodality of G we have

$$G(x) \leq \frac{U(B_{|x|})}{|B_{|x|}|} \leq \frac{c(d)}{\psi^*(1/|x|)|x|^d}.$$

The lower bound in the same form was proven under WLSC for ψ.

Theorem 21 (Theorem 3 in [H2]). *Let $d \geq 3$ and let $\{X_t\}_{t\geq 0}$ be an isotropic unimodal Lévy process in $\mathbb{R}^d$. If $\psi \in WLSC(\underline{\alpha}, \theta, \underline{c})$, then there are constants $c = c(d, \underline{\alpha}, \underline{c}) < 1$ and $C = C(d, \underline{\alpha}, \underline{c})$ such that*

$$\frac{C}{|x|^d\psi^*(|x|^{-1})} \leq G(x), \qquad \theta\,|x| \leq c.$$

Moreover under the assumption that $\{X_t\}_{t\geq 0}$ is a subordinated Brownian motion governed by a special subordinator we obtained that $G(x) \approx \frac{1}{\psi^*(1/|x|)|x|^d}$ if and only if ψ satisfies WLSC (see [H2, Theorem 5]). Recently we improved it in the paper [Pre2](Theorem 5.11) for general isotropic unimodal processes in dimension $d \geq 6$. Using [Pre3, Theorem 1.2] instead of [Pre2, Theorem 5.9] it can also be shown that in fact the claim is true for $d \geq 3$.

Harnack inequality describes behaviour of harmonic functions inside sets - the quotient of the supremum and the infimum is bounded. While the boundary Harnack inequality says that for harmonic functions which vanish outside the set the decay near the boundary is the same for all of them. We finish this section with the boundary Harnack inequality with the explicit decay near the boundary of the smooth set, described by the characteristic exponent.

Proposition 22 (Proposition 7.6 in [H5]). *Let ν be continuous in $\mathbb{R}^d \setminus \{0\}$. Assume that ψ satisfies the global weak lower and upper scaling conditions, D is $C^{1,1}$ at scale $\rho > 0$, $z \in \partial D$, $0 < r < \rho$ and $u \geq 0$ is regular harmonic in $D \cap B(z,r)$ and vanishes in $B(z,r) \setminus D$. Then positive $c = c(d,\psi)$, $c_1 = c_1(d,\psi)$ exist such that*

$$\frac{u(x)}{u(y)} \leq c\frac{V(\delta_D(x))}{V(\delta_D(y))} \leq c_1\sqrt{\frac{\psi^*(1/\delta_D(y))}{\psi^*(1/\delta_D(x))}}, \qquad x,y \in D \cap B(z,r/2).$$

This proposition is a consequence of the boundary Harnack inequality for general sets ([40]), where the decay is described by the expected exit time from the set and estimates of this mean value for smooth set obtained in [H5]. The research on this topic was continued in the paper [Pre3] and we can drop assumption on continuity of the Lévy kernel, because instead of the result of [40] we can use [Pre3, Theorem 1.9].

IV. Expected exit time and survival probability

In the proofs of the results described in this section we used the scale invariant Harnack inequality and estimates of Green function of the whole $\mathbb{R}^d$, which are discussed in the previous section. The main objects of this section are $s_D(x) = \mathbb{E}^x\tau_D$ expected exit time from the set D and $\mathbb{P}^x(\tau_D > t)$ the survival probability for a fixed time $t > 0$, considered as functions of the starting point of the process. The mean value of the first exit time is naturally important from probabilistic point of view and applications in mathematical finance and insurance. Since due to the boundary Harnack inequality the mean value of the first exit time describes the decay of harmonic functions near the boundary, therefore its estimates are also important for Analysis. This is due to the fact that the expected exit time is a (mild) superharmonic function for infinitesimal generator of the semigroup of the process and in some sense, $\mathcal{L}s_D = -1$ on D (see discussion below of the Dynkin like operator). Therefore, not only the decay of harmonic functions, but also the decay of any continuous solutions of the Dirichlet problem

$$\begin{cases} \mathcal{L}u = f, & \text{on } D, \\ u = 0, & \text{on } D^c, \end{cases}$$

for a bounded function f, is not slower than the decay of the expected exit time from D (it is easy consequence of the maximum principle of the generator $\mathcal{L}$).

The basic object of interest in our study described in this section is $\mathbb{E}^x\tau_{B_r}$, the expected exit time from the ball B_r centred at the origin and with radius $r > 0$, for arbitrary starting point $x \in \mathbb{R}^d$ of $\{X_t\}_{t\geq 0}$. For every Lévy processes there are estimates when $x = 0$ ([52]). When $\{X_t\}_{t\geq 0}$ is symmetric, there are constants $c = c(d)$ and $C = C(d)$ such that

$$\frac{c}{h(r)} \leq \mathbb{E}^0\tau_{B_r} \leq \frac{C}{h(r)}, \qquad r > 0. \tag{24}$$

In the one-dimensional case (that is for intervals) we extended the above Pruitt estimates to every starting point x and to every symmetric Lévy process which is not a compound Poisson process

$$\mathbb{E}^x\tau_{(-r,r)} \approx \frac{C}{\sqrt{h(r-|x|)h(r)}}, \qquad r > 0,\ |x| < r, \tag{25}$$

where the comparability constant is absolute. We conjecture that the same estimates hold true for every isotropic Lévy processes in $\mathbb{R}^d$, but we proved it only for a large class of isotropic unimodal processes.

Let $\{X_t\}_{t\geq 0}$ be an isotropic unimodal Lévy process. Recall that the renewal function V and its derivative V' are strictly positive on $(0,\infty)$. Below we state the assumption under which we obtained in [H4] the estimates of the expected exit time of the process from the set. An explanation of the reason for using this assumption is given below together with the discussions of the Dynkin like operator.

Definition 23. *We say that condition* (**H**) *holds if for every $r > 0$ there is $H_r \geq 1$ such that*

$$V(z) - V(y) \leq H_r V'(x)(z-y) \quad \textit{whenever} \quad 0 < x \leq y \leq z \leq 5x \leq 5r. \tag{26}$$

We say that (**H***) *holds if $H_\infty = \sup_{r>0} H_r < \infty$.*

We consider (**H**) and (**H***) as Harnack type conditions because (**H**) is implied by the following property:

$$\sup_{x\le r, y\in[x,5x]} V'(y) \le H_r \inf_{x\le r, y\in[x,5x]} V'(y), \qquad r>0. \tag{27}$$

Both conditions control relative growth of V. If (**H**) holds, then we may and do chose H_r non-decreasing in r. Each of the following situations imply (**H**):

1. $\{X_t\}_{t\ge 0}$ is a subordinated Brownian motion governed by a special subordinator (see Lemma 7.5 in [H5]).
2. $d \ge 3$ and the characteristic exponent of X satisfies WLSC (see (5.1) and Lemma 7.2 in [H5]).
3. $d \ge 1$ and the characteristic exponent of X satisfies WLSC and WUSC (see (5.2) and Lemma 7.3 in [H5]).
4. $\{X_t\}_{t\ge 0}$ has the non trivial Gaussian part (see Lemma 7.4 in [H5]).

