Summary of scientific achievements

1. **Name:** Łukasz Płociniczak.
2. **Scientific degrees:**
   - Ph. D. in Mathematics, Wroclaw University of Science and Technology 2013 (April), *Summa Cum Laude*. Thesis topic: *Mathematical Analysis of a New Corneal Topography Model*. Supervisor: prof. dr hab. Wojciech Okrasiński
   - M. Sc. in Mathematics, speciality: pure mathematics, Wroclaw University of Science and Technology 2011 (July), *Summa Cum Laude*. Thesis topic: *Mathematical Models of the Corneal Topography*. Supervisor: prof. dr hab. Wojciech Okrasiński
3. **Previous employment.**
   - 2015–: Wroclaw University of Science and Technology, Faculty of Pure and Applied Mathematics, *Assistant Professor*[1]
   - 2014–2015: Wroclaw University of Science and Technology, FacultyFundamental Problems in Technology, *Assistant Professor*
   - 2013–2014: Wroclaw University of Science and Technology, Faculty of Fundamental Problems in Technology, *Assistant*
4. **An indication of the scientific achievement**

   (a) Title:

   On a nonlinear subdiffusion equation: existence, uniqueness, inverse problems and numerical methods

   (b) The list of papers constituting scientific achievement.

[H1] Ł. Płociniczak, H. Okrasińska, *Approximate self-similar solutions to a nonlinear diffusion equation with time-fractional derivative*, Physica D 261 (2013), 85–91.

[H2] Ł. Płociniczak, *Analytical studies of a time-fractional porous medium equation. Derivation, approximation and application*s, Communications in Nonlinear Science and Numerical Simulation 24 (1–3) (2015), 169–183.

[H3] Ł. Płociniczak, *Approximation of the Erdélyi-Kober fractional operator with application to the time-fractional porous medium equation*, SIAM Journal of Applied Mathematics 74(4) (2014), 1219–1237.

[H4] Ł. Płociniczak, *Diffusivity identification in a nonlinear time-fractional diffusion equation*, Fractional Calculus and Applied Analysis 19(4) (2016), pp. 843-866.

[H5] Ł. Płociniczak, Sz. Sobieszek, *Numerical schemes for integro-differential equations with Erdélyi-Kober fractional operator*, Numerical Algorithms 76(1) (2016), pp. 125-150.

[H6] Ł. Płociniczak, M. Świtała, *Existence and uniqueness results for a time-fractional nonlinear diffusion equation*, Journal of Mathematical Analysis and Applications 462(2) (2018), pp. 1425-1434.

[1] The change in employment has been a result of a transformation of the Institute of Mathematics and Computer Science on the Faculty of Fundamental Problems in Technology into the Faculty of Pure and Applied Mathematics.

(c) A discussion of the aforementioned papers and obtained results

The scientific achievement constitutes of 6 research papers concerning differential equations with derivative of an arbitrary real order. From the historical reasons this operator is now called *fractional derivative* and the mathematical field investigating its properties - *fractional calculus*. Thanks to many features associated with the memory and nonlocality, fractional derivatives find successively broader field of applications in many other sciences such as physics, chemistry and biology.

The main aim of the aforementioned papers was the analysis of a nonlinear anomalous diffusion equation where the fractional derivative is taken as the time-evolution operator (in our case we are concerned with the subdiffusion). Because the fractional derivative is a nonlocal operator, it is very rare to find an exact solution to a given fractional differential equation. Mostly, this is possible only in the simplest linear cases, where often the solutions are given in terms of the special functions. Because fractional differential equations are very widely used in applications, even knowing approximate forms of their solutions is valuable. This is one of the reasons that were the motivation for obtaining the results given in the achievement. Apart from approximate solutions, we are concerned in proving existence, uniqueness and solving several inverse problems associated with an identification of the diffusivity. We also state some results about the numerical methods for discretization of the Erdélyi-Kober operator and its use in integro-differential equations.

In the first part of this summary we present some prelimiaries in fractional calculus. The purpose of this exposition is to lay mathematical ground for the results to follow and to gather classical theorems and definitions needed in the subsequent sections. Moreover, we state the relationship between our results and those present in the literature. In the next part we give a discussion of the papers included in the scientific achievement. The last section consists of a short presentation of works not included in the achievement.

Contents

4.1 Fractional operators

In this section we will presents some preliminary information concerning fractional calculus - the field of mathematics that investigates generalizations of differentiation and integration.

4.1.1 Notation, definition and basic properties

We start by introducing some of the basic definitions, notations and theorems used in fractional calculus. They will serve us in the following sections as a background for our results. A very thorough review of the theory and application of the fractional derivatives and integrals can be found in the monographs [96, 91, 35]. An interesting historical introduction is given in [81].

Definition 1 *Let $a \in \mathbb{R}$ and $\alpha > 0$. The* ***Riemann-Liouville fractional integral*** *of order α of a locally integrable function f is defined by*

$$I_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad x \geq a. \tag{4.1.1}$$

When $a = 0$ we write $I_0^\alpha \equiv I^\alpha$. Moreover, for $\alpha = 0$ we put $I_a^0 = Id$.

Immediately, we can make a few simple remarks. First, the above operator is well-defined and maps the space of locally integrable functions to itself (it follows from Fubini's Theorem). An interesting paper concerning defining fractional integral on several function spaces is [76]. Second, notice that if $\alpha = n \in \mathbb{N}$, then

$$I_a^n f(x) := \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} f(t) dt = \int_a^x dt_1 \int_a^{t_1} dt_2 ... \int_a^{t_{n-1}} f(t_n) dt_n, \tag{4.1.2}$$

that is, the Riemann-Liouville integral reduces to the well-known formula for the n-th antiderivative. The above definition can also be generalized to take complex orders such that Re $\alpha > 0$.

Now, we are in position to define the Riemann-Liouville fractional derivative.

Definition 2 *Let $a \in \mathbb{R}$ and $\alpha \geq 0$. The* ***Riemann-Liouville fractional derivative*** *of order $\alpha > 0$ of a locally integrable function f is defined by the formula*

$$D_a^\alpha f(x) := \frac{d^n}{dx^n} I_a^{n-\alpha} f(x), \quad x \geq a, \tag{4.1.3}$$

for $n = [\alpha] + 1$. When $a = 0$ we write $D^\alpha \equiv D_0^\alpha$.

We can see that if $\alpha = n \in \mathbb{N}$ then $D_a^\alpha = \frac{d^n}{dx^n} I_a^0 = \frac{d^n}{dx^n}$, that is the Riemann-Liouville fractional derivative is indeed the generalization of the classical derivative. Moreover, for $\alpha = 0$ we have $D_a^0 = \frac{d^1}{dx^1} I^1 = \mathrm{Id}$.

It is important to notice that (4.1.3) is a nonlocal operator that is to say, in order to find its value for a given function at a specific point x it is needed to know the values of that function on the whole interval $t \in (a, x)$. This property is in a strong contrast with a local character of the usual derivative, and is a foundation of the great success of fractional calculus in modelling many phenomena possessing a long memory.

Its can be shown in a simple way that the formulas for fractional derivative and integral of a power function have a very natural form. Notice that the fractional derivative of a constant does not vanish.

Proposition 1 *Let $a \in \mathbb{R}$, $\alpha \geq 0$ and $\beta > -1$. For $f(x) = (x-a)^{\beta}$ we have*

$$I_a^{\alpha} f(x) = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta+\alpha)} (x-a)^{\beta+\alpha}, \quad \alpha > 0, \tag{4.1.4}$$

and

$$D_a^{\alpha} f(x) = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)} (x-a)^{\beta-\alpha}, \quad \alpha \geq 0. \tag{4.1.5}$$

In particular, if $\beta = 0$ then

$$(D_a^{\alpha} 1)(x) = \frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)}. \tag{4.1.6}$$

In the further part of this summary we describe some results concerning differential equations with fractional derivative. Below we define all the necessary objects. In order not to be carried away from the main topic of the dissertation, we will discuss only the simplest instances. Some of the many generalizations will be introduced further in the text.

Definition 3 *Let $\alpha \in (0, \infty) \setminus \mathbb{N}$ and $n := [\alpha] + 1$. The **Cauchy problem for a equation with Riemann-Liouville derivative** is the following equation*

$$D_a^{\alpha} y = f(x, y(x)), \tag{4.1.7}$$

along with the initial conditions

$$D_a^{\alpha-k} y(a) = y_k, \quad k = 1 ... n. \tag{4.1.8}$$

If $y = y(x)$ is a continuous function satisfying (4.1.7)-(4.1.8) we say that it is a solution.

Notice that the initial conditions in the above definition are not of a standard type, i.e. we do not fix the values of the function or its integer-order derivatives. The existence and uniqueness of the solution to fractional equation is usually shown by transforming the investigated problem into an equivalent Volterra integral equation. Then, the result follows from the classical theory. Without pursuing the case thoroughly, we can only state that the sufficient condition is, similarly as for the classical case, that f is y-Lipschitz (a very detailed exposition is given in [96]).

There are several methods of finding explicit solutions of the fractional equations. The most common are based on reduction to the integral equation or using the Laplace transform (for linear problems). Obviously, this task can only be conducted successfully only for f in (4.1.7) which is of a simple form. Even for linear equations, the solution is often expressed in terms of special functions.

The form of initial conditions in (4.1.8) makes their physical (or geometrical) interpretation very difficult. Although the fractional dynamics finds many successful applications in the real world, there still is a problem of imposing a set of adequate initial conditions. One resolution of this difficulty has been proposed by Caputo [16], who modified the definition of the fractional derivative. It differs from the Riemann-Liouville's version (4.1.3) by the interchanged order of differentiation and fractional integration. In particular, the Caputo derivative of a constant is zero.

The fractional calculus investigates also a number of different operators which can be though as generalizations of the differentiation or integration. One of the most important are: Riesz derivative [96, 24] and associated with it fractional Laplacian [15, 104], Hadamard's operator [48], Weyl's [80] and Grünwald-Letnikov's derivative [91]. We will not present the definitions here because there are not relevant for our further exposition. However, we have to mention another operator which will be the central object for us in [H1-H6].

Definition 4 *Let* $a, b, c > 0$ *and* f *be locally integrable. The* ***Erdélyi-Kober operator*** *is defined by*

$$I_c^{a,b} f(x) = \frac{1}{\Gamma(b)} \int_0^1 (1-s)^{b-1} s^a f(x s^{\frac{1}{c}}) ds. \tag{4.1.9}$$

In the case when f *is compactly supported we allow that* $a, c \in \mathbb{R}$.

The Erdélyi-Kober (E-K) operator appeared in the literature in the first half of the twentieth century in the complex analysis [27, 53]. It has a number of interesting properties [100, 90] while its applications in fractional calculus are described in the monograph [51]. Moreover, many explicit solutions of the integral equations with E-K operator can be found in [97, 52] and the references therein.

4.1.2 Applications of the fractional calculus

During the last half of the century the fractional calculus has been broadening its area of applications. Some causes of this phenomenon are: development of new technologies used to investigate the dynamical behaviour of various particles, construction of new materials and inquires in complex systems. Below, we will review some important and modern applications of fractional calculus. Additional information can be found for example in [95, 42, 71].

One of the first applied fields, where fractional calculus was very successful is the viscoelasticity. In the simplest view, we can think about fractional derivative to impose a constitutive law lying somewhere between the Hooke's Law for elastic materials and Newton's Law for viscous forces. This and other approaches have been undertaken by many authors [5, 54] (some interesting experimental results are given in [78]). A more modern and mathematical exposition of the fractional viscoelasticity theory is given in [72].

Another important application of the fractional calculus is the anomalous diffusion. To be very brief, anomalous diffusion is a diffusive phenomenon which undergoes a slower (subdiffusion) or faster (superdiffusion) dynamics than the classical diffusion (a precise definition will be given below). Very often this anomalous behaviour is visualized as a random walk of a particle which trajectory is based on a stochastic process constructed on the basis of some heavy-tailed distribution. A very thorough and readable review of this construction is given in [79, 101], where Continuous Time Random Walk is used as the governing process. There is a constant inflow of new investigations that discuss new places where anomalous diffusion is present. Superdiffusion has been observed for example in: Richardson turbulent diffusion in the atmosphere [93, 8], in some rotating flows [106], in polymers [87], quantum optics [98], physics of plasma [21] and in the movement of amoeba and bacteria [56, 83]. Subdiffusion, on the other hand, takes place in: transport on fractals [10, 41], some polymers [28, 49], convection in a fluid with a large Péclet number [109, 107], charge transport in amorphous semiconductors [99, 38] or infiltration in porous media [25, 92, 103, 55, 4]. The latter field has been the main inspiration and motivation for this dissertation.

4.2 A detailed summary of the results

4.2.1 The aim of the dissertation

The main purpose of this dissertation is a summary of different results concerning the nonlinear subdiffusion equation, where diffusivity is of a power type. These rigorous results are motivated mainly by applications in hydrology and material science. They concern some more practical aspects such as estimates, numerical methods and inverse problems as well as more basic questions like existence, uniqueness and properties of nonlinear operators. The bulk of discussed results constitutes a theory for time-fractional generalization of the porous medium equation, which in the classical case has been investigated for a few decades. On the other hand, the anomalous diffusion has been a object of a very vigorous investigations almost only in the linear case. Our works are one of the first ones entering the regime of nonlinearity.

The main results of this dissertation can be summarized as follows.

- Investigation of the properties of Erdély-Kober operator defined by (4.1.9).
- A proof of existence and uniqueness of solution to the nonlinear subdiffusion equation.
- Finding several approximations and bounds on the exact solution.
- Regularization of the inverse problem concerning identification of the diffusivity in the investigated equation.
- Introducing a numerical method for solving integro-differential equations with Erdélyi-Kober operator.
- A construction of the discretizatyion of E-K operator with exact leading-order asymptotics of the error term.

All of the above results are original (apart, maybe, from some simple properties of E-K operator) and have not been investigated in the literature before. The scientific achievement stated in this summary expands the theory of the nonlinear subdiffusion equation and given many rigorous results motivated by their use in the material science.

The author's interest in the considered topic has a two-fold nature. First, as has been mentioned before, the main equation constantly finds its broad applications in hydrology and material science. Second, during the course of investigations it appeared that the literature does not contain many of the anticipated results. This cased a need to undertake the work in that field. As it turned out, even the first published results focused the attention of many researchers from all over the world. In particular, we give some bibliometric statistics (without self-citations; source WoS)

[H1] (2013) 3 citations in: Journal of Mathematical Physics, Journal of Inverse and Ill-posed Problems, International Journal of Numerical Analysis and Modelling.

[H2] (2015) 10 citations in: Communication in Nonlinear Science and Numerical Simulation, Boundary Value Problems, Applied Mathematics and Computation, Dynamic Systems and Applications, International Journal of Heat and Mass Transfer, Materials and Structures.

[H3] (2014) 7 citations in: SIAM Journal on Applied Mathematics, Boundary Value Problems, Dynamic Systems and Applications, Journal of Inverse and Ill-posed Problems.

The remaining papers have not been cited yet because of their relatively recent publication date.

4.2.2 Model formulation and the self-similar for of the equation (papers [H1-H3])

In all papers constituting the achievement [H1-H6] we present mathematical results concerning the time-fractional subdiffusion equation

$$\partial_t^\alpha u = \left(D(u)u_x\right)_x, \quad 0 < \alpha < 1, \tag{4.2.1}$$

where the derivative is of the Riemann-Liouville type (4.1.3). The physical derivation of this formula and its meaning in the modelling of particular porous media has been given in [H2]. We have to remark that when devising the subdiffusion equation we arrive at the Caputo derivative rather that the R-L version. However, when we impose the vanishing initial condition, what is consistent with the experiment, we can use Theorem **??** and conclude that both versions of the fractional derivative are equivalent. This assumption does not narrow the generality of our approach.

The case when $\alpha \to 1^-$ will be called **classical** because then the equation reduces to the well-known porous medium equation. The diffusivity D can be of any form but, in the major part of the dissertation, we will take $D(u) \propto u^m$ for $m \geq 1$. If $D(u) = \text{const.}$ we will be talking about the **linear** case.

The (sub)diffusion anomalous diffusion equation has been present in the literature for several years especially in the linear setting [73, 34, 62]. In that case it is possible to find the exact form of the solution in terms of the Wright special functions [75]. Some very interesting and general results concerning the fundamental solution of the space- and time-fractional diffusion equation has been given in [74]. Moreover, the existence and uniqueness of this Cauchy problem has been proved in [64, 58]. The fractional maximum principle has been investigated in [67, 66, 45], and it has been used in [68] for an existence and uniqueness proof.

The main motivation for undertaking our investigations was the infiltration in some porous media which showed an anomalous behaviour [25, 4, 55]. The experiment which we are trying to model is designed as to measure the dynamics of water inhibition in an essentially one-dimensional sample of material [4, 25] (say, 12mm $\times$ 12mm $\times$ 120mm). The influence of the gravity is reduced by a horizontal mount of the brick. One boundary of the sample is then put under influence of the water and further, the whole sample is being observed by a nuclear magnetic resonance (NMR). Next, all the data points are being rescaled according to the self-similar transformation $xt^{-\frac{1}{2}}$. In the classical case all measurements should collapse onto one curve [25], but recently some new materials showed that this not necessarily occur. This motivated many reserachers to seek for a different mathematical models for observed phenomena. Proposed equations were mostly solved numerically without any substantial mathematical analysis. The purpose of this dissertation is to provide a rigorous investigation of the one of the most important models of aforementioned anomalous phenomena.

In this and two subsequence sections we will take

$$D(u) = u^m, \quad m \geq 1. \tag{4.2.2}$$

This form of the diffusivity is one of the most common in the literature [9, 11]. In the works from our achievement [H1-H3,H6] we also investigated the diffusion equation along with the above formula. Moreover, from the very beginning we use the nondimensional variables which could be introduced by a proper scaling (see [H2]). Whence, our main equation takes the form

$$\partial_t^\alpha u = \left(u^m u_x\right)_x, \quad x > 0, \quad t > 0, \quad 0 < \alpha < 1. \tag{4.2.3}$$

It is a nonlocal in time nonlinear diffusion equation considered on the half-line. As for the boundary conditions we impose two sets of them

$$u(0,t) = 1, \quad u(\infty,t) = 0, \quad t > 0, \tag{4.2.4}$$

or

$$-u^m(0,t)u_x(0,t)=1, \quad u(\infty,t)=0, \quad t>0. \tag{4.2.5}$$

The first one describes a typical experimental set-up given above in which the water can infiltrate through one of the interfaces. The condition (4.2.5) imposes a constant influx of water into the medium. In the applications one is interested mostly in finding the rate of infiltration. Because the first boundary condition (4.2.4) has a great significance in the application, most of our results concern precisely this case.

Let us make the final initial assumption of the model. In all of the given works we have been investigating the various properties of *compactly supported* solutions of (4.2.3). The motivation behind this assumption is twofold. First, the existence of the compact support is directly related with the finite speed of water front propagation. This speed is an important factor which can be easily measured in the experiment. On the other hand, the research done on the classical porous medium equation clearly shows that for a particular class of the diffusivities, the solution of the corresponding evolution equation is always compactly supported (see [3, 20]). Therefore, in our work we assume that for any time $t>0$ there exists a point $x^*(t)>0$, such that

$$u(x,t)=0 \quad \text{for} \quad x\geq x^*(t). \tag{4.2.6}$$

We are interested in the self-similar solutions of (4.2.3) hence, we can make the transformation

$$u(x,t)=t^a U(\eta), \quad \eta=\frac{x}{t^b}. \tag{4.2.7}$$

where a and b are to be found. The fractional derivative transforms as follows

$$\begin{aligned}\partial_t^\alpha u(x,t) &= \frac{\partial}{\partial t}\left(I_t^{1-\alpha}\left(t^a U(xt^{-b})\right)\right) = \frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial t}\int_0^t (t-z)^{-\alpha} z^a U(xz^{-b})dz \\ &= \frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial t}\left(t^{a-\alpha+1}\int_0^1 (1-s)^{-\alpha} s^a U(\eta s^{-b})ds\right).\end{aligned} \tag{4.2.8}$$

The right-hand side contains the E-K operator, thus

$$\partial_t^\alpha u(x,t) = \frac{\partial}{\partial t}\left(t^{a-\alpha+1} I_{-\frac{1}{b}}^{a,1-\alpha} U(\eta)\right) = t^{a-\alpha}\left[(a-\alpha+1)-b\eta\frac{d}{d\eta}\right] I_{-\frac{1}{b}}^{a,1-\alpha} U(\eta), \tag{4.2.9}$$

where we have used the chain rule

$$\frac{\partial}{\partial t}=\frac{\partial \eta}{\partial t}\frac{d}{d\eta}=-b\eta t^{-1}\frac{d}{d\eta}. \tag{4.2.10}$$

Moreover, the other part of (4.2.3) transforms into

$$\left(u^m(x,t)u_x(x,t)\right)_x = t^{a(m+1)-2b}\left(U^m(\eta)U'(\eta)\right)', \tag{4.2.11}$$

the prime denoted the derivative with respect to η. Now, comparing (4.2.9) with (4.2.11) we can ascertain that the t variable can be cancelled

$$2b-ma=\alpha. \tag{4.2.12}$$

Because we have two unknown constants a and b, we have to look for another equation which will uniquely determine them. This can be done by using the boundary conditions (the details are given in [H2]). Eventually, we arrive at

$$(U^m U')' = \left[(a-\alpha+1)-b\eta\frac{d}{d\eta}\right] I_{-\frac{1}{b}}^{a,1-\alpha} U, \quad 0<\alpha<1, \quad m\geq 1, \tag{4.2.13}$$

along with a set of transformed boundary conditions

$$U(0) = 1, \quad U(\eta) = 0 \quad \text{for} \quad \eta \geq \eta^* > 0 \quad \text{and} \quad a = 0, \quad b = \frac{\alpha}{2}, \tag{4.2.14}$$

or

$$-U(0)^m U'(0) = 1, \quad U(\eta) = 0 \quad \text{for} \quad \eta \geq \eta^* > 0 \quad \text{and} \quad a = \frac{\alpha}{m+2}, \quad b = \frac{m+1}{m+2}\,\alpha, \tag{4.2.15}$$

for $\eta^* := \sup \operatorname{supp} U$. The number η^* has to be found as a part of the solution. Because the equation (4.2.13) is singular at a point at which $U = 0$ hence, we expect that the solution can loose its regularity near η^*. Therefore we have to interpret it as a weak solution (see [86]).

4.2.3 Existence and uniqueness (paper [H6])

In this section we will discuss the results concerning existence and uniqueness of the solution to (4.2.13) with conditions (4.2.14).

The experimental results clearly show that the linear diffusion is not sufficient to describe moisture dispersion in some materials (e.g. certain bricks or zeolite). According to the author's best knowledge, the majority of published results concerning (4.2.1) is of a physical or experimental, rather than mathematical, nature (see [30, 89, 102, 77, 61]). In the cited works it is common to numerically solve the considered equation and to fit its solution to the set of data. One of the first approaches to the rigorous analysis of our problem has been based on looking for a classes of self-similar solutions using the Lie group methods [29, 23, 69, 70].

All of referenced mathematical papers proof only the existence of some sets of exact solution without taking into account the imposed boundary conditions. Our reasoning leading to the existence and uniqueness proof is based on the classical theory for Volterra equations [85, 3] and according to the author's knowledge is the first such approach to the nonlinear anomalous diffusion equation.

To end this short review of the literature we will mention some fundamental results form the classical theory of porous medium. The existence and uniqueness of its solution has been proved in [3] and [20] (but see other approach in [84, 85]). The topic of regularity has been undertaken in [2, 1]. On the other hand, a very thorough classification of the self-similar solutions has been given in the series of papers [31, 32, 33]. In those works authors also prove necessary and sufficient conditions for the existence of a compact support. In the lecture notes [2] one can find a vast summary of the known results and somewhat modern approach is presented in [105]. The problem considered by us in this dissertation add a nonlocal element to the porous medium equation producing a need for other proof techniques than those used in the classical case.

Before we discuss our main results we need a slightly rescaled definition of the E-K operator

Notation 1 *Let $U = U(\eta)$ be a locally integrable function, which vanishes for $\eta \leq 0$. The operator G_α is defined by the formula*

$$G_\alpha U(\eta) = \frac{1}{\Gamma(1-\alpha)} \int_{(1-\eta)^{\frac{\alpha}{2}}}^{1} (1-s)^{-\alpha} U\left(1 - s^{-\frac{2}{\alpha}}(1-\eta)\right) ds, \quad \eta \in (0,1). \tag{4.2.16}$$

The above operator enjoys some interesting properties which are summarized in the following lemma.

Lemma 1 ([H6], Lemma 1) *Let $0 < \alpha < 1$ and $U(\eta) := \eta^\beta$ for some $\beta > 0$. Define also*

$$A := \left(\frac{\alpha}{2}\right)^{1-\alpha} \frac{\Gamma(1+\beta)}{\Gamma(2-\alpha+\beta)}, \quad B := \frac{1}{\Gamma(2-\alpha)}. \tag{4.2.17}$$

Then we have

(i) (Behaviour at zero)

$$G_\alpha U(\eta) \sim C\eta^{\beta+1-\alpha} \quad gdy \quad \eta \to 0^+, \tag{4.2.18}$$

Morover, the above formula can be differentiated with respect to η.

(ii) (Estimates)

$$A\eta^{\beta+1-\alpha} \le G_\alpha U(\eta) \le B\eta^{\beta+1-\alpha}, \quad 0 \le \eta \le 1. \tag{4.2.19}$$

The proof of the above results is based on the analysis of the following integral

$$\frac{1}{\Gamma(1-\alpha)} \int_{(1-z)^{\frac{\alpha}{2}}}^{1} (1-s)^{-\alpha} \left(1 - s^{-\frac{2}{\alpha}}(1-z)\right)^\beta ds, \tag{4.2.20}$$

for which we are looking for the behaviour near $z \to 0^+$. We can see that for small z the mass of the integral becomes concentrated near $s = 1$. This suggests that we can Taylor expand the integrand near that point. Unfortunately, this does not work since the function becomes nonsmooth there. The resulting integral would then be divergent. This difficulty can be overcame by a transformation

$$s = \left(1 - (1-z)^{\frac{\alpha}{2}}\right) t + (1-z)^{\frac{\alpha}{2}}, \tag{4.2.21}$$

which fixes the integration range and allows us to conduct a successful expansion. The estimates follow from further analysis.

In order to make a fundamental step that brings us closer to the proof of existence and uniqueness it is important that there exists a transformation changing the free-boundary problem into an initial value one.