A more detailed discussion of (**H**) and further examples are given in Section 7 in [H5].

One of the main result presented in this section is estimate for the expected exit time from a ball. Recall that by $\delta_D(x)$ we denote the distance of $x \in \mathbb{R}^d$ to D^c.

Theorem 24 (Thorem 4.1 in [H5]). *Let $\{X_t\}_{t\ge0}$ be an isotropic unimodal Lévy process in $\mathbb{R}^d$. If* (**H**) *holds, then there is $C = C(d)$ such that for $r > 0$,*

$$\frac{C}{H_r} V(\delta_{B_r}(x))V(r) \le \mathbb{E}^x \tau_{B_r} \le 2V(\delta_{B_r}(x))V(r), \quad x \in \mathbb{R}^d. \tag{28}$$

Recall that instead of $V(r)$ we can use $1/\sqrt{h(r)}$ or $1/\sqrt{\psi^*(1/r)}$ but we decided to choose V for simplicity. The upper bound is true for all isotropic Lévy processes by the one-dimensional estimates and a domain monotonicity (it is enough to consider an appropriate strip). To prove the lower bound we use the Dynkin like operator

$$\mathcal{A}f(x) = \limsup_{s\to0^+} \frac{\mathbb{E}^x f(X_{\tau_{B(x,s)}}) - f(x)}{\mathbb{E}^x \tau_{B(x,s)}}.$$

A very useful property of this operator is that the harmonic functions in a set D are in the domain of this operator on D and it is equal to 0 for such functions. This property allows for calculating the Dynkin like operator for the composition of V with the distance to the complement of the ball or to the ball and the calculations only minimally depend on the differential regularity of V via the condition (**H**). We proved that these compositions are super- and subharmonic, respectively. Next we use the maximum principle for $\mathcal{A}$ for these functions. Using the above theorem, the Ikeda-Watanabe formula and again the maximum principle we obtained the expected exit time from more general domains.

Theorem 25 (Theorem 4.6 in [H5]). *Let $\{X_t\}_{t\ge0}$ be an isotropic unimodal Lévy process in $\mathbb{R}^d$. If* (**H**) *holds and $D \subset \mathbb{R}^d$ is open, bounded and $C^{1,1}$ at scale $r > 0$, then $c = c(d)$ and $C = C(d)$ exist such that*

$$\frac{c}{H_r} V(\delta_D(x))V(r) \le \mathbb{E}^x \tau_D \le C \frac{H_r}{(\mathcal{J}(r))^2} \frac{V^2(\operatorname{diam} D)}{V^2(r)} V(\delta_D(x))V(r), \quad x \in \mathbb{R}^d,$$

where $\mathcal{J}(r) = \inf_{0<\rho\le r}\left[V^2(\rho)\int_{B_r^c}\nu(y)dy\right]$

By [H5, Proposition 5.2] $\mathcal{J}$ is positive on the neighbourhood of 0 if and only if ψ satisfies WUSC. Moreover for ψ satisfying global WLSC and WUSC there is a constant $C>0$ such that $H_r<C$ and $\mathcal{J}(r)>C^{-1}$ for every $r>0$. If an open set D additionally is convex then we do not use the Ikeda-Watanabe formula and $\mathcal{J}$ does not appear in the upper bound (see [H5, Corollary 4.2]). Moreover, we can treat more general isotropic processes with the absolutely continuous Lévy measure with the density comparable to a unimodal function ([H5, Corollary 4.3]).

We present results concern the survival probabilities, first for the half-line and general symmetric Lévy processes.

Proposition 26 (Proposition 2.6 in [H5]). *For every symmetric Lévy process in* $\mathbb{R}$ *which is not compound Poisson,*

$$\mathbb{P}^x(\tau_{(0,\infty)} \ge t) \approx 1 \wedge \frac{1}{\sqrt{t\psi^*(1/x)}}, \quad t, x > 0, \tag{29}$$

and the comparability constant is absolute.

As a corollary we have upper bound for survival probability in bounded convex domain D for every isotropic processes

$$\mathbb{P}^x(\tau_D \ge t) \le C\left(1 \wedge \frac{V(\delta_D(x))}{\sqrt{t}}\right), \quad t>0,\, x\in D.$$

In particular this is true for balls. Analogously for sets which a complement is convex we obtained the lower bound for the survival probability

$$\mathbb{P}^x(\tau_D \ge t) \ge c\left(1 \wedge \frac{V(\delta_D(x))}{\sqrt{t}}\right), \quad t>0,\, x\in D.$$

In particular we can use it for the complement of the ball. Under global WLSC and WUSC for the symbol we obtained the following bounds.

Theorem 27 (Theorem 6.3 in [H5]). *Let* $\{X_t\}_{t\ge 0}$ *be an isotropic unimodal Lévy process in* $\mathbb{R}^d$. *Suppose that* $\psi \in WLSC(\underline{\alpha}, 0, \underline{c}) \cap WUSC(\overline{\alpha}, 0, \overline{C})$. *Let* $R>0$ *and* $D=\overline{B}_R^c$.

(i) *There is a constant* $C^* = C^*(d, \underline{\alpha}, \underline{c}, \overline{\alpha}, \overline{C})$ *such that,*

$$\mathbb{P}^x(\tau_D > t) \le C^*\left(\frac{V(\delta_D(x))}{\sqrt{t}\wedge V(R)} \wedge 1\right), \qquad t>0.$$

(ii) *If* $d>\overline{\alpha}$, *then*

$$\mathbb{P}^x(\tau_D > t) \approx \frac{V(\delta_D(x))}{\sqrt{t}\wedge V(R)} \wedge 1, \qquad t>0,$$

where the comparability constant depends only on d, $\underline{\alpha}$, $\underline{c}$, $\overline{\alpha}$, $\overline{C}$.

Noteworthy the constants in the above bounds do not depend on the radius. Apart from the half-spaces and the complement of a ball we obtained in [H4] also sharp estimates for bounded $C^{1,1}$ domains, half-space like domains and the $C^{1,1}$ exterior domains (D^c is bounded).

In the article [H1] we considered the isotropic stable processes and we prove the following estimates for more general sets.

Theorem 28 (Theorem 2 in [H1]). *Let $\alpha \in (0,2)$ and let $\{X_t\}_{t\geq 0}$ be an isotropic α-stable Lévy process in $\mathbb{R}^d$. There is a constant $C = C(d,\alpha,\kappa)$ such that if D is $(\kappa, T^{1/\alpha})$-fat then*

$$C^{-1}\frac{s_D(x)}{s_D(A_{t^{1/\alpha}}(x))} \leq \mathbb{P}^x(\tau_D > t) \leq C\frac{s_D(x)}{s_D(A_{t^{1/\alpha}}(x))}, \quad 0 < t \leq T, \; x \in \mathbb{R}^d \tag{30}$$

where $s_D(x) = \mathbb{E}^x\tau_D$ if D is bounded and s_D is the Martin kernel at infinity otherwise.