Theorem 1 ([H6]) *The solution of the* free boundary *problem (4.2.13)-(4.2.14) can be expressed as*

$$U(\eta) = \left(m(\eta^*)^2\right)^{\frac{1}{m}} y(z), \quad z = 1 - \frac{\eta}{\eta^*}, \tag{4.2.22}$$

where $y = y(z)$ is a solution of the initial value *problem*

$$m(y^m y')' = \left[(1-\alpha) + \frac{\alpha}{2}(1-z)\frac{d}{dz}\right] G_\alpha y, \quad 0 < \alpha < 1, \quad m \ge 1, \quad 0 \le z \le 1, \tag{4.2.23}$$

with conditions

$$y(0) = 0, \tag{4.2.24}$$

and

$$\lim_{z\to 0} y(z)^m y'(z) = 0. \tag{4.2.25}$$

Moreover, the support location can be calculated by the formula

$$\eta^* = \frac{1}{\sqrt{m y(1)^m}}. \tag{4.2.26}$$

We can see that one of the difficulties can be bypassed by a sensible change of the variables. Thanks to it, one boundary condition transforms into the initial condition while the other lets us to compute the exact position of the wetting front η^*. The other initial condition (4.2.25) is a result of a integration of (4.2.23) and using a fact that $G_\alpha U$ is a continuous function. A much more stronger results is also true which gives the exact behaviour of the solution to (4.2.23) with $z \to 0^+$.

Theorem 2 ([H6], Theorem 1) *The solution $y = y(z)$ of the equation (4.2.23) with a condition (4.2.24) satisfies*

$$y(z) \sim \left(\frac{\alpha}{2}\right)^{2-\alpha} \frac{\Gamma\left(\frac{2-\alpha}{m}\right)}{\Gamma\left(1-\alpha+\frac{2-\alpha}{m}\right)} \frac{z^{\frac{2-\alpha}{m}}}{(2-\alpha)\left(1+\frac{1}{m}\right)-1} \quad dla \quad z \to 0^+, \quad 0 < \alpha \leq 1. \tag{4.2.27}$$

For the proof it is sufficient to observe that since y is a twice-differentiable function we can write

$$y(z) = Cz^\beta + R(z), \quad \beta \geq 1, \tag{4.2.28}$$

for some differentiable R. After plugging it to the (4.2.23) we can find the exact forms of C and β. A key point is to use Lemma 1 giving the behaviour of G_α acting on a polynomial. We can see that the solution y not only vanishes at $z = 0$ but also has a vanishing derivative.

In terms of the fractional derivative this situation can be described as

$$D_0^{\frac{2-\alpha}{m}} y(0) = \left(\frac{\alpha}{2}\right)^{2-\alpha} \frac{\Gamma\left(\frac{2-\alpha}{m}\right)}{\Gamma\left(1-\alpha+\frac{2-\alpha}{m}\right)} \frac{\Gamma(1+\frac{2-\alpha}{m})}{(2-\alpha)\left(1+\frac{1}{m}\right)-1}. \tag{4.2.29}$$

In order to show the main result, i.e. the existence and uniqueness of the solution to (4.2.23) with (4.2.24) and (4.2.25) we make yet another transformation changing our differential equation into the nonlinear integral equation. Then, the fixed point theorem will grant the assertion.

Proposition 2 ([H6]) *The solution of the following integral equation*

$$y(z) = \left(\frac{m+1}{m}\right)^{\frac{1}{m+1}} \left(\frac{\alpha}{2}\int_0^z (1-t) G_\alpha y(t)\,dt + \left(1-\frac{\alpha}{2}\right)\int_0^z (z-t) G_\alpha y(t)\,dt\right)^{\frac{1}{m+1}}, \tag{4.2.30}$$

is twice-differentiable for $z \in (0,1)$ and satisfies (4.2.23) along with (4.2.24)-(4.2.25).

The expression (4.2.30) is a fixed-point equation for the following operator

$$T(y)(z) := \left(\frac{m+1}{m}\right)^{\frac{1}{m+1}} \left(\int_0^z \left(\frac{\alpha}{2} + \left(1-\frac{\alpha}{2}\right)z - t\right) G_\alpha y(t)\,dt\right)^{\frac{1}{m+1}}. \tag{4.2.31}$$

For the domain of T we take

$$K := \{y \in C[0,1] : y \geq 0\}, \tag{4.2.32}$$

and as for its subset in which the solution will be sought

$$K_0 := \{y \in C[0,1] : y(0) = 0,\ y(z) > 0 \text{ dla } z \in (0,1]\}. \tag{4.2.33}$$

Immediately we can notice that T is monotone

$$y_1 \leq y_2 \quad \Longrightarrow \quad T(y_1) \leq T(y_2), \tag{4.2.34}$$

and homogeneous of degree $\frac{1}{m+1}$

$$T(\lambda y) = \lambda^{\frac{1}{m+1}} T(y), \quad \lambda > 0. \tag{4.2.35}$$

Moreover, the representation (4.2.30) can be used to show that y and y' are positive ([H6], Proposition 1).

We are now in the position to formulate the main result.

Theorem 3 ([H6], Corollary 1) *The integro-differential equation (4.2.13) with (4.2.14) has exactly one solution being a decreasing and positive function.*

Using the Proposition 2 along with some other previous results included in Theorem 1 it is only needed to show that (4.2.30) has exactly one fixed point belonging to the space K_0. The classical approach based on invoking the fixed point theorem does not work since the nonlinearity in our problem is not Lipschitz. Similar problems has been considered by a number of mathematicians who showed a series of sufficient conditions for the solution to exist (zob. [36, 82, 84, 14]). We will use the Bushell's theorem which uses the Hilbert projective metric (see [13, 14]).

Theorem 4 (Bushell, 1976) *Assume that* $T : K \to K$ *is an increasing operator and homogeneous of order* p *for* $0 < p < 1$. *If there exists a function* $\varphi \in K_0$ *such that*

$$\gamma_1 \varphi \leq T(\varphi) \leq \gamma_2 \varphi, \tag{4.2.36}$$

for some constants $\gamma_{1,2}$, *then* T *as a unique fixed point* $y \in K_0$ *and*

$$\gamma_1^{\frac{1}{1-p}} \varphi \leq y \leq \gamma_2^{\frac{1}{1-p}} \varphi. \tag{4.2.37}$$

It is then sufficient to find the sub- and supersolution for the equation $Ty = y$. The class of such φ can be guessed from the Theorem 2 which gives the asymptotic form of the solution.

Lemma 2 ([H6], Lemma 2) *Let* $\varphi(z) := z^{\frac{2-\alpha}{m}}$, *then*

$$\gamma_1 \varphi \leq T(\varphi) \leq \gamma_2 \varphi, \tag{4.2.38}$$

where

$$\gamma_1 = \begin{cases} \left(\left(\frac{\alpha}{2}\right)^{1-\alpha} \frac{\Gamma\left(\frac{2-\alpha}{m}\right)}{\Gamma\left(2-\alpha+\frac{2-\alpha}{m}\right)} \frac{1}{2-\alpha+m(3-\alpha)} \right)^{\frac{1}{m+1}}, & 0 < \alpha \leq 1 - \frac{1}{m+1}; \\ \left(\left(\frac{\alpha}{2}\right)^{2-\alpha} \frac{\Gamma\left(1+\frac{2-\alpha}{m}\right)}{\Gamma\left(2-\alpha+\frac{2-\alpha}{m}\right)} \frac{1}{2-\alpha} \right)^{\frac{1}{m+1}}, & 1 - \frac{1}{m+1} < \alpha \leq 1, \end{cases} \tag{4.2.39}$$

$$\gamma_2 = \Gamma(3-\alpha)^{-\frac{1}{m+1}}.$$

The proof is technical and uses several estimates that have been obtained previously in Lemma 1. It is important to note that the solution of (4.2.23) can be bounded by a similar power-type functions.

The function φ devised in Lemma 2 can now be used in Bushell's theorem in orger to guarantee the existence of a unique fixed point for (4.2.30). This concludes the sketch of the proof.

Finally, we can state a short but useful corollary about the estimates.

Corrolary 1 *Let* $U = U(\eta)$ *be the unique solution of (4.2.13) with (4.2.14). For* $0 \leq \eta \leq \eta^*$ *we have*

$$\frac{1}{\sqrt{m\gamma_2^{m+1}}} \leq \eta^* \leq \frac{1}{\sqrt{m\gamma_1^{m+1}}}, \tag{4.2.40}$$

and

$$\left(\frac{\gamma_2}{\gamma_1}\right)^{\frac{m+1}{m}} \left(1 - \sqrt{m\gamma_1^{m+1}}\eta\right)^{\frac{2-\alpha}{m}} \leq U(\eta) \leq \left(\frac{\gamma_1}{\gamma_2}\right)^{\frac{m+1}{m}} \left(1 - \sqrt{m\gamma_2^{m+1}}\eta\right)^{\frac{2-\alpha}{m}}, \tag{4.2.41}$$

where $\gamma_{1,2}$ *are defined in (4.2.39).*

4.2.4 Approximate solutions (papers [H1-H3])

We know that there is a unique compactly supported solution of our problem. In this section we will discuss some more applied aspects of the developed theory. Because the exact solution of (4.2.13) cannot be obtained in a closed, analytical form, we will devise a systematic method of finding approximate solutions.

As a side product of existence proof we have obtained the estimates (4.2.41). We can immediately notice that these cannot be very accurate since they do not satisfy the boundary condition at $\eta = 0$. In what follows we will devise an asymptotic expansion as $\alpha \to 1^-$ of the E-K operator and use it to obtain several very accurate approximations.

Main results are proved in [H2], however their earlier versions have been published in [H1,H3]. Let us consider the equation (4.2.13) but this time along with conditions (4.2.14) or (4.2.15). The main idea is to derive a series representation of the nonlocal E-K operator.

Theorem 5 ([H3], Theorem 1) *Let U be analytic and $a > -1$, $b > 0$, $c > 0$. For the Erdélyi-Kober operator (4.1.9) we have*

$$I_c^{a,b}U(\eta) = \sum_{k=0}^{\infty} \lambda_k U^{(k)}(\eta)\frac{\eta^k}{k!}, \tag{4.2.42}$$

where

$$\lambda_k = \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} \frac{\Gamma\left(a + \frac{j}{c} + 1\right)}{\Gamma\left(a + b + \frac{j}{c} + 1\right)}. \tag{4.2.43}$$

Moreover, for $k \to \infty$ we have

$$\lambda_k \sim (-1)^k \frac{\Gamma\left(c(a+1)\right)}{\Gamma(b)} \frac{c}{k^{c(a+1)}}. \tag{4.2.44}$$

The main tool in showing the above theorem is the Faá di Bruno formula for a derivative of a composite function

$$\frac{d^n}{ds^n}U(g(s)) = \sum_{k=0}^{n} B_{n,k}\left(g'(s), g''(s), \ldots, g^{(n-k+1)}(s)\right) U^{(k)}(g(s)), \tag{4.2.45}$$

where $B_{n,k}$ are Bell's polynomials. Take $g(s) = \eta s^{\frac{1}{c}}$ and expand $U(\eta s^{\frac{1}{c}})$ into the Taylor series. Using the properties of Bell's polynomials (see [H3], eq. (2.23) and [110]) we obtain

$$U(\eta s^{\frac{1}{c}}) = \sum_{k=0}^{\infty} \left(s^{\frac{1}{c}} - 1\right)^k U^{(k)}(\eta) \frac{\eta^k}{k!}, \tag{4.2.46}$$

what after plugging into the definition of the E-K operator (4.1.9) and integrating gives us (4.2.42). The asymptotic form of λ_k can be inferred from the integral form

$$\lambda_k = (-1)^k \frac{c}{\Gamma(b)} \int_0^1 \left(1 - (1-s)^c\right)^{b-1} (1-s)^{c(a+1)-1} s^k ds, \tag{4.2.47}$$

where $\left(1 - (1-s)^c\right)^{b-1}$ can be expanded into more simple function with known asymptotic behaviours.

Knowing the exact form of the coefficients λ_k we can show a stronger result that Proposition 6. The following result can be showed by some sensible transformations and using Lebesgue's theorem.

	A	B
BCs (4.2.14)	$\frac{1}{\Gamma(1-\alpha)}$,	$\frac{\alpha}{2\Gamma(2-\alpha)}$;
BCs (4.2.15)	$\frac{\Gamma\left(1+\frac{\alpha}{2+m}\right)}{\Gamma\left(1-\alpha+\frac{\alpha}{2+m}\right)}$,	$\alpha\,\frac{m+1}{m+2}\frac{\Gamma\left(1+\frac{\alpha}{2+m}\right)}{\Gamma\left(2-\alpha+\frac{\alpha}{2+m}\right)}$;

Table 1: Coefficients for (4.2.53).

Proposition 3 ([H2], Proposition 1) *We have*

$$\lambda_k \xrightarrow{b\to 0^+} \begin{cases} 1, & k=0; \\ 0, & k>0, \end{cases} \quad \textit{for} \quad a>-1,\ c>0. \tag{4.2.48}$$

Notice also the very restrictive assumption of analyticity stated in Theorem 5. It is needed in order to show the full expansion but in applications, especially in porous media, we have to deal with much less regular functions. For function which are not analytic we have the following result.

Proposition 4 ([H2], Proposition 2) *Let U be a $C^N(0,\infty)$ function with uniformly bounded derivatives. We then have*

$$\left| I_c^{a,b}U(\eta) - \sum_{k=0}^{N-1} \lambda_k U^{(k)}(\eta)\frac{\eta^k}{k!}\right| \le C\frac{|\eta|^N}{N^{c(a+1)}N!}, \tag{4.2.49}$$

where λ_k is defined in (4.2.43) and C depends on U, a, b and c.