In particular we have, for $D = \overline{B^c}_1$,

$$\mathbb{P}^x(\tau_D > t) \approx \begin{cases} 1 \wedge \frac{\delta_D^{\alpha/2}(x)}{(1\wedge t^{1/\alpha})^{\alpha/2}}, & d > \alpha, \\ 1 \wedge \frac{\log(1+\delta_D^{1/2}(x))}{\log(1+t^{1/2})} & d = 1 = \alpha, \\ \frac{\delta_D^{\alpha-1}(x)\wedge\delta_D^{\alpha/2}(x)}{(t^{1/\alpha}\vee\delta_D(x))^{\alpha-1}\wedge(t^{1/\alpha}\vee\delta_D(x))^{\alpha/2}}, & d = 1 < \alpha. \end{cases} \tag{31}$$

By the scaling property of the stable processes we obtain the estimates for the ball with arbitrary radius.

V. Dirichlet heat kernel

The main result presented in this section is the approximate factorization (32), which involves the transition density of the free process described in Section II and the survival probabilities discussed in Section IV. Dirichlet heat kernel provides direct access to properties of operators with Dirichlet conditions. For instance the harmonic measure and the Green function are expressed by the kernel. We recall that precise estimates for the heat kernel of the Laplacian (and the Brownian motion) were given for $C^{1,1}$ domains in 2002 by Zhang [63]. In 2006 Siudeja [56] gave upper bounds for the heat kernel of the fractional Laplacian (the generator of the isotropic stable Lévy process) in convex sets. In 2010 Chen, Kim and Song [15] gave sharp (two-sided) explicit estimates for the heat kernel of the fractional Laplacian in bounded $C^{1,1}$ open sets. Gradual extensions were then obtained for generators of many subordinate Brownian motions satisfying scaling conditions [15, 17, 16, 18], and for processes with comparable Lévy measure [41]. Rather precise but less explicit bounds of $p_D(t,x,y)$ are also known to hold for Lipschitz sets in a number of situations. Such bounds were first obtained for the Laplacian in 2003 by Varopoulos [61]. In this section we present the main results of articles [H1] and [H4]. In [H1] we proved that the following approximate factorization holds for the fractional Laplacian.

Theorem 29 (Theorem 1 in [H1]). *Let $\alpha \in (0,2)$ and let $\{X_t\}_{t\geq 0}$ be an isotropic α-stable Lévy process in $\mathbb{R}^d$. If D is $(\kappa, T^{1/\alpha})$-fat then there is $C = C(\alpha, D)$ such that for $0 < t < T$ and all $x, y \in \mathbb{R}^d$,*

$$C^{-1}\mathbb{P}^x(\tau_D > t)\mathbb{P}^y(\tau_D > t) \leq \frac{p_D(t,x,y)}{p(t,x,y)} \leq C\mathbb{P}^x(\tau_D > t)\mathbb{P}^y(\tau_D > t)\,. \tag{32}$$

The comparison (32) is uniform in time and space for cones, homogeneous Lipschitz domains and exterior $C^{1,1}$ sets. For these sets, (32) is made rather explicit by approximating the survival probability with superharmonic functions of $\{X_t\}_{t\geq 0}$ (Theorem 28). The approximate factorization of the heat kernel and the estimates of the survival probability in the above non-smooth setting are closely related to the boundary Harnack inequality. The setting offers a structured approach to the heat kernel estimates of non-local operators. Since for the fractional Laplacian the Green function of many domains are obtained or estimated due to Theorem 28 we get numerous examples of sets where the estimates are sharp and explicit. Unfortunately in the general unimodal setting there were no estimates for the Green function of the bounded $C^{1,1}$ sets, except for a class of the subordinated Brownian motion. Moreover the boundary Harnack inequality (the main tool used in the proof of Theorem (29)) does not hold in this general. For instance, the so-called truncated stable Lévy process is manageable by [H4] but cannot be resolved by previous methods because the boundary Harnack inequality fails in this case.

To bound the heat kernel $p_D(t,x,y)$ of the unimodal Lévy process $\{X_t\}_{t\geq 0}$ and the $C^{1,1}$ set D we use the estimates of the free transition density $p(t,x,y)$ from [H3] and the estimates of superharmonic functions of $\{X_t\}_{t\geq 0}$ at the boundary of D from [H5].

First we present the estimates for bounded sets. Let D be an open bounded set. In the remainder of the section we assume that $p_t(0)$ is finite for every $t > 0$. Then the semigroup of integral operators on $L^2(D)$ with kernels $p_D(t,x,y) \leq p_t(0)$ is compact, in fact Hilbert-Schmidt. General theory yields eigenvalues $0 < \lambda_1(D) < \lambda_2(D) \leq \ldots$ and orthonormal basis of eigenfunctions $\phi_1 \geq 0, \phi_2, \phi_3 \ldots$:

$$\phi_k(x) = e^{\lambda_k(D)t} \int p_D(t,x,z)\phi_k(z)dz.$$

In what follows, we interchangeably write $\lambda_1(D) = \lambda_1$.

Our estimates are generally expressed in terms of V, the renewal function of the ladder-height process of one-dimensional projections of $\{X_t\}_{t\geq 0}$, but they could equivalently be expressed in terms of the more familiar Lévy-Khintchine exponent ψ of $\{X_t\}_{t\geq 0}$ or the Pruitt function h. Accordingly, we observed a wide range of power-like asymptotics of heat kernels. The derivative of V is the *éminence grise* of the papers [H4] and [H5]. It is quite delicate to control V', but under a mild Harnack-type condition (**H**), V' only influences the comparability constants, not the *structure* of the estimates, thus allowing for the present generality of results.

The following theorem gives sharp estimates for general bounded $C^{1,1}$ sets. Our estimates are *global*, that is hold with a uniform constant for all $t > 0$ and $x, y \in \mathbb{R}^d$.

Theorem 30 (Theorem 4.5 in [H4]). *Let $\{X_t\}_{t\geq 0}$ be an isotropic unimodal Lévy process. Suppose that $\psi \in$ WLSC $\cap$ WUSC. There is $r_0 = r_0(d,\psi) > 0$ such that if $0 < r < r_0$ and open $D \subset \mathbb{R}^d$ is bounded and $C^{1,1}$ at scale r, and $\nu(\operatorname{diam} D) > 0$, then for all $x, y \in \mathbb{R}^d$, $t > 0$,*

$$p_D(t,x,y) \approx \mathbb{P}^x(\tau_D > t/2)\mathbb{P}^y(\tau_D > t/2)p\left(t \wedge V^2(r), x, y\right) \tag{33}$$

and

$$\mathbb{P}^x\left(\tau_D > t\right) \approx e^{-\lambda_1 t}\left(\frac{V(\delta_D(x))}{\sqrt{t} \wedge V(r)} \wedge 1\right). \tag{34}$$

If the scalings are global, then we may take $r_0 = \infty$ and comparability constants depending only on d, $\operatorname{diam} D/r$ and scaling characteristics of ψ.

In addition, we obtained sharp estimates for the first eigenvalue λ_1 in [H4, Proposition 4.4]. Due to intrinsic ultracontractivity obtained in [D2] for very large class of symmetric Lévy processes we have a sharp bounds for the ground state for the generator $\mathcal{L}$ with the Dirichlet condition. For instance we obtained bounds for an arbitrary ball with the comparability constant that does not depend on the radius.

Corollary 31 (Corollary 4.7 in [H4]). *Let ψ satisfy global* WLSC *and* WUSC. *Let $r > 0$ and $\phi_1^{(r)}$ be the (positive) eigenfunction corresponding to $\lambda_1(r) = \lambda_1(B_r)$. Then there is $c = c(d,\psi)$ such that*

$$c^{-1}\frac{V(\delta_D(x))}{r^{d/2}V(r)} \leq \phi_1^{(r)}(x) \leq c\frac{V(\delta_D(x))}{r^{d/2}V(r)}, \qquad x \in \mathbb{R}^d.$$

The proof of the above theorem is done in tree steps. First we proved the upper bound for small time, next the lower bound also for small time, and finally we used the spectral theory to prove estimates for large time. We start with the following upper bound, which elaborates Lemma 3 for the complement of the ball.

Theorem 32 (Theorem 2.1 in [H4]). *Let $\{X_t\}_{t\geq 0}$ be an isotropic unimodal Lévy process. Suppose that $p(t,x) \leq tF(|x|)$, $t>0$, $x\neq 0$, with non-increasing $F\geq 0$. Let* (**H**) *hold, $R>0$ and $D=\overline{B}_R^c$. Then we have, for $0<t\leq V^2(|x-y|)$ and $x,y\in D$,*

$$p_D(t,x,y) \leq C\frac{H_R^2}{\mathcal{J}(R)^4}\left(\frac{V(\delta_D(x))}{\sqrt{t}\wedge V(R)}\wedge 1\right)\left(\frac{V(\delta_D(y))}{\sqrt{t}\wedge V(R)}\wedge 1\right) tF(|x-y|/9). \tag{35}$$

Due to Proposition 11 we can always take $F(r)=\psi^*(1/r)$, but recently in [Pre2] we obtained better estimates that holds for every isotropic unimodal process and are sharp for a larger class of symbols than WLSC and WUSC. For instance we obtained sharp estimates for gamma variance process for small time and space. Moreover in [Pre3] we proved the scale invariant Harnack inequality for this type processes, so in particular (**H**) holds for them. Therefore we can use the above theorem to such processes.

The consequence of scalings and Theorem 32 is the following upper bound.

Theorem 33 (Theorem 2.6 in [H4]). *Let $\{X_t\}_{t\geq 0}$ be an isotropic unimodal Lévy process. Let $R>0$ and let D be an open set satisfying the outer ball condition at scale R. Suppose that global* WLSC *and* WUSC *hold for ψ. Then there is a constant $C=C(d,\psi)$ such that for all $t>0$ and $x,y\in D$,*

$$p_D(t,x,y) \leq C\left(\frac{V(\delta_D(x))}{\sqrt{t}\wedge V(R)}\wedge 1\right)\left(\frac{V(\delta_D(y))}{\sqrt{t}\wedge V(R)}\wedge 1\right) p(t,x,y).$$

Another consequence of Lemma 3 is the lower bound for the Dirichlet heat kernel. Below we state the estimates of the heat kernel of the union of two balls with the same radius. We used it in [H4] to obtain the lower bound for general $C^{1,1}$ sets.

Theorem 34 (Theorem 3.3 in [H4]). *Let $\{X_t\}_{t\geq 0}$ be an isotropic unimodal Lévy process. Let $R>0$ and $\psi\in WLSC(\underline{\alpha},R^{-1},\underline{c})\cap WLSC(\overline{\alpha},R^{-1},\overline{C})$. Let $D=B(z_1,R)\cup B(z_2,R)$. There exist $c=c(d,\psi)$ and $C=C(d,\psi)$ such that*

$$p_D(t,x,y) \geq C\left(\frac{V(\delta_D(x))}{\sqrt{t}}\wedge 1\right)\left(\frac{V(\delta_D(y))}{\sqrt{t}}\wedge 1\right)\left(p_{t/2}(0)\wedge [t\,\nu(2|x-y|\wedge \mathrm{diam}(D))]\right),$$

provided $0<t\leq cV^2(R)$, $x,y\in D$, $\delta_D(x)=\delta_{B(z_1,R)}(x)$ and $\delta_D(y)=\delta_{B(z_2,R)}(y)$.

Apart from the bounded sets we considered also the *exterior* sets, which are the complement of a *bounded* set, and a *half-space-like* set which is the one included between two translates of a half-space. We thus covered bounded and some unbounded $C^{1,1}$ sets. Unbounded sets are especially challenging: the $C^{1,1}$ condition does not specify their geometry at infinity, whereas the geometry strongly influences the asymptotics of the heat kernel. We note that the exterior $C^{1,1}$ sets and the half-space-like sets were studied for the fractional Laplacian in [H1] and [20]. The case of the subordinate Brownian motions with global scalings is resolved in [41] for the halfspace, and [P1, H1] handle the fractional Laplacian in cones. The presented estimates for the heat kernel of exterior sets in Theorem 35 were new even for the sum of two independent isotropic stable Lévy processes. Noteworthy, the comparability constants in the estimates do not change upon dilation of D if the scalings of the Lévy-Khintchine exponent of $\{X_t\}_{t\geq 0}$ are global, which is an added bonus of our approach. In general we strived to control comparability constants because they may be important in scaling arguments and applications to more general Markov processes.

The next theorem may be considered as the main result discussed in this section. For the isotropic α-stable proceeses a similar result was proven in [H1] (Theorem 3).

Theorem 35 (Theorem 5.4 in [H4]). *Let $\{X_t\}_{t\geq 0}$ be an isotropic unimodal Lévy process. Let $\psi \in WLSC(\underline{\alpha}, 0, \underline{c}) \cap WUSC(\overline{\alpha}, 0, \overline{C})$ and $d > \overline{\alpha}$. Let D be a $C^{1,1}$ at scale R_1 and $D^c \subset \overline{B_{R_2}}$. Constants $c_* = c_*(d,\psi), c^* = c^*(d,\psi)$ exist such that for all $x, y \in \mathbb{R}^d$, $t > 0$,*

$$c_* \left(\frac{R_1}{R_2}\right)^{4+2d} \left(\frac{V(\delta_D(x))}{\sqrt{t} \wedge V(R_1)} \wedge 1\right) \left(\frac{V(\delta_D(y))}{\sqrt{t} \wedge V(R_1)} \wedge 1\right) p(t,x,y) \leq p_D(t,x,y)$$

and

$$p_D(t,x,y) \leq c^* \left(\frac{V(\delta_D(x))}{\sqrt{t} \wedge V(R_1)} \wedge 1\right) \left(\frac{V(\delta_D(y))}{\sqrt{t} \wedge V(R_1)} \wedge 1\right) p(t,x,y).$$

We also obtained the following estimates for the half-space like domains.