In order to find the asymptotic behaviour of the approximation error it is usefil to use (4.2.44) for λ_j. Because for $N\to\infty$ the remainder of the series

$$I_\delta^{\beta,\gamma}U(\eta) - \sum_{k=0}^{N-1} \lambda_k U^{(k)}(\eta)\frac{\eta^k}{k!} = \lambda_N U^{(N)}(\eta)\frac{\eta^N}{N!} + \cdots, \tag{4.2.50}$$

is dominated by its first term

$$I_c^{a,b}U(\eta) - \sum_{k=0}^{N-1} \lambda_k U^{(k)}(\eta)\frac{\eta^k}{k!} \sim (-1)^N c\frac{\Gamma\left(c(a+1)\right)}{\Gamma(b)}\frac{U^{(N)}(\eta)}{N^{c(a+1)}}\frac{\eta^N}{N!} \quad \text{gdy} \quad N\to\infty. \tag{4.2.51}$$

In [H2-H3] we give some concrete examples of the above results.

The main idea for approximation of the solution to the main equation is to replace the operator $I_{-\frac{2}{\alpha}}^{0,1-\alpha}$ with a few terms of its series representation. Taking too many terms is not efficient since it requires more regularity that supposedly our function could have. Let us then use the simplest approximation

$$I_{-\frac{2}{\alpha}}^{0,1-\alpha}U(\eta) \approx \lambda_0 U(\eta). \tag{4.2.52}$$

It can be shown that this approximation is equivalent to the application of Laplace's method for finding the asymptotic behaviour of the integral for $\alpha\to 1^-$. The nonlocal equation (4.2.13) can thus be approximated by

$$(U^m U')' \approx AU - B\eta U', \tag{4.2.53}$$

where the constants A and B depend on the boundary condition Tab. 1.

The ordinary equation (4.2.53) with imposed boundary conditions still cannot be exactly solved in an analytic form (with the exception of the linear case considered in [H3]). Moreover, it is still a free-boundary problem. In order to transform it into the initial value problem we will use the substitution similar to the one used before (4.2.22). In this case it is [50]

$$U(\eta) = \left(m(\eta^*)^2 y(z)\right)^{\frac{1}{m}}, \quad z = 1 - \frac{\eta}{\eta^*}. \tag{4.2.54}$$

The difference between (4.2.22) is the m-th root of y. This makes the resulting equation somewhat simpler

$$\frac{1}{m} y'^2 + y y'' = A y + \frac{B}{m}(1-z)\, y', \quad y(0) = 0, \quad y'(0) = B, \tag{4.2.55}$$

where the second initial condition has been determined from the structure of the equation. The solution of the above can be obtained as the Taylor series.

Theorem 6 ([H2,H3]) *Let $y = y(z)$ be the solution of (4.2.55). Then*

$$y(z) = \sum_{k=1}^{\infty} a_k z^k, \tag{4.2.56}$$

where the coefficients can be calculated via the recurrence relations

$$\begin{gathered} a_0 = 0, \quad a_1 = B, \quad a_{n+1} = \frac{1}{Bn(n+1)} \left[\left(A - \frac{Bn}{m} - \frac{2n}{m} a_2 \right) a_n \right. \\ \left. - \sum_{k=2}^{n} \frac{1}{m}(k+1)(n-k+1) a_{k+1} a_{n-k+1} + (n-k+1)(n-k+2) a_k a_{n-k+2} \right], \quad n \geq 1. \end{gathered} \tag{4.2.57}$$

Moreover, the solution of (4.2.53) can be expressed as

$$\begin{aligned} U(\eta) &= \left(m(\eta^*)^2 \sum_{k=0}^{\infty} a_k \left(1 - \frac{\eta}{\eta^*}\right)^k \right)^{\frac{1}{m}} = \left(m(\eta^*)^2 \left(1 - \frac{\eta}{\eta^*}\right) \sum_{k=0}^{\infty} a_{k+1} \sum_{j=0}^{k} \binom{k}{j} (-1)^j \left(\frac{\eta}{\eta^*}\right)^j \right)^{\frac{1}{m}} \\ &= \left(m(\eta^*)^2 \left(1 - \frac{\eta}{\eta^*}\right) \sum_{j=0}^{\infty} \left(\sum_{k=0}^{j} \binom{k}{j} a_{k+1} \right) \left(-\frac{\eta}{\eta^*}\right)^j \right)^{\frac{1}{m}}, \end{aligned} \tag{4.2.58}$$

where the front position is given by

$$\eta^* = \frac{1}{\sqrt{m y(1)}}, \tag{4.2.59}$$

in the case of (4.2.14) or

$$\eta^* = \left(m^{\frac{1}{m}} y^{\frac{1}{m}}(1) y'(1) \right)^{-\frac{m}{m+2}}, \tag{4.2.60}$$

for (4.2.15).

For instance, we can quickly give some initial coefficients

$$a_2 = \frac{Am - B}{2(1+m)}, \quad a_3 = \frac{(A+B)m(B-Am)}{6B(1+m)^2(1+2m)}, \quad a_4 = \frac{(A+B)m(B-Am)(B(3+m) - Am(5+3m))}{24B^2(1+m)^3(1+m(5+6m))}. \tag{4.2.61}$$

Taking only the first terms in the expansion we get

$$
\begin{aligned}
U_1(\eta) &= \left(m(\eta_1^*)^2 a_1 \left(1-\frac{\eta}{\eta_1^*}\right)\right)^{\frac{1}{m}}, \quad U_2(\eta) = \left(m(\eta_2^*)^2\left(1-\frac{\eta}{\eta_2^*}\right)\left(a_1+a_2-a_2\frac{\eta}{\eta_2^*}\right)\right)^{\frac{1}{m}}, \\
U_3(\eta) &= \left(m(\eta_3^*)^2\left(1-\frac{\eta}{\eta_3^*}\right)\left(a_1+a_2+a_3-(a_2+2a_3)\frac{\eta}{\eta_3^*}+a_3\left(\frac{\eta}{\eta_3^*}\right)^2\right)\right)^{\frac{1}{m}},
\end{aligned}
\tag{4.2.62}
$$

Moreover, we can calculate the approximation to the cumulative moisture intake

$$
I(t) := \int_0^\infty u(x,t)dx = \int_0^\infty t^a U\left(xt^{-b}\right)dx = t^{a+b}\int_0^{\eta^*} U(\eta)d\eta. \tag{4.2.63}
$$

The approximations are

$$
I_1(t) = t^{a+b}\frac{m\eta_1^*}{1+m}\left(m(\eta_1^*)^2 a_1\right)^{\frac{1}{m}}, \ I_2(t) = t^{a+b}\frac{m\eta_2^*}{1+m}\left(m(\eta_2^*)^2 a_1\right)^{\frac{1}{m}} {}_2F_1\left(1+\frac{1}{m}, -\frac{1}{m}; 2+\frac{1}{m}; -\frac{a_2}{a_1}\right). \tag{4.2.64}
$$

Another approach to obtain some approximations to the solution of (4.2.13) is to use the perturbation theory. To this end, we substitute

$$
y(z) = y_0(z) + \frac{1}{m}y_1(z) + \cdots, \tag{4.2.65}
$$

and plug the above into the main equation. Whence

$$
\begin{aligned}
y_0 y_0'' &= A y_0, \quad y_0(0)=0, \quad y_0'(0)=B, \\
y_0'^2 + y_0 y_1'' + y_1 y_0'' &= A y_1 + B(1-z)y_0', \quad y_1(0)=0, \quad y_1'(0)=0.
\end{aligned}
\tag{4.2.66}
$$

What immediately gives us

$$
\begin{aligned}
y_0(z) &= Az^2/2 + Bz, \\
y_1(z) &= -2(A+B)\left[\frac{z^2}{2} - \frac{B}{A}\left(\left(z+\frac{2B}{A}\right)\left(\ln\left(1+\frac{A}{2B}z\right)-1\right)+\frac{2B}{A}\right)\right].
\end{aligned}
\tag{4.2.67}
$$

4.2.5 Inverse problems (paper [H4])

In this section we will present a discussion of the results associated with inverse problems for (4.2.13). Usually, in the experiments it is the moisture distribution that is being measured [25]. In that case we know the function $U = U(\eta)$ which is the solution of (4.2.13). On this basis, experimentalists try to determine the material properties of the sample encoded in the diffusivity $D = D(U)$. This statement of the problem is the so-called *inverse problem*: we know the solution of the differential equation and we have to find of its coefficients. Inverse problems are usually ill-posed in the Hadamard sense [39].

Definition 5 *A mathematical problem (for example differential or algebraic equation) is* ***well-posed*** *if it possesses each of the following properies*

- *it has a solution,*
- *it has a unique solution,*
- *small changes in the initial data yield small changes in the outcome (stability).*

*If the problem is not well-posed its is **ill-posed**.*

The first paper that had undertaken the topic of inverse problems in anomalous diffusion was [17], where the problem of existence and uniqueness of the parameter α was investigated along with the determination of the space- and time dependent diffusivity. A similar problem was considered in [47]. The identification of the source function has been proposed in [111]. Moreover, a relatively recent work is [94], where authors find the initial conditions. There is also an interesting paper [63] where the existence and uniqueness was proved by using the maximum principle. A very broad summary of recent developments in the field of inverse problems was given in [46], in which the main and striking differences between classical and anomalous diffusion were given.

We want to point out that the majority of the previous results concern only the linear case of anomalous diffusion. However, a step in the direction of nonlinear problems has been made in [65], where the nonlinearity was allowed to appear in the source function. A similar problem has been undertaken in [44]. According to author's knowledge, most papers consider diffusivity being a function of the *independent* variables. Our approach discusses the case of the diffusivity related to the *dependent* variables.

Very often, the fundamental difficulty in treating inverse problems is the lack of the stability. Existence and uniqueness can be imposed by considering the least-squares solution on a larger function space [37, 26]. In applications, the lack of stability is a serious drawback of the problem since it can lead to a complete loss of the information about the solution. There are certain techniques, known as regularizations, that allow for a partial recovery of the stability.

Notation 2 *The operator F_α defined on a locally integrable functions is defined via the formula*

$$F_\alpha U(\eta) := I_{-\frac{2}{\alpha}}^{0,1-\alpha} U(\eta) = \frac{1}{\Gamma(1-\alpha)} \int_0^1 (1-s)^{-\alpha} U(\eta s^{-\frac{\alpha}{2}}) ds, \tag{4.2.68}$$

where $I_{-\frac{2}{\alpha}}^{0,1-\alpha}$ is the E-K operator (4.1.9).

Our inverse problem is to find $D(U)$ from the knowledge of U. By a straightforward integration of (4.2.13) we can obtain the formula

$$D(U(\eta)) = \frac{1}{U'(\eta)} \left[D_s U'(0) + \left(1 - \frac{\alpha}{2}\right) \int_0^\eta F_\alpha U(z) dz - \frac{\alpha}{2} \eta F_\alpha U(\eta) \right]. \tag{4.2.69}$$

This grants us the existence and uniqueness (up to a constant) of the solution. We are thus left with a problem of stability (since the differentiation is not a stable operation). Moreover, using the above formula can be very expensive in terms of the computer power. We can solve this issue by the approximation of F_α which follows from Theorem 5. The approximate formula for the diffusivity takes then the form

$$\widetilde{D}(U(\eta)) = \frac{1}{U'(\eta)} \left[D_s U'(0) + \left(\frac{1}{\Gamma(1-\alpha)} + \frac{\alpha}{2\Gamma(2-\alpha)} \right) \int_0^\eta U(z) dz - \frac{\alpha}{2\Gamma(2-\alpha)} \eta U(\eta) \right]. \tag{4.2.70}$$

Evaluating this formula requires much less computing power than the former but, from the construction, is not accurate. This rises a question about how much error do we make when using the above formula instead of the exact one.

Before we proceed to our main results we will show some auxiliary lemmas and propositions concerning the operator F_α.

Proposition 5 ([H4], Proposition 2.1) *Let U be a bounded function defined on $\mathbb{R}$. Then $F_\alpha U$ is a well-defined and bounded function for $\alpha \in (0,1)$. Moreover, we have*

(a) $F_\alpha U(0) = \frac{U(0)}{\Gamma(2-\alpha)}$.

(b) If U is increasing (decreasing), then $F_\alpha U$ is increasing (decreasing).

(c) If $\lim_{\eta\to\infty} U(\eta) = g$, then $\lim_{\eta\to\infty} F_\alpha U(\eta) = \frac{g}{\Gamma(2-\alpha)}$.

The proof that F_α maps bounded function set to itself follows from the definition (4.2.68) since $|F_\alpha U| \leq \frac{1}{\Gamma(2-\alpha)} \|U\|_\infty$. Using this fact and Lebesgue's theorem helps us to show the assertion (c).

As it can be inferred from the circumstances of the F_α appearance, it should somehow approach the identity operator when $\alpha \to 1^-$. This is the object of the next result.

Proposition 6 ([H4], Theorem 1) *Let U be differentiable. Then*

$$\lim_{\alpha\to 1^-} F_\alpha U(\eta) = U(\eta). \tag{4.2.71}$$

The proof of the above is based on a integration by parts in the formula (4.2.68). The assumption of differentiability can be weakened but we will not demonstrate that. In order to show our main results we need a somewhat stronger result on the rate of the aforementioned convergence. As it appears, it depends not only on the function U but also on its derivative.

Lemma 3 ([H4], Lemma 1) *Let $U : [0,\infty) \to \mathbb{R}$ be bounded, decreasing and vanishing at infinity. We then have*

$$|F_\alpha U(\eta) - U(\eta)| \leq \left(\frac{1}{\Gamma(2-\alpha)} - 1\right) U(\eta s_0^{-\alpha/2}) + \Gamma(2-\alpha)^{\frac{1}{1-\alpha}} \left(1 - \frac{1}{2-\alpha}\right) \max_{s\in[s_0,1]} \frac{d}{ds}\left(U\left(\eta s^{-\frac{\alpha}{2}}\right)\right), \tag{4.2.72}$$

where $s_0 := 1 - \Gamma(2-\alpha)^{\frac{1}{1-\alpha}}$.

Once again, the key-point in the proof is a integration by parts in the definition of F_α. Then, a sensible estimation is necessary starting from the obvious fact that

$$U(\eta) = U(\eta) - U(\infty) = \int_0^1 \frac{d}{ds}\left(U\left(\eta s^{-\alpha/2}\right)\right) ds. \tag{4.2.73}$$

We can then write

$$|F_\alpha U(\eta) - U(\eta)| \leq \left|\int_0^1 \left(\frac{(1-s)^{1-\alpha}}{\Gamma(2-\alpha)} - 1\right) \frac{d}{ds}\left(U\left(\eta s^{-\alpha/2}\right)\right) ds\right|. \tag{4.2.74}$$

because the integrand is decreasing and has a zero in s_0 we can separate the integral into two parts and estimate each in a different way.

The above local estimated will now be used in order to show some global results.

Definition 6 *For $\eta_0 > 0$ we define the semi-norm $\|\cdot\|_{\infty,\eta_0}$ of the function $U : [0,\infty] \to \mathbb{R}$ by*

$$\|U\|_{\infty,\eta_0} := \sup_{\eta\in[0,\eta_0]} |U(\eta)|. \tag{4.2.75}$$

We will also write $\|U\|_\infty := \|U\|_{\infty,\infty}$.

The introduction of η_0 is necessary for the proof of some future results. In the real experiment as η_0 we can always take η^* defined above as a bound of the support of U (see (4.2.14)-(4.2.15)). In this case we of course have $\|U\|_{\infty,\eta^*} = \|U\|_\infty$.

Theorem 7 ([H4], Theorem 1) *Let $U : [0,\infty) \to \mathbb{R}$ be bounded, decreasing and vanishing at infinity. Then*

$$\|F_\alpha U - U\|_{\infty,\eta_0} \leq A(\alpha)\, \|U\|_\infty + B(\alpha)\eta_0\, \|U'\|_\infty . \tag{4.2.76}$$

Moreover, if U and U' are in $L^1(0,\infty)$, then

$$\|F_\alpha U - U\|_1 \leq C(\alpha)\, \|U\|_1 \tag{4.2.77}$$

when $A(\alpha)$, $B(\alpha)$ and $C(\alpha)$ are defined by

$$\begin{aligned} A(\alpha) &:= \frac{1}{\Gamma(2-\alpha)} - 1, \quad B(\alpha) := \frac{\alpha}{2}\frac{\Gamma(2-\alpha)^{\frac{1}{1-\alpha}}}{\left(1-\Gamma(2-\alpha)^{\frac{1}{1-\alpha}}\right)^{1+\frac{\alpha}{2}}}\left(1-\frac{1}{2-\alpha}\right), \\ C(\alpha) &:= \frac{1}{\Gamma(2-\alpha)} - 1 + \frac{\alpha}{2}\frac{\Gamma(2-\alpha)^{\frac{1}{1-\alpha}}}{\left(1-\Gamma(2-\alpha)^{\frac{1}{1-\alpha}}\right)^{1-\frac{\alpha}{2}}}\left(1-\frac{1}{2-\alpha}\right). \end{aligned} \tag{4.2.78}$$

Here, $A(\alpha) = \gamma(1-\alpha) + O((1-\alpha)^2)$, $B(\alpha) = \frac{e^{\gamma/2}}{2(e^\gamma-1)^{3/2}}(1-\alpha) + O((1-\alpha)^2)$ oraz $C(\alpha) = \left(\gamma + \frac{1}{e^{\gamma/2}(e^\gamma-1)^{1/2}}\right)(1-\alpha) + O((1-\alpha)^2)$ when $\alpha \to 1^-$, where γ Euler-Mascheroni constant.

The above gives us the convergence of $F_\alpha U$ to U when $\alpha \to 1^-$ in the supremum and L^1 norms. Moreover, we can see that this convergence is of the first order.

The next theorem gives the estimate on the difference between (4.2.69) and (4.2.70).

Theorem 8 ([H4], Theorem 2) *Let U be bounded, decreasing and vanishing at infinity. Fix $\eta_0 > 0$ such that $E_{\eta_0} := \sup_{\eta\in[0,\eta_0]} |\eta/U'(\eta)|$ is finite. Then*

$$\left\|D(U) - \widetilde{D}(U)\right\|_{\infty,\eta_0} \leq E_{\eta_0}\left(2A(\alpha)\, \|U\|_\infty + B(\alpha)\eta_0\, \|U'\|_\infty\right), \tag{4.2.79}$$

where $A(\alpha)$ and $B(\alpha)$ are defined in Theorem 7.

The proof is based on some estimates based on the point-wise difference

$$\begin{aligned} \left|D(U(\eta)) - \widetilde{D}(U(\eta))\right| \leq \frac{1}{|U'(\eta)|}&\left[\left|\left(1-\frac{\alpha}{2}\right)\int_0^\eta F_\alpha U(z)dz - \left(\frac{1}{\Gamma(1-\alpha)} + \frac{\alpha}{2\Gamma(2-\alpha)}\right)\int_0^\eta U(z)dz\right|\right. \\ &\left. +\eta\left|\frac{\alpha}{2}F_\alpha U(\eta) - \frac{\alpha}{2\Gamma(2-\alpha)}U(\eta)\right|\right], \end{aligned} \tag{4.2.80}$$

and treating each of the absolute values one at a time. The first part can be estimates as follows

$$\begin{aligned} &\left|\left(1-\frac{\alpha}{2}\right)\int_0^\eta F_\alpha U(z)dz - \left(\frac{1}{\Gamma(1-\alpha)} + \frac{\alpha}{2\Gamma(2-\alpha)}\right)\int_0^\eta U(z)dz\right| \\ &\leq \eta\left[\left(1-\frac{\alpha}{2}\right)\|F_\alpha U - U\|_{\infty,\eta_0} + \left(\frac{\alpha}{2} + \frac{1}{\Gamma(1-\alpha)} + \frac{\alpha}{2\Gamma(2-\alpha)} - 1\right)\|U\|_\infty\right], \end{aligned} \tag{4.2.81}$$

while the second

$$\begin{aligned}\left|\frac{\alpha}{2}F_\alpha U(\eta)-\frac{\alpha}{2\Gamma(2-\alpha)}U(\eta)\right| &= \frac{\alpha}{2}\left|F_\alpha U(\eta)-\frac{1}{\Gamma(2-\alpha)}U(\eta)\right| \\ &\leq \frac{\alpha}{2}\left[|F_\alpha U(\eta)-U(\eta)|+\left(\frac{1}{\Gamma(2-\alpha)}-1\right)U(\eta)\right].\end{aligned} \tag{4.2.82}$$

Taking the norms, using the Theorem 7 and a manipulation of the Gamma functions implies the assertion.

Let us now consider the problem os stability of (4.2.69). The only source of the instability is the differentiation operator. Let $\delta > 0$ be the error (noise) level in the initial data, i.e.

$$\left\|U-U^\delta\right\| \leq \delta, \tag{4.2.83}$$

where U is the exact (unknown) function, U^δ its measured (known) value while $\|\cdot\|$ is some norm. Notice that in *every* experiment any measurement is always taken with some noise added. Hence, we never can know the exact value of U. How does then $D(U)$ differs from the known noisy function $D(U^\delta)$? The answer is very quick: since our procedure is not stable, $D(U^\delta)$ can be arbitrarily different from its exact value. Hence, a regularization strategy is necessary. We need a family of *stable* operators which approximate the first derivative arbitrarily close. Denote the regularized diffusivity by $D_h(U^\delta)$, where $h > 0$ is a parameter. We ask how much it is different from $D(U)$. Assume also that $D_h(U) \to D(U)$ for $h \to 0^+$.

In order to regularize the diffusivity (4.2.69) we choose the family according to

$$\left\|U'-(U^\delta)'_h\right\|_\infty \leq R(h,\delta), \tag{4.2.84}$$

for which there exists a $h_0 = h_0(\delta)$, such that $R(h_0(\delta),\delta) \to 0$ when $\delta \to 0$. We also defined the regularized diffusivity $D_h(U)$ by the one calculated via the formula (4.2.69), where in place of U' we have U'_h. In a similar way we define $D_h(U^\delta)$.

Theorem 9 ([H4], Theorem 3) *Assume that* $\left\|U-U^\delta\right\|_\infty \leq \delta$ *and let* D_h *be the family of regularization operators for the derivative chosen according to (4.2.84). Moreover, fix* $\eta_0 > 0$*. Then, for* U *bounded, decreasing and vanishing at infinity such that there exists an* $\epsilon > 0$ *for which* $\epsilon \leq \inf_{\eta\in[0,\eta_0]}|U'(\eta)| < \infty$*, we have*

$$\begin{aligned}&\left\|D(U)-D_h(U^\delta)\right\|_{\infty,\eta_0} \leq \\ &\frac{1}{\epsilon}\left[D_s\left(1+\frac{R(h,\delta)+\|U'\|_\infty}{\epsilon}\right)R(h,\delta)+\frac{\eta_0}{\Gamma(2-\alpha)}\left(\delta+\frac{\delta+\|U\|_\infty}{\epsilon}R(h,\delta)\right)\right].\end{aligned} \tag{4.2.85}$$

In applications, the following corollary is often more useful.

Corrolary 2 *Let the assumptions of the Theorem 9 be satisfied. If the regularization strategy (4.2.84) is such that* $R(h_0(\delta),\delta) = O(\delta^\mu)$ *when* $\delta \to 0$*, then*

$$\left\|D(U)-D_h(U^\delta)\right\|_{\infty,\eta_0} = O(\delta^\mu) \quad \textit{gdy} \quad \delta \to 0. \tag{4.2.86}$$

The proof of the theorem is based on several estimates. If we write $D(U(\eta)) = G(U(\eta))/U'(\eta)$, then

$$\begin{aligned}\left|D(U(\eta))-D_h(U^\delta(\eta))\right| &= \left|\frac{G(U(\eta))}{U'(\eta)}-\frac{G_h(U^\delta(\eta))}{U'(\eta)}+\frac{G_h(U^\delta(\eta))}{U'(\eta)}-\frac{G_h(U^\delta(\eta))}{(U^\delta)'_h(\eta)}\right| \\ &\leq \frac{1}{|U'(\eta)|}\left|G(U(\eta))-G_h(U^\delta(\eta))\right|+|G_h(U^\delta))|\left|\frac{1}{U'(\eta)}-\frac{1}{(U^\delta)'_h(\eta)}\right|.\end{aligned} \tag{4.2.87}$$

The first part can be estimated with the use of the assumption of the noise level and the definition of F_α

$$\left|G(U(\eta)) - G_h(U^\delta(\eta))\right| \leq D_s \left\|U' - (U^\delta)'_h\right\|_\infty + \frac{\delta}{\Gamma(2-\alpha)}\eta. \tag{4.2.88}$$

The second part can be dealt with in a standard way

$$\left|\frac{1}{U'(\eta)} - \frac{1}{(U^\delta)'_h(\eta)}\right| \leq \frac{1}{|U'(\eta)(U^\delta)'_h(\eta)|}\left|U'(\eta) - (U^\delta)'_h(\eta)\right| \leq \frac{1}{\epsilon^2}\left\|U' - (U^\delta)'_h\right\|_\infty. \tag{4.2.89}$$

Combining these two bounds and taking the norm $\|\cdot\|_{\infty,\eta_0}$ grants us the assertion.

We have shown that in order for a unique and stable determination of the diffusivity we have to use a regularization strategy. What is more important, the optimal regularization of D does not worsen the asymptotic convergence with δ.

4.2.6 Numerical methods (papers [H2-H3,H5])

In this section we will discuss some results of numerical simulations verifying the accuracy of the devised approximations. We will also derive several discretization schemes for the E-K operator. Moreover, convergence proofs will be given.

As we mentioned before, usually the first approach to the investigation of the nonlinear fractional equations is the use of numerical methods. Because of their nonlocality, this approach has to be different than for the differential equations. There is a wealth of literature concerning numerical methods for integro-differential equations, in particular - fractional equations. Here, we will mention only the review papers. The classical monographs which undertake the full breadth of the topic are [12, 60, 6]. A survey of numerical methods for fractional equations can be found for example in [7]. Moreover, schemes tailored particularly for the time-fractional diffusion have been given in [57]. Similar results can be found in [59].

The main result of this section gives a construction of a convergent numerical scheme used to discretize the E-K operator. According to the best knowledge of the author, this topic has not been previously investigated in the literature. However, there is a large number of works about numerical methods for other operators associated with fractional calculus, namely fractional derivatives and integrals [22] and fractional Laplacian [108, 43].