Theorem 36 (Theorem 5.8 in [H4]). *Let $\{X_t\}_{t\geq 0}$ be an isotropic unimodal Lévy process. Let ψ satisfy global* WLSC *and* WUSC*, D be $C^{1,1}$ at scale R and $\mathbb{H}_a \subset D \subset \mathbb{H}_b$. Then for all $x, y \in \mathbb{R}^d$ and $t > 0$,*

$$p_D(t,x,y) \approx \mathbb{P}^x(\tau_D > t)\mathbb{P}^y(\tau_D > t)p(t,x,y) \qquad \text{and} \qquad \mathbb{P}^x(\tau_D > t) \approx \frac{V(\delta_D(x))}{\sqrt{t}} \wedge 1,$$

and constants in the comparisons may be so chosen to depend only on d, ψ, $a-b$ and R.

In particular we have $\mathbb{P}^x(\tau_D > t) \approx \mathbb{P}^x(\tau_D > t/2)$ in the above two cases of unbounded D, hence the approximate factorizations in all the three cases above may be considered identical in bounded time. In fact, Theorem 32 and Theorem 34 give estimates which essentially resolve the asymptotics of the heat kernels in bounded time and space for every $C^{1,1}$ open set D, regardless of the geometry of D at infinity.

5. Description of other scientific achievements

Besides the five papers, which constitute mono-thematic series of publications, after Ph.D., I published five articles, two was accepted and they are waiting for publication, another three were submitted to journals. Total number of my publications is 18, the number of citations, according to the Web of Science database ('Sum of the Times Cited' on 2017-02-16), is 132 (118 without self-citations), and the h-index (Hirsh index) is 7. Total *impact factor* of the journals for five publications included in the *scientific achievement*, according to the Journal Citation Reports, is 7,044; total *impact factor* of the journals for all publications is 13,008, see Table 1.

Table 1: Impact factor of the journals according to Journal Citation Report from the publication year (or 2015 for publications from 2016 and 2017)

article	journal	publication year	impact factor
[H1]	Annals Prob.	2010	1.470
[H2]	Potential Anal.	2014	0.992
[H3]	J. Funct. Anal.	2014	1.322
[H4]	Stoch. Pr. Appl.	2014	1.056
[H5]	Prob. Th. Rel. Fields	2015	2.204
[P1]	Colloq. Math.	2010	–
[P2]	Colloq. Math	2012	0.403
[P3]	Potential Anal.	2016	0.956
[P4]	J. Math. Anal. Appl.	2017	1.014
[P6]	Trans. Amer. Math. Soc.	2015	1.196*
[P7]	Prob. Math. Stat.	2016	0.315**
[P5]	Potential Anal.	2017	0.956
[D1]	Illinois J. Math	2007	0.558
[D2]	Prob. Math. Stat.	2008	–
[D3]	Potential Anal.	2008	0.566
		Sum:	13,008

* - article [P6] is waiting for publication since September 2015.
** - article [P7] is waiting for publication since February 2016.

[P1] K. Bogdan, T. Grzywny. *Heat kernel of fractional Laplacian in cones,* Colloquium Mathematicum 118(2), 365–377 (2010).

[P2] T. Grzywny, M. Ryznar. *Potential theory of one-dimensional geometric stable processes*, Colloquium Mathematicum 129(1), 7–40 (2012).

[P3] T. Grzywny, M. Ryznar. *Hitting times of points and intervals for symmetric Lévy processes*, Potential Analysis, 1–39, DOI:10.1007/s11118-016-9600-z (2016).

[P4] W. Cygan, T. Grzywny. *Heat content for convolution semigroups*, Journal of Mathematical Analysis and Applications 446(2). 1393–1414 (2017).

[P5] T. Grzywny, K. Szczypkowski. *Kato classes for Lévy processes*, Potential Analysis, 1–32, DOI:10.1007/s11118-017-9614-1 (2017).

[P6] W. Cygan, T. Grzywny, B. Trojan. *Asymptotic behavior of densities of unimodal convolution semigroups* , w druku w Transactions of the American Mathematical Society, 1–22, DOI: 10.1090/tran/6830.

[P7] T. Grzywny, T. Jakubowski, G. Żurek. *Green function for gradient perturbation of unimodal Lévy processes*, w druku w Probability and Mathematical Statistics, 1–25, (`http://www.math.uni.wroc.pl/~pms/forthcoming.php`).

[Pre1] T. Grzywny, K.-Y. Kim, P. Kim. *Estimates of Dirichlet heat kernel for symmetric Markov processes*, 1–40, arxiv: http://arxiv.org/abs/1512.02717.

[Pre2] T. Grzywny, M. Ryznar and B. Trojan. *Asymptotic behaviour and estimates of slowly varying convolution semigroups*, 1–35, arxiv: http://arxiv.org/abs/1606.04178.

[Pre3] T. Grzywny, M. Kwaśnicki. *Potential kernels, probabilities of hitting a ball, harmonic functions and the boundary Harnack inequality for unimodal Lévy processes*, 1–34, arxiv: https://arxiv.org/abs/1611.10304.

Before PhD I published the following tree papers, which will not be discussed here.

[D1] T. Grzywny, M. Ryznar, *Estimates of Green Function for some perturbations of fractional Laplacian*, Illinois Journal of Mathematics 51(4), 1409–1438 (2007).

[D2] T. Grzywny, *Intrinsic ultracontractivity for Lévy processes*, Probability and Mathematical Statistics 28, 91–106 (2008).

[D3] T. Grzywny, M. Ryznar, *Two-sided optimal bounds for Green functions of half-spaces for relativistic α-stable process*, Potential Analysis 28(3), 201–239 (2008).

I will now discuss the results obtained in the papers [P1]-[P5] and [Pre1]-[Pre3].

Hitting probability

The main results of the paper [P3] are estimates and asymptotics for the tail function for hitting times of points and intervals for the symmetric Lévy processes in the real line, provided the process hits every point with positive probability. For example we showed that if ψ has the weak lower scaling property with index $\underline{\alpha} > 1$ then

$$\mathbb{P}^x(T_{\{0\}} > t) \approx \frac{1}{t\psi^{-1}(1/t)|x|\psi(1/x)} \wedge 1, \quad x \in \mathbb{R}, t > 0.$$

Under the assumption that the process is unimodal we do not need to assume scaling and we obtained the following estimates

$$\mathbb{P}^x(T_{\{0\}} > t) \approx \frac{K(x)}{K(1/\psi^{-1}(1/t))} \wedge 1,$$

where K is the compensated potential kernel

$$K(x) = \int_0^\infty (p_t(0) - p_t(x))dt.$$

In addition to hitting times of points, we provided estimates for the tail function for hitting time of the interval for any symmetric Lévy process, whose characteristic exponent satisfies the condition WLSC with the exponent $\underline{\alpha} > 1$. To prove this result we obtained the global Harnack inequality for the considered processes, which was quite surprising, because we assume only scaling for ψ and not anything more. In particular, the Lévy measure may be purely atomic.