Simulations (papers [H2-H3])

In order to verify the accuracy of all derived approximations of the solution to (4.2.3), we have to solve the exact equation numerically. Since our problem is both nonlocal and nonlinear, we expect that the calculations can be very demanding on computer power. This is mainly due to the significance of the history of the process which has to be taken into account in each of the iteration steps. Let us introduce the grid (x_j, t_i), where $x_j = jk$, $t_i = ih$ for $k = X/M$ and $h = T/N$. Here, X and T are the maximal space and time values. If by u^i_n we denote the numerical approximation of $u(x_j, t_i)$ and use the rectangle rule for calculating the fractional integral, we will arrive at

$$(\partial^\alpha_t)\, u^i_j \approx \frac{h^{-\alpha}}{\Gamma(2-\alpha)}\left(u^{i+1}_j + \sum_{k=1}^{i} a_{k,i} u^k_j\right) \tag{4.2.90}$$

with weights

$$a_{k,i} = (i-k+2)^{1-\alpha} - 2(i-k+1)^{1-\alpha} + (i-k)^{1-\alpha}. \tag{4.2.91}$$

In order to discretize the spatial part of the equation we use θ-weighted method which reduces to explicit ($\theta = 0$), implicit ($\theta = 1$) and Crank-Nicolson ($\theta = 1/2$) schemes. Finally, we obtain

$$
\begin{aligned}
&u_j^{i+1} - (1-\theta)\frac{h^\alpha}{k^2\Gamma(2-\alpha)}\left(D_{j-1/2}^{i+1}u_{j-1}^{i+1} + \left(D_{j-1/2}^{i+1} + D_{j+1/2}^{i+1}\right)u_j^{i+1} + D_{j+1/2}^{i+1}u_{j+1}^{i+1}\right) \\
&= -\sum_{k=1}^{i} a_{k,i}u_j^k + \theta\frac{h^\alpha}{k^2\Gamma(2-\alpha)}\left(D_{j-1/2}^{i}u_{j-1}^{i} + \left(D_{j-1/2}^{i} + D_{j+1/2}^{i}\right)u_j^{i} + D_{j+1/2}^{i}u_{j+1}^{i}\right),
\end{aligned}
\tag{4.2.92}
$$

where

$$
D_{j\pm1/2}^i = \frac{1}{2}\left((u^m)_j^i + (u^m)_{j\pm1}^i\right). \tag{4.2.93}
$$

In order to effectively calculate the nonlinearity we can use the linerization technique given in [88]

$$
(u^m)_j^{i+1} = (u^m)_j^i + m(u^{m-1})_j^i\left(u_j^i - u_j^{i-1}\right) + O(h^2). \tag{4.2.94}
$$

The diffusivity can now be approximated by

$$
\begin{aligned}
D_{j\pm1/2}^{i+1} \approx \frac{1}{2}\Big((u^m)_j^i + m(u^{m-1})_j^i\left(u_j^i - u_j^{i-1}\right) + \\
(u^m)_{j\pm1/2}^i + m(u^{m-1})_{j\pm1/2}^i\left(u_{j\pm1/2}^i - u_{j\pm1/2}^{i-1}\right)\Big),
\end{aligned}
\tag{4.2.95}
$$

Each iteration of (4.2.92) requires to take into account all of the previous steps and solve a linear system of equations. Since the system matrix is tridiagonal we can use specialistic algorithms but still, the computational cost can be significant.

For the numerical simulations we will use (4.2.3) with (4.2.4). An exemplary calculation is given of Fig. 1, where we present a moisture profile for a fixed time. Immediately we can see that estimates (4.2.41) give rather poor results. This is not unexpected since these approximations had been a side-product of our existence and uniqueness theorem. However, they give a rather good indication of the position of the wetting front η^*. Notice also that the approximation U_3 differs very little from the exact solution almost on the whole domain. Despite the lack of rigorous estimates, but rather being an asymptotic representation as $\alpha \to 1$, this approximation is good enough in order to fit the model to he real data with ease.

The fitting with the real data from [25] is given on Fig. 2. We have used the dimensional variables, that is the U_3 approximation to the

$$
\partial_t^\alpha u = \left(D_0 u^m u_x\right)_x, \tag{4.2.96}
$$

with

$$
u(0,t) = C, \quad u(\infty,t) = 0, \quad u(x,0) = 0, \quad x,t > 0. \tag{4.2.97}
$$

We can see that the model (or its approximate solution) reproduces the data with good accuracy both in self-similar and temporal form. The only discrepancy arises near the wetting front and this can be a result of the measuring technique or different that power-type diffusivity. However, the approximation of (4.2.3) is gives a very good model of the moisture percolation in the porous medium and the fitting can be done with very large speed and low computational expenses. The fitting parameters derived in the calculations are very close to the ones obtained by other researchers.

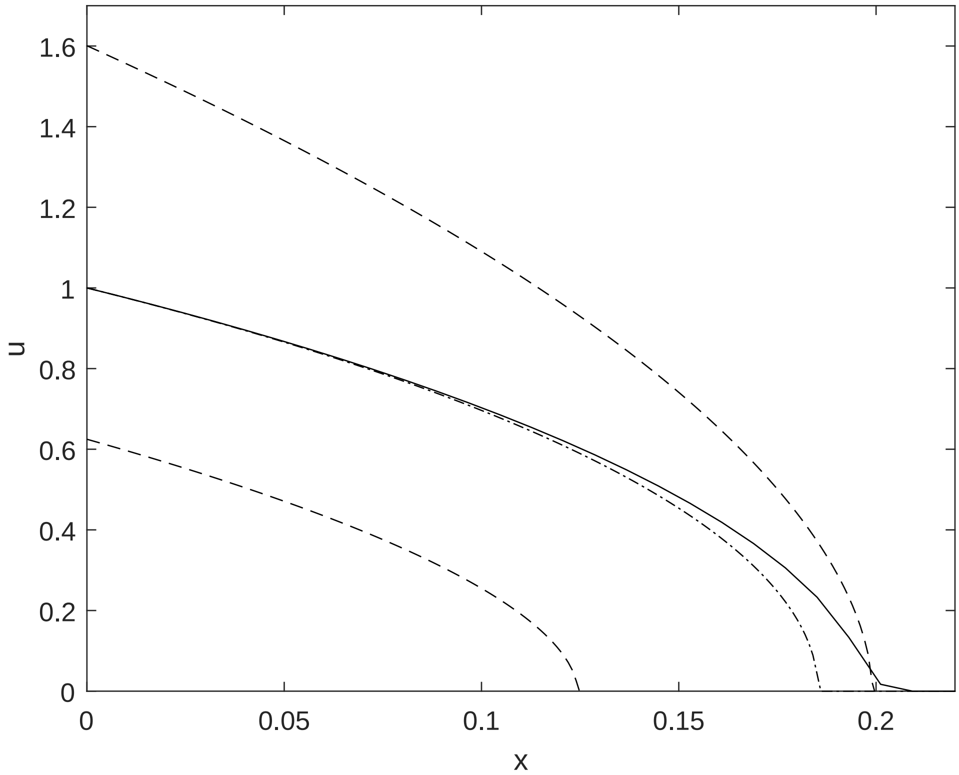


Figure 1: Estimates and approximation to the solution of (4.2.3) with (4.2.4). The solid line represents the numerical solution according to (4.2.92), dashed lines are estimates (4.2.41), while dot-dashed lines represent U_3 defined in (4.2.62). The parameters are $\alpha = 0.9$ and $m = 2$. The lot is drawn for $t = 0.02$.

Finite difference schemes for the E-K operator (paper [H5])

In this part of the dissertation we will discuss some results concerning discretization of the E-K operator which, according to our knowledge, are one of the first ones in the literature. In order to state the matters more clearly, we slightly change the notation

$$I_{a,b,c}y(x) := \frac{1}{\Gamma(b)} \int_0^1 (1-s)^{b-1} s^a y\left(s^{1/c} x\right) ds, \quad x \in (0, X). \tag{4.2.98}$$

We can also write

$$I_{a,b,c}y(x) = \frac{1}{\Gamma(b)} \sum_{i=0}^{n-1} \int_{s_i}^{s_{i+1}} (1-s)^{b-1} s^a y\left(s^{1/c} x\right) ds, \tag{4.2.99}$$

where

$$0 = s_0 < s_1 < s_2 < \cdots < s_i < \cdots < s_n = 1, \tag{4.2.100}$$

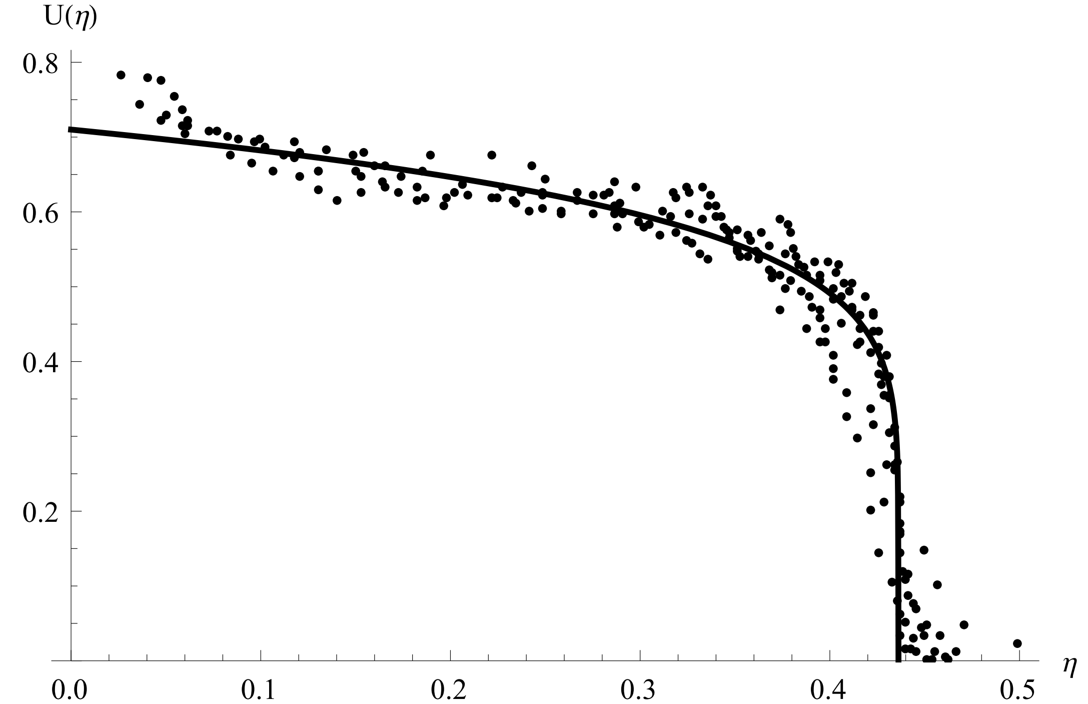
is the partition of the interval $[0, 1]$. The discretization method depends on the method of approximating the above integral. Now, we will present some of the typical and most useful discretizations. All the detailed calculations can be found in [H5].

- **Rectangle rule.**

$$L^r_{a,b,c}y(x) := \frac{1}{\Gamma(b)} \sum_{i=0}^{n-1} y(s_i^{1/c} x) \int_{s_i}^{s_{i+1}} (1-s)^{b-1} s^a ds = \sum_{i=0}^{n-1} v_i^r(a, b)\, y(s_i^{1/c} x), \tag{4.2.101}$$

with weights

$$v_i^r(a, b) := \frac{1}{\Gamma(b)} \int_{s_i}^{s_{i+1}} (1-s)^{b-1} s^a ds = \frac{B(s_{i+1}; a+1, b) - B(s_i; a+1, b)}{\Gamma(b)}, \tag{4.2.102}$$



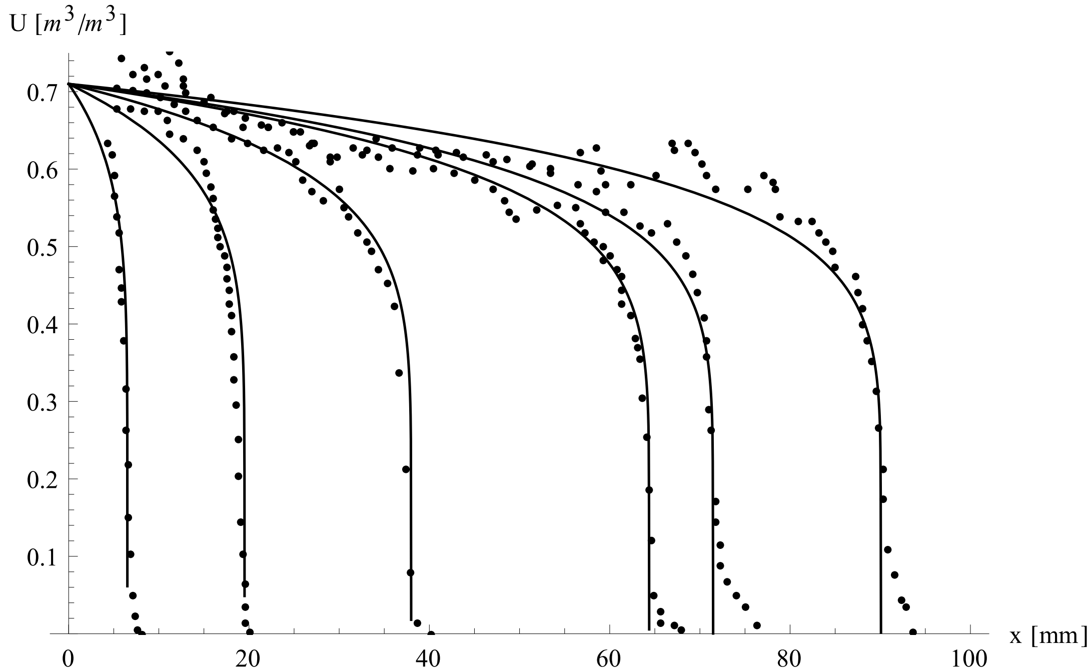


Figure 2: Fitting of U_3 as in (4.2.62) to the experimental data from [25]. On the top: self-similar form $U_3(\eta)$. On the bottom: time evolution of $u_3(x,t)$. Fitted parameters are $\alpha = 0.855$, $C = 0.71$ m^3/m^3, $m = 6.98$, $D_0 = 5.36$ mm/$s^{0.855}$.

and we used the Incomplete Beta function.