Heat content

In the article [P4] we considered the quantity

$$H_\Omega(t) = \int_\Omega \mathbb{P}^x(X_t \in \Omega)dx$$

related to an arbitrary Lévy process $\{X\}_{t\geq 0}$ in $\mathbb{R}^d$, which is called the *heat content*. We obtained estimates of $H_\Omega(t)$ in terms of the Pruitt function h and the asymptotic behaviour as t goes to zero. IN particular we provided asymptotics for all Lévy processes with the finite variation in full generality. For this purpose, we used the semigroup theory and that Lipschitz functions vanishing at infinity are in the domain of the infinitesimal generator. We proved that

$$|\Omega| - H_\Omega(t) \sim tC(\Omega),$$

where $C(\Omega)$ is an explicit constant depending on the process. In addition, we described the asymptotic behaviour of the heat content for isotropic processes whose characteristic exponent ψ is a regularly varying function at infinity with index $\alpha > 1$. In this situation

$$|\Omega| - H_\Omega(t) \sim 1/\psi^{-1}(1/t)c(d,\alpha)Per(\Omega),$$

where $Per(\Omega)$ is a variational measure of the boundary of the set Ω. For regular sets, for instance Lipschitz, $Per(\Omega)$ is equal to a Hausdorff measure of the boundary of Ω.

Heat kernels asymptotics

The classic result obtained by Pólya ($d = 1$) and Blumenthal and Getoor ($d \geq 2$) gives the asymptotic behaviour of the heat kernel of the isotropic α-stable processes, $\alpha \in (0,2)$,

$$\lim_{t|x|^{-\alpha}\to 0} \frac{p_t(x)}{t|x|^{-d-\alpha}} = \mathcal{A}_{d,\alpha}, \tag{36}$$

where $\mathcal{A}_{d,\alpha}$ is some constant. In the paper [P6] we considered isotropic unimodal Lévy processes, whose symbols vary regularly at 0 or at infinity. We obtained an analogous limit as in the stable case, if the exponent of regularity of the symbol is strictly between 0 and 2. It should be emphasized that the limit is uniform when time t and position x satisfy an appropriate limiting condition. In addition, we proved that conversely the existence of such limit necessitates regular variation of the characteristic exponent. The main result of [P6] gives an analogue of Theorem 13, replacing inequalities by limits. Besides studying the off-diagonal behaviour of the heat kernel we provided limit on the on-diagonal, too. We proved the strong ratio limit theorem.

In the paper [Pre2] we improved methods developed in [P6]. We obtained the asymptotic behaviour of the heat kernel both on and off the diagonal, in the case when the symbol belongs to the so-called de Haan class, which is a subclass of slowly varying functions. We also proved that this de Haan condition is equivalent to the fact that the Lévy kernel is regularly varying function with index $-d$.

Kato classes

The Kato class plays an important role in the theory of stochastic processes and in the theory of operators that emerge as generators of stochastic processes. The definition of the Kato class may differ according to the underlying probabilistic or analytical problem. In the first case the primary definition of the Kato condition is

$$\lim_{t\to 0^+}\left[\sup_x \mathbb{E}^x\left(\int_0^t |q(X_u)|\,du\right)\right]=0\,. \tag{37}$$

Here q is a Borel function on the state space of the process $\{X_t\}_{t\geq 0}$. By the Khas'minskii Lemma the condition yields sufficient local in time regularity of the corresponding Schrödinger (Feynman-Kac) semigroup

$$\widetilde{P}_t f(x)=\mathbb{E}^x\left[\exp\left(-\int_0^t q(X_u)\,du\right)f(X_t)\right].$$

The condition (37) can be understood as a smallness condition with respect to time. The alternative definition of the Kato condition is given by the following space smallness,

$$\lim_{r\to 0^+}\left[\sup_x \mathbb{E}^x\left(\int_0^\infty e^{-\lambda u}\mathbf{1}_{B(x,r)}(X_u)|q(X_u)|\,du\right)\right]=0\,, \tag{38}$$

for some $\lambda > 0$ (equivalently for every $\lambda > 0$). If we denote the resolvent by G^λ, then the above limit is of the following form

$$\lim_{r\to 0^+}\left[\sup_x G^\lambda(\mathbf{1}_{B(x,r)}|q|)(x)\right]=0.$$

The main result of [P5] answers the question if the definitions of the Kato class through the semigroup and through the resolvent of the Lévy process in $\mathbb{R}^d$ coincide.

Theorem 37. *Let $\{X_t\}_{t\geq 0}$ be a Lévy process in $\mathbb{R}^d$. The conditions* (37) *and* (38) *are NOT equivalent if and only if* 0 *is regular for* $\{0\}$, *which means* $\mathbb{P}^0(T_{\{0\}}=0)=1$.

We also give an analytic reformulation of these results by means of the characteristic exponent of the process. Furthermore, when these definitions are not equivalent, we fully describe (37) and (38). We should emphasise that the above theorem holds for *arbitrary* Lévy process and the proof depends on the structure of the process, especially for the asymmetric processes.

Slowly varying symbols

Most results obtained in papers [H1]-[H5] concern the isotropic unimodal Lévy processes with characteristic exponents satysfying weak scaling conditons WLSC and WUSC or at least WLSC. The processes with logaritmically growing symbol are not well studied and understood. In the paper [P2] we studied the geometric stable processes ($\psi(\xi) = \ln(1 + |\xi|^\alpha)$, $\alpha \in (0, 2]$) on the real line, and proved among other things, the scale invariant Harnack inequality and estimates for the Green function and the Poisson kernel for intervals and the half-line. It was the first result of this type for the process with the slowly varying symbol. In the paper [Pre2], we focused on estimates of the heat kernel and the Green function for the free processes which characteristic exponents that do not satisfy WLSC. Among other things we obtained sharp estimates for the heat kernel of processes with the Lévy kernel form $\frac{\ell(1/|x|)}{|x|^d}$, where ℓ is a bounded function slowly varying at infinity. The symbol of this process varies slowly as well and

$$\psi(\xi) \sim \int_1^{|\xi|} \frac{\ell(s)}{s} ds.$$

Under this assumption the heat kernel has the following estimate

$$p_t(x) \approx t \frac{\ell(1/|x|)}{|x|^d} e^{-t\psi(1/|x|)}.$$

We should emphasize that in this case the function p_t is not bounded and cannot be expressed by the inverse Fourier transform, therefore we cannot use the standard estimates obtained by Fourier analysis discussed the previous sections.

The paper [Pre3] mainly concerns the processes with the symbol ψ not satisfying WLSC. In the first part of this work, we proved two-sided estimates of hitting probabilities of balls, and we improved estimates obtained in [H2] and [Pre2] for the Green function of $\mathbb{R}^d$. In the second part we proved the scale invariant boundary Harnack inequality and Martin representation of harmonic functions, in particular the scale invariant Harnack inequality, under mild assumptions on the Lévy kernel. As an application, we provided sharp two-sided estimates of the Green function of a half-space.

Heat kernels of killed Markov processes

The heat kernels with the Dirichlet condition were also the topic of papers [P1], [P3] and [Pre1]. In [P1] we studied the heat kernels for cones for the fractional Laplacian, where we obtained sharp estimates. It was the first such result for the unbounded sets for non-local operator. These work was also the first stage of the study developed later in the paper [H1]. The article [P3] mainly concerned the hitting probability for points and intervals for general symmetric Lévy processes on the real line. As an applications of the obtained estimates we also found estimates for the heat kernels of processes killed after hitting the interval for isotropic unimodal Lévy processes, whose symbols satisfy global weak scaling conditions with exponents between 1 and 2. We thus extended Theorem 35 to $1 = d < \underline{\alpha}$. In this situation, the factorization (32) also holds, by the estimates of the survival probabilities are more complex than for transient processes, as discussed in Theorem

35 above. We should emphasize that the recurrent processes, except the isotropic stable processes [H1], were not explored with regard to the heat kernel estimates for exterior sets, and to the best of our knowledge our result was the first one with such generality.

Apart form the Lévy processes we considered more general Markov processes in the work [Pre1]. We obtained sharp estimates for the heat kernels of killed Markov processes, whose jump measures are comparable with the Lévy measure of some isotropic unimodal Lévy process. The results appear to a wide class of Markov processes and they are also new for the Lévy processes, because in this paper we considered more general sets than $C^{1,1}$.

Gradient perturbation

The paper [P7] concerns the study of the operator $\mathcal{L} + b(x) \cdot \nabla$ on bounded subsets of $\mathbb{R}^d$, $d \geq 2$, where $\mathcal{L}$ is the generator of an isotropic unimodal Lévy process, whose the symbol satisfies scaling conditions WLSC and WUSC with exponents greater than 1, and b is a vector field from an appropriate Kato class. The operator $\mathcal{L}$ is of higher order than ∇, therefore $\mathcal{L} + b(x) \cdot \nabla$ is an operator $\mathcal{L}$ perturbed by an operator of the lower order. The main result of the paper is comparability of the Green function of $\mathcal{L}$ and the Green function of the perturbation on bounded $C^{1,1}$ sets, under additional mild assumption on the Lévy kernel. In particular, the subordinated Brownian motions satisfy the additional assumption. The basic tool was the Duhamel formula for Green functions, which we proved in [P7]

$$\tilde{G}_D(x,y) = G_D(x,y) + \int_D \tilde{G}_D(x,z) b(z) \cdot \nabla_z G_D(z,y) dz.$$

Here $\tilde{G}_D(x,y)$, $G_D(x,y)$ are Green functions for a set D for the operator $\mathcal{L}$ and its gradient perturbation respectively. The immediate consequence of this result was the scale invariant boundary Harnack inequality for harmonic functions with respect to $\mathcal{L} + b(x) \cdot \nabla$.

References

[1] S. Aljančić, D. Arandelović. 0-regularly varying functions. *Publ. Inst. Math. (Beograd) (N.S.)*, 22(36):5–22, 1977.

[2] D. Applebaum. Lévy processes—from probability to finance and quantum groups. *Notices Amer. Math. Soc.*, 51(11): 1336–1347, 2004.

[3] M. T. Barlow, A. Grigor'yan, T. Kumagai. Heat kernel upper bounds for jump processes and the first exit time. *J. Reine Angew. Math.*, 626:135–157, 2009.

[4] R. F. Bass, M. Kassmann. Harnack inequalities for non-local operators of variable order. *Trans. Amer. Math. Soc.*, 357(2):837–850, 2005.

[5] R. F. Bass, D. Levin. Harnack inequalities for jump processes. *Potential Anal.*, 17:375–388, 2002.

[6] M. Berger, P. Gauduchon, E. Mazet. *Le spectre d'une variété riemannienne.* Lecture Notes in Mathematics, Vol. 194. Springer-Verlag, Berlin-New York, 1971.

[7] J. Bertoin. *Lévy processes*, volume 121 of *Cambridge Tracts in Mathematics.* Cambridge University Press, Cambridge, 1996.

[8] N. H. Bingham, C. M. Goldie, J. L. Teugels. *Regular variation*, volume 27 of *Encyclopedia of Mathematics and its Applications.* Cambridge University Press, Cambridge, 1987.

[9] R. M. Blumenthal, R. K. Getoor. Some theorems on stable processes. *Trans. Amer. Math. Soc.*, 95:263–273, 1960.

[10] K. Bogdan, K. Burdzy, Z.-Q. Chen. Censored stable processes. *Probab. Theory Related Fields*, 127(1):89–152, 2003.

[11] L. A. Caffarelli, A. Vasseur. Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation. *Ann. of Math. (2)*, 171(3):1903–1930, 2010.

[12] Z.-Q. Chen, P. Kim. Global Dirichlet Heat Kernel Estimates for Symmetric LA©vy Processes in Half-Space. *Acta Appl. Math.*, 146:113–143, 2016.

[13] Z.-Q. Chen, P. Kim, T. Kumagai. Global heat kernel estimates for symmetric jump processes. *Trans. Amer. Math. Soc.*, 363(9):5021–5055, 2011.

[14] Z.-Q. Chen, P. Kim, T. Kumagai. On heat kernel estimates and parabolic Harnack inequality for jump processes on metric measure spaces. *Acta Math. Sin. (Engl. Ser.)*, 25(7):1067–1086, 2009.

[15] Z.-Q. Chen, P. Kim, R. Song. Heat kernel estimates for the Dirichlet fractional Laplacian. *J. Eur. Math. Soc. (JEMS)*, 12(5):1307–1329, 2010.

[16] Z.-Q. Chen, P. Kim, R. Song. Dirichlet heat kernel estimates for $\Delta^{\alpha/2} + \Delta^{\beta/2}$. *Illinois J. Math.*, 54(4):1357–1392, 2012.

[17] Z.-Q. Chen, P. Kim, R. Song. Global heat kernel estimate for relativistic stable processes in exterior open sets. *J. Funct. Anal.*, 263(2):448–475, 2012.

[18] Z.-Q. Chen, P. Kim, R. Song. Dirichlet heat kernel estimates for rotationally symmetric Lévy processes. *Proc. London Math. Soc. (3)*, 109(1):90–120, 2014.

[19] Z.-Q. Chen, T. Kumagai. Heat kernel estimates for jump processes of mixed types on metric measure spaces. *Probab. Theory Related Fields*, 140(1-2):277–317, 2008.

[20] Z.-Q. Chen, J. Tokle. Global heat kernel estimates for fractional Laplacians in unbounded open sets. *Probab. Theory Related Fields*, 149(3-4):373–395, 2011.

[21] R. Cont, P. Tankov. Financial modelling with jump processes. Chapman & Hall/CRC Financial Mathematics Series. Chapman & Hall/CRC, Boca Raton, FL, 2004.

[22] E. B. Davies. *Heat kernels and spectral theory*, volume 92 of *Cambridge Tracts in Mathematics.* Cambridge University Press, Cambridge, 1989.

[23] R. A. Doney. *Fluctuation theory for Lévy processes*, volume 1897 of *Lecture Notes in Mathematics.* Springer, Berlin, 2007. Lectures from the 35th Summer School on Probability Theory held in Saint-Flour, July 6–23, 2005, Edited and with a foreword by Jean Picard.