- **Midpoint rule.**

$$L^m_{a,b,c}y(x) := \frac{1}{\Gamma(b)}\sum_{i=0}^{n-1} y(s_{i+1/2}^{1/c}x)\int_{s_i}^{s_{i+1}}(1-s)^{b-1}s^a ds = \sum_{i=0}^{n-1} v_i^r(a,b)\, y(s_{i+1/2}^{1/c}x), \tag{4.2.103}$$

where $v^r(a,b)$ are the same as above.

- **Trapezoid rule.**

$$\begin{aligned} L^t_{a,b,c}y(x) : &= \frac{1}{\Gamma(b)}\sum_{i=0}^{n-1}\frac{y(s_{i+1}^{1/c}x)-y(s_i^{1/c}x)}{s_{i+1}-s_i}\int_{s_i}^{s_{i+1}}(1-s)^{b-1}s^a(s-s_i)ds + y(s_i^{1/c}x)\int_{s_i}^{s_{i+1}}(1-s)^{b-1}s^a ds \\ &= \sum_{i=0}^{n} v_i^t(a,b)\, y(s_i^{1/c}x), \end{aligned} \tag{4.2.104}$$

with

$$v_i^t(a,b) := \begin{cases} B_0, & i=0; \\ A_{i-1}+B_i, & 0<i<n; \\ A_{n-1}, & i=n, \end{cases} \tag{4.2.105}$$

$$\begin{aligned} A_i &:= \frac{1}{\Gamma(b)}\frac{\delta_i B(a+2,b) - s_i\ \delta_i B(a+1,b)}{s_{i+1}-s_i}, \\ B_i &:= \frac{1}{\Gamma(b)}\left(\delta_i B(a+1,b) - \frac{\delta_i B(a+2,b) - s_i\ \delta_i B(a+1,b)}{s_{i+1}-s_i}\right), \end{aligned} \tag{4.2.106}$$

oraz

$$\delta_i B(a,b) := B(s_{i+1};a,b) - B(s_i;a,b). \tag{4.2.107}$$

It is easy to propose higher order methods but due to nonlocality of the E-K, theirs computational cost might be too high for the usual needs.

As it will appear, it is much more convenient to discretize the integral in (4.1.9) after the substitution $t = s^{1/c}x$. Then

$$I_{a,b,c}y(x) = \frac{cx^{-c(a+b)}}{\Gamma(b)}\int_0^x (x^c - t^c)^{b-1}t^{c(a+1)-1}y(t)dt. \tag{4.2.108}$$

If we introduce the subdivision of the interval $[0,x]$

$$0 = t_0 < t_1 < t_2 < \cdots < t_i < \cdots < t_n = x. \tag{4.2.109}$$

we than have

$$I_{a,b,c}y(x) = \frac{cx^{-c(a+b)}}{\Gamma(b)}\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} (x^c - t^c)^{b-1}t^{c(a+1)-1}y(t)dt. \tag{4.2.110}$$

The standard discretizations have now the form

- **Rectangle rule.**

$$K^r_{a,b,c}y(x) = \frac{1}{\Gamma(b)}\sum_{i=0}^{n-1} y(t_i)\int_{(t_i/x)^c}^{(t_{i+1}/x)^c}(1-s)^{b-1}s^a ds = \sum_{i=0}^{n-1} w_i^r(a,b,c)y(t_i), \tag{4.2.111}$$

with weights

$$w_i^r(a,b,c) := \frac{B((t_{i+1}/x)^c; a+1,b) - B((t_i/x)^c; a+1,b)}{\Gamma(b)}. \tag{4.2.112}$$

- **Midpoint rule.**

$$K^m_{a,b,c}y(x) = \frac{1}{\Gamma(b)}\sum_{i=0}^{n-1} y(t_{i+1/2})\int_{(t_i/x)^c}^{(t_{i+1}/x)^c}(1-s)^{b-1}s^a ds = \sum_{i=0}^{n-1} w^r_i(a,b,c)y(t_{i+1/2}), \tag{4.2.113}$$

  with the weight given above.

- **Trapezoid rule.**

$$K^t_{a,b,c}y(x) = \sum_{i=0}^{n} w^t_i(a,b,c)y(t_i), \tag{4.2.114}$$

  with

$$w^t_i(a,b) := \begin{cases} D_0, & i=0; \\ C_{i-1}+D_i, & 0<i<n; \\ C_{n-1}, & i=n, \end{cases} \tag{4.2.115}$$

  where

$$\begin{aligned} C_i &:= \frac{1}{\Gamma(b)}\frac{x\,\Delta_i B(a+1/c+1,b,c) - t_i\,\Delta_i B(a+1,b,c)}{t_{i+1}-t_i}, \\ D_i &:= \frac{1}{\Gamma(b)}\left(\Delta_i B(a+1,b,c) - \frac{x\,\Delta_i B(a+1/c+1,b,c) - t_i\,\Delta_i B(a+1,b,c)}{t_{i+1}-t_i}\right), \end{aligned} \tag{4.2.116}$$

  and

$$\Delta_i B(a,b,c) := B((t_{i+1}/x)^c;a,b) - B((t_i/x)^c;a,b). \tag{4.2.117}$$

Using the uniform grid spacing we can prove theorems of the exact form of the discretization error. Before that, however, we need a technical lemma.

Lemma 4 ([H5], Lemma 1) *For* $a,b\in\mathbb{R}$ *and* $c>0$ *we have*

$$\frac{1}{n}\sum_{i=1}^{n-1}\left(\frac{i}{n}\right)^{a-1}\left(1-\left(\frac{i}{n}\right)^c\right)^{b-1} \sim \begin{cases} \frac{1}{c}B\left(\frac{a}{c},b\right), & a>0 \textit{ oraz } b>0; \\ \ln n, & a=0 \textit{ oraz } b>0; \\ c^{b-1}\ln n, & a>0 \textit{ oraz } b=0; \\ \zeta(1-a)n^{-a}, & a<0 \textit{ oraz } b>a \\ c^{b-1}\zeta(1-b)n^{-b}, & b<0 \textit{ oraz } a>b \\ (1+c^{a-1})\zeta(1-a)n^{-a}, & a=b<0, \end{cases} \tag{4.2.118}$$

where $n\to\infty$.

The proof is based on a proper estimates on the sum behaviour. For $a,b>0$ the problems boils down to the evaluation of the Riemann sum. The other cases are more difficult are require careful estimations procedures from which one of the most useful is to bound certain part of the sum by a convergent integral.

Lemma 4 is the key tool in finding the exact form of the discretization error. Thanks to this, it is possible to find the *exact* values of the error constants in the difference between $I_{a,b,c}$ and $L^\mu_{a,b,c}$ or $K^\mu_{a,b,c}$. Here, $\mu\in\{r,m,t\}$. Introduce the uniform grid $s_i=i/n$ and $t_i=xi/n$, where n is the partition number. We obviously have $s_{i+1}-s_i=1/n$ and $t_{i+1}-t_i=x/n$.

Theorem 10 ([H5], Theorem 1) *Fix $a, b, c > 0$ and assume that $y \in C^2(0, X)$. Then, for a fixed $x \in (0, X)$ the discretizations of $I_{a,b,c}$ have the following asymptotic behaviour as $n \to \infty$.*

- *Rectangle rule*

$$I_{a,b,c}y(x) - L^r_{a,b,c}y(x) \sim \tfrac{x}{c}\tfrac{y'(\sigma^{\frac{1}{c}}x)}{2\Gamma(b)} \begin{cases} n^{-1}B\left(\frac{1}{c}+a, b\right), & \frac{1}{c}+a>0; \\ n^{-1}\ln n, & \frac{1}{c}+a=0; \\ n^{-(1+\frac{1}{c}+a)}\zeta\left(1-\frac{1}{c}-a\right), & \frac{1}{c}+a<0, \end{cases} \tag{4.2.119}$$

$$I_{a,b,c}y(x) - K^r_{a,b,c}y(x) \sim \tfrac{1}{n}\tfrac{x}{2\Gamma(b)}y'(\tau)B(a+1,b).$$

- *Trapezoid rule*

$$I_{a,b,c}y(x) - L^t_{a,b,c}y(x) \sim$$
$$\begin{cases} -\frac{x}{c}\left(\frac{1}{c}-1\right)\frac{y'(\sigma^{\frac{1}{c}}x)}{12\Gamma(b)} \begin{cases} n^{-2}B\left(\frac{1}{c}+a-1, b\right), & \frac{1}{c}+a>1; \\ n^{-2}\ln n, & \frac{1}{c}+a=1; \\ n^{-(1+\frac{1}{c}+a)}\zeta\left(2-\frac{1}{c}-a\right), & \frac{1}{c}+a<1, \end{cases} & c \neq 1; \\ -\frac{1}{n^2}\frac{x^2y''(\sigma x)}{12\Gamma(b)}B(a+1,b), & c=1. \end{cases} \tag{4.2.120}$$

$$I_{a,b,c}y(x) - K^t_{a,b,c}y(x) \sim -\tfrac{1}{n^2}\tfrac{x^2}{12\Gamma(b)}y''(\tau)B(a+1,b). \tag{4.2.121}$$

- *Midpoint rule*

$$I_{a,b,c}y(x) - L^m_{a,b,c}y(x) = \begin{cases} O(n^{-2}), & a+\frac{1}{c}>1 \text{ and } b \geq 1; \\ O(n^{-2}\ln n), & a+\frac{1}{c}=1 \text{ or } b \geq 1; \\ O(n^{-(1+\min\{b, a+\frac{1}{c}\})}), & -1<a+\frac{1}{c}<1 \text{ or } 0<b<1, \end{cases} \tag{4.2.122}$$

$$I_{a,b,c}y(x) - K^m_{a,b,c}y(x) = \begin{cases} O(n^{-2}), & c(a+1) \geq 1 \text{ and } b \geq 1; \\ O(n^{-(1+\min\{c(a+1),b\})}), & 0<c(a+1)<1 \text{ or } 0<b<1. \end{cases}$$

Here $\sigma \in (0,1)$ and $\tau \in (0,x)$ depend on a, b, c, the function y and can be different between the discretizations.

Each of the above cases can be showed by the estimates of the remainder in the Taylor-Lagrange series. For example, in the $K^r_{a,b,c}$ case we have

$$I_{a,b,c}y(x) = K^r_{a,b,c}y(x) + \frac{cx^{-c(a+b)}}{\Gamma(b)}\sum_{i=0}^{n-1} y'(\widetilde{\tau}_i)\int_{t_i}^{t_{i+1}} (x^c-t^c)^{b-1}t^{c(a+1)-1}(t-t_i)dt. \tag{4.2.123}$$

If we define the function

$$G(z) := \int_{t_i}^{z} (x^c-t^c)^{b-1}t^{c(a+1)-1}(t-t_i)dt, \tag{4.2.124}$$

we can expand it at $t=t_i$ and evaluate it at $z=t_{i+1}$

$$G(t_{i+1}) = (x^c-t_i^c)^{b-1}t_i^{c(a+1)-1}\frac{x^2}{2n^2} + \frac{d^2}{dt^2}\left[(x^c-t^c)^{b-1}t^{c(a+1)-1}(t-t_i)\right]|_{t=\widehat{\tau}_i}\frac{x^3}{6n^3}. \tag{4.2.125}$$

The main difficulty is to find the asymptotic order of the remainder when $n \to \infty$. In [H5] we show that the second part of the above is $o(n^{-2})$. The proof is technical and requires a careful counting of the powers of n present in the expansion. Then, we can use Lemma 4 to obtain the assertion $K^r_{a,b,c}$. In a similar way we prove the rest cases. For the midpoint operators, $L^m_{a,b,c}$ and $K^m_{a,b,c}$ we cannot use the mean-value theorem in a critical part of the proof. This causes a somewhat weaker assertion limited only to the convergence rate (but still, this is completely sufficient in applications).

In the discretization $L^\mu_{a,b,c}$ we can see that the kernel's singularity causes the convergence rate to weaken. This is an unpleasant property which indicates that the operator $K^\mu_{a,b,c}$ is much more reliable in numerical analysis. Notice also the following result concerning the uniform estimates.