[24] C. Dupuis. *Mesure de Hausdorff de la trajectoire de certains processus à accroissements indépendants et stationnaires*, Séminaire de Probabilités, VIII (Univ. Strasbourg, année universitaire 1972-1973), Lecture Notes in Math., vol. 381, Springer, 1974, pp. 37–77.

[25] B. Fristedt. Sample functions of stochastic processes with stationary, independent increments. In *Advances in probability and related topics, Vol. 3*, pages 241–396. Dekker, New York, 1974.

[26] R. K. Getoor. First passage times for symmetric stable processes in space. *Trans. Amer. Math. Soc.*, 101:75–90, 1961.

[27] P. B. Gilkey. *Invariance theory, the heat equation, and the Atiyah-Singer index theorem.* Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, second edition, 1995.

[28] A. Grigor'yan. Heat kernels on metric measure spaces with regular volume growth. In *Handbook of geometric analysis, No. 2*, volume 13 of *Adv. Lect. Math. (ALM)*, pages 1–60. Int. Press, Somerville, MA, 2010.

[29] A. Grigor'yan, J. Hu, K.-S. Lau. Estimates of heat kernels for non-local regular Dirichlet forms. *Trans. Amer. Math. Soc.*, 366(12):6397–6441, 2014.

[30] A. Grigor'yan, L. Saloff-Coste. Heat kernel on connected sums of Riemannian manifolds. *Math. Res. Lett.*, 6(3-4):307–321, 1999.

[31] A. Grigor'yan, L. Saloff-Coste. Dirichlet heat kernel in the exterior of a compact set. *Comm. Pure Appl. Math.*, 55(1):93–133, 2002.

[32] A. Grigor'yan, L. Saloff-Coste. Heat kernel on manifolds with ends. *Ann. Inst. Fourier (Grenoble)*, 59(5):1917–1997, 2009.

[33] F. Hiroshima, J. Lörinczi. Lieb-Thirring bound for Schrödinger operators with Bernstein functions of the Laplacian. *Commun. Stoch. Anal.*, 6(4):589–602, 2012.

[34] N. Jacob. *Pseudo differential operators and Markov processes. Vol. I.* Imperial College Press, London, 2001. Fourier analysis and semigroups.

[35] N. Jacob, V. Knopova, S. Landwehr, R.L. Schilling. A geometric interpretation of the transition density of a symmetric Lévy process. *Sci. China Math.*, 55(6):1099–1126, 2012.

[36] K. Kaleta, M. Kwaśnicki, J. Małecki. One-dimensional quasi-relativistic particle in the box. *Rev. Math. Phys.*, 25(8):1350014, 52, 2013.

[37] K. Kaleta, P. Sztonyk. Upper estimates of transition densities for stable-dominated semigroups. *J. Evol. Equ.*, 13(3):633–650, 2013.

[38] G. Karch, W. A. Woyczyński. Fractal Hamilton-Jacobi-KPZ equations. *Trans. Amer. Math. Soc.*, 360(5):2423–2442, 2008.

[39] P. Kim, R. Song, Z. Vondraček. Potential theory of subordinate Brownian motions revisited. In *Stochastic analysis and applications to finance*, volume 13 of *Interdiscip. Math. Sci.*, pages 243–290. World Sci. Publ., Hackensack, NJ, 2012.

[40] P. Kim, R. Song, Z. Vondraček. Uniform boundary Harnack principle for rotationally symmetric Lévy processes in general open sets. *Sci. China Math.*, 55(11):2317–2333, 2012.

[41] P. Kim, R. Song, Z. Vondraček. Global uniform boundary Harnack principle with explicit decay rate and its application. *Stochastic Process. Appl.*, 124(1):235–267, 2014.

[42] V. Knopova, A. M. Kulik. Exact asymptotic for distribution densities of Lévy functionals. *Electron. J. Probab.*, 16:no. 52, 1394–1433, 2011.

[43] V. Knopova, R. L. Schilling. Transition density estimates for a class of Lévy and Lévy-type processes. *J. Theoret. Probab.*, 25(1):144–170, 2012.

[44] T. Kulczycki, M. Ryznar. Gradient estimates of harmonic functions and transition densities for Lévy processes. *Trans. Amer. Math. Soc.*, 368(1): 281–318, 2016.

[45] T. Kulczycki, B. Siudeja. Intrinsic ultracontractivity of Feynman-Kac semigroup for relativistic stable processes. Trans. Amer. Math. Soc., 358: 5025–5057, 2006.

[46] M. Kwaśnicki, J. Małecki, M. Ryznar. Suprema of Lévy processes. *Ann. Probab.*, 41 (3B):2047–2065, 2013.

[47] E. H. Lieb, R. Seiringer. *The stability of matter in quantum mechanics.* Cambridge University Press, Cambridge, 2010.

[48] A. Mimica. Heat kernel estimates for subordinate Brownian motions. *Proc. Lond. Math. Soc.*, 113(5):627–648, 2016.

[49] S. A. Molčanov. Diffusion processes, and Riemannian geometry. *Uspehi Mat. Nauk*, 30(1(181)):3–59, 1975.

[50] J. R. Norris. Heat kernel asymptotics and the distance function in Lipschitz Riemannian manifolds. *Acta Math.*, 179(1):79–103, 1997.

[51] G. Pólya. On the zeros of an integral function represented by Fourier's integral. *Messenger of Math.*, 52:185–188, 1923.

[52] W. E. Pruitt. The growth of random walks and Lévy processes. *Ann. Probab.*, 9(6):948–956, 1981.

[53] K.-i. Sato. *Lévy processes and infinitely divisible distributions*, volume 68 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 1999.

[54] R. L. Schilling, R. Song, Z. Vondraček. *Bernstein functions. Theory and applications.*, volume 37 of *de Gruyter Studies in Mathematics.* Walter de Gruyter & Co., Berlin, second edition, 2012.

[55] M. L. Silverstein. Classification of coharmonic and coinvariant functions for a Lévy process. *Ann. Probab.*, 8(3):539–575, 1980.

[56] B. Siudeja. Symmetric stable processes on unbounded domains. *Potential Anal.*, 25(4):371–386, 2006.

[57] R. Song, Z. Vondraček. Harnack inequalities for some classes of Markov processes. *Math. Z.* 246:177–202, 2004.

[58] P. Sztonyk. Approximation of stable-dominated semigroups. *Potential Anal.*, 33(3):211–226, 2010.

[59] P. Sztonyk. Transition density estimates for jump Lévy processes. *Stochastic Process. Appl.*, 121(6):1245–1265, 2011.

[60] S. R. S. Varadhan. On the behavior of the fundamental solution of the heat equation with variable coefficients. *Comm. Pure Appl. Math.*, 20:431–455, 1967.

[61] N. Th. Varopoulos. Gaussian estimates in Lipschitz domains. *Canad. J. Math.*, 55(2):401–431, 2003.

[62] T. Watanabe. The Isoperimetric Inequality for Isotropic Unimodal Lévy Processes. *Probab. Theory Relat. Fields* 63: 487–499, 1983.

[63] Q. S. Zhang. The boundary behavior of heat kernels of Dirichlet Laplacians. *J. Differential Equations*, 182(2):416–430, 2002.