Corrolary 3 ([H5], Corollary 1) *Let* $a, b, c > 0$ *and assume that* $y \in C^2(0, X)$. *Let* M *be the common bound for* y' *and* y'', *that is* $|y'(x)| \le M$ *and* $|y''(x)| \le M$ *for* $x \in (0, X)$. *We then have*

- *rectangular rule*

$$|I_{a,b,c}y(x) - L^r_{a,b,c}y(x)| \le \frac{X}{c}\frac{M}{2\Gamma(b)} \begin{cases} n^{-1}B\left(\frac{1}{c}+a, b\right), & \frac{1}{c}+a > 0; \\ n^{-1}\ln n, & \frac{1}{c}+a = 0; \\ n^{-(1+\frac{1}{c}+a)}\zeta\left(1-\frac{1}{c}-a\right), & \frac{1}{c}+a < 0, \end{cases} \tag{4.2.126}$$

$$|I_{a,b,c}y(x) - K^r_{a,b,c}y(x)| \le \frac{1}{n}\frac{X}{2\Gamma(b)}MB(a+1, b).$$

- *Trapezoid rule*

$$|I_{a,b,c}y(x) - L^t_{a,b,c}y(x)| \le \begin{cases} \frac{X}{c}\left(\frac{1}{c}-1\right)\frac{M}{12\Gamma(b)} \begin{cases} n^{-2}B\left(\frac{1}{c}+a-1, b\right), & \frac{1}{c}+a > 1; \\ n^{-2}\ln n, & \frac{1}{c}+a = 1; \\ n^{-(1+\frac{1}{c}+a)}\zeta\left(2-\frac{1}{c}-a\right), & \frac{1}{c}+a < 1, \end{cases} & c \neq 1; \\ \frac{1}{n^2}\frac{X^2 M}{12\Gamma(b)}B(a+1, b), & c = 1. \end{cases} \tag{4.2.127}$$

$$|I_{a,b,c}y(x) - K^t_{a,b,c}y(x)| \le \frac{1}{n^2}\frac{X^2}{12\Gamma(b)}MB(a+1, b). \tag{4.2.128}$$

In order to numerically illustrate Theorem 10 we will explicitly calculate the error constants. More specifically, for $K^r_{a,b,c}$ we evaluate the expression

$$\frac{|I_{a,b,c}y(x) - K^r_{a,b,c}y(x)|}{\frac{x}{2\Gamma(b)n}B(a+1, b)}, \tag{4.2.129}$$

for increasing n. This sequence has to converge to 1. Similar expressions can be devised for other discretizations. The trial function y is chosen with respect to the method: $y(x) = x$ for rectangular or $y(x) = x^2/2$ for trapezoid. The results of numerical simulations are given in Fig. 3. We can see that all the quantities converge to 1 which verifies the assertion of the theorem.

The next test is to compute the empirical convergence rates. For the trial function we choose $y(x) = e^x$. Since we cannot calculate the explicit form of $I_{a,b,c}y(x)$ we use the Richardson extrapolation technique applied to finding the convergence rate (Aitken's method, see [40]). Results re given in Tab. 2-3. We can see that they are identical to the theoretically predicted values 10. Notice the weakening of the convergence rate for $L_{a,b,c}$ with the increase of the kernel's singularity.

To end this summary we give a result concerning a numerical solution of the integro-differential equation with the E-K operator

$$y' = f(x, y, I_{a,b,c}y), \quad y(0) = y_0, \quad x \in (0, X), \tag{4.2.130}$$

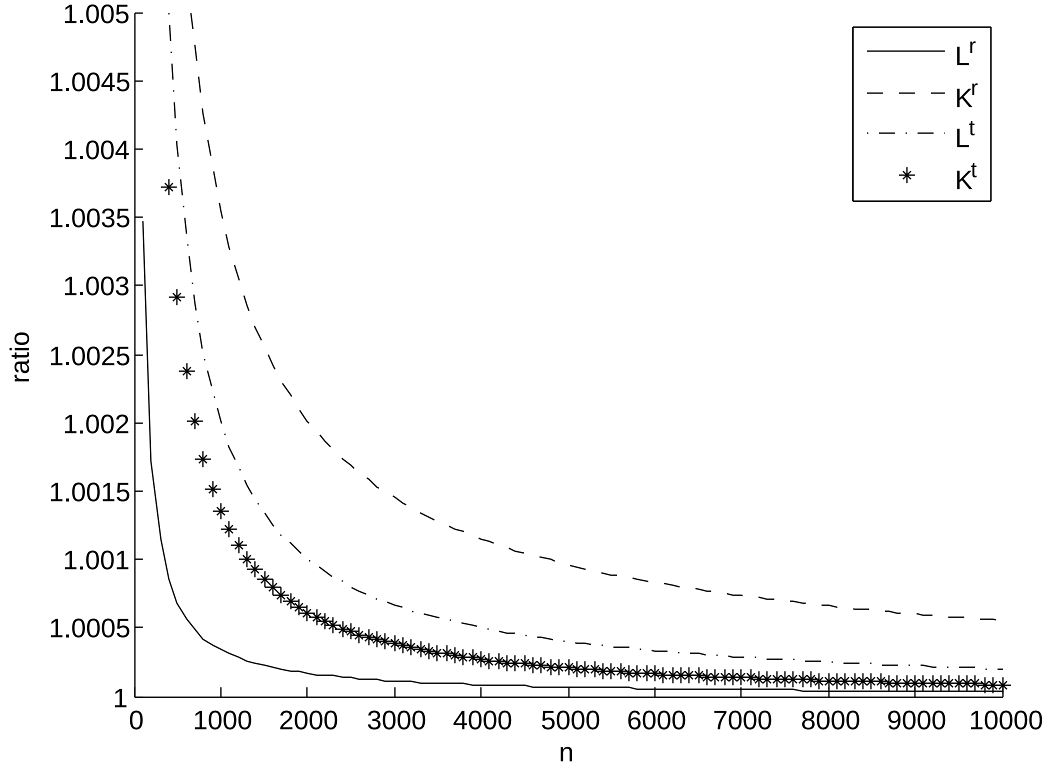


Figure 3: The ratios $|I_{a,b,c}y(x) - A_{a,b,c}y(x)|$ and its asymptotic form given in (4.2.119)-(4.2.121) for $n \to \infty$. Here $A_{a,b,c}$ is one of the operators given in the legend. Parameters are $a = 0.5$, $b = 1.5$ and $c = 0.5$ for $x = 1$.

	$a = 1, b = 1.5, c = 0.5$	$a = -0.9, b = 0.5, c = 2$
Rectangle	1.0000	0.6050
Trapezoid	2.0002	0.6000
Midpoint	1.9934	0.6008

Table 2: Simulated convergence rates for $L_{a,b,c}$.

	$a = 1, b = 1.5, c = 0.5$	$a = -0.9, b = 0.5, c = 2$
Rectangle	1.0001	1.0088
Trapezoid	2.0001	1.9977
Midpoint	1.9985	1.1955

Table 3: Simulated convergence rates for $K_{a,b,c}$.

where f is Lipschitz

$$|f(x,u,p) - f(x,v,p)| \leq L_1|u-v|, \quad |f(x,u,p) - f(x,u,q)| \leq L_1|p-q|. \tag{4.2.131}$$

As a discretization we choose the trapezoid rule for both the equation and the E-K operator. It has the form

$$y_{n+1} = y_n + \frac{h}{2}\left(f(x_n, y_n, K^t_{a,b,c}y_n) + f(x_{n+1}, y_{n+1}, K^t_{a,b,c}y_{n+1})\right). \tag{4.2.132}$$

The method (4.2.132) is convergent with order 2.

Theorem 11 ([H5], Theorem 3) *Assume that $f \in C^2(\mathbb{R}^3)$ for $x \in [0, X]$. Then, for $a > -1$, $b \geq 1$ and $c > 0$ the numerical scheme (4.2.132) is convergent with order 2, that is*

$$|y(x_n) - y_n| = O(h^2) \quad \textit{gdy} \quad h \to 0 \quad \textit{dla} \quad nh = \mathrm{const}. \tag{4.2.133}$$

When we denote the error by

$$e_n := y(x_n) - y_n, \tag{4.2.134}$$

we can write

$$\begin{aligned} e_{n+1} = e_n &+ \frac{h}{2}\left(f(x_n, y(x_n), Iy(x_n)) - f(x_n, y_n, Ky_n)\right) \\ &+ \frac{h}{2}\left(f(x_{n+1}, y(x_{n+1}), I_{a,b,c}y(x_{n+1})) - f(x_{n+1}, y_{n+1}, Ky_{n+1})\right) + Ch^3, \end{aligned} \tag{4.2.135}$$

where we have used the error form of the trapezoid quadrature. We can estimate further with the use of the definition of $K^t_{a,b,c}$ and the Lipschitz condition. We arrive at the recurrence inequality

$$\begin{aligned} |e_{n+1}| &\leq \frac{1}{1 - L_1\frac{h}{2} - WL_2\frac{h^2}{2}}\left[\left(1 + L_1\frac{h}{2}\right)|e_n| + WL_2h^2\sum_{i=1}^{n}|e_i| + (DL_2 + C)h^3\right] \\ &\leq (1 + C_1h)\,|e_n| + C_2h^2\sum_{i=1}^{n}|e_i| + C_3h^3, \end{aligned} \tag{4.2.136}$$

where all the capital letters denote the positive constants. By the use of the mathematical induction we can show that

$$|e_{n+1}| \leq \frac{\left(1 + C_1h + C_2(n+1)h^2\right)^{n+1} - 1}{C_1h}C_3h^3. \tag{4.2.137}$$

Since we have $(1 + C_1h + C_2(n+1)h^{1+\delta})^{n+1} \leq (1 + C_4h)^{n+1} \leq e^{C_4}$ then $|e_{n+1}| \leq O(h^2)$.

We have shown that the E-K operator can be efficiently approximated using the numerical methods. These discretizations can then be used in order to solve integro-differential equations. The relevant issue is to appropriately choose the discretization scheme since the singularity of the kernel can reduce the convergence rate.

5. **A discussion of the other papers.**

(a) List of papers which have not been included into the scientific achievement. All works are listed according to their topic (and not chronologically)

[P1] W. Okrasiński, Ł.Płociniczak, *A Nonlinear Mathematical Model of the Corneal Shape*, Nonlinear Analysis: Real World Applications 13 (2012), 1498-1505.

[P2] W. Okrasiński, Ł.Płociniczak, *Bessel function model of corneal topography*, Applied Mathematics and Computation 223 (2013), 436-443.

[P3] W.Okrasiński, J.J.Nieto, Ł.Płociniczak, O.Dominguez, *On a nonlinear boundary value problem modeling corneal shape*, Journal of Mathematical Analysis and Applications 414 (1) (2014), 461–471.

[P4] W.Okrasiński, Ł.Płociniczak, *Regularization of an Ill-posed Problem in Corneal Topography*, Inverse Problems in Science and Engineering 21 (6) (2013), 1090–1097.

[P5] W.Okrasiński, Ł.Płociniczak, *Nonliear Parameter Identification in Corneal Geometry Model*, Inverse Problems in Science and Engineering 23 (3) (2015), 443-456.

[P6] G. Griffiths, Ł.Płociniczak, W. Schiesser, *Analysis of cornea curvature using radial basis functions - Part I: Methodology*, Computers in Biology and Medicine, 77 (2016), pp. 274-284.

[P7] G. Griffiths, Ł.Płociniczak, W. Schiesser, *Analysis of cornea curvature using radial basis functions - Part II: Fitting to data-set*, Computers in Biology and Medicine 77 (2016), pp. 285-296.

[P8] G. Griffiths, Ł.Płociniczak, W. Schiesser, *ODE/PDE Analysis of Corneal Curvature*, Computers in Biology and Medicine 53 (2014), 30-41.

[P9] W. Okrasiński, Ł.Płociniczak, *Solution estimates for a system of nonlinear integral equations arising in optometry*, Journal of Integral Equations and Applications, in press

[P10] Ł.Płociniczak, *On asymptotics of some fractional differential equations*, Mathematical Modelling and Analysis 18 (3) (2013), 358-373.

[P11] Ł.Płociniczak, *Eigenvalue asymptotics of a fractional boundary-value problem*, Applied Mathematics and Computation 241 (2014), 125-128.

[P12] W.Okrasiński, Ł.Płociniczak, *A note on fractional Bessel function and its asymptotics*, Fractional Calculus and Applied Analysis 16 (3) (2013), 559-572.

[P13] Ł.Płociniczak, A. Popiołek-Masajada, M. Szatkowski, J. Masajada, *High order vortex beam in the optical vortex microscope*, SPIE Optical Engineering+ Applications, International Society for Optics and Photonics (2015) p. 95810N-95810N-8.

[P14] Ł.Płociniczak, A. Popiołek-Masajada, J. Masajada, M. Szatkowski, *Analytical model of the optical vortex microscope*, Applied Optics 55(12) (2016), pp. B20-B27.

[P15] Ł.Płociniczak, A. Popiołek-Masajada, M. Szatkowski, D. Wojnowski, *Transformation of the vortex beam in the optical vortex scanning microscope*, Optics & Laser Technology 81 (2016), pp. 127–136.

[P16] M. Maciejewska, Ł.Płociniczak, A. Szczurek, *Regularization and the inflection point method for a sensor signal in gas concentration measurement*, Inverse Problems in Science and Engineering 25(4) (2017), pp. 555-579.

[P17] Ł.Płociniczak, M.Świtała, *Monotonicity, oscillations and stability of a solution to a nonlinear equation modelling the capillary rise*, Physica D 362, pp. 1-8

(b) A discussion of the results included in the aforementioned papers

5.1 Corneal topography models (papers [P1-P9])

Author's interest in mathematical modelling of the human cornea has arisen during his PhD studies and yielded a thesis. The problem that emerged during these studies happened to be interesting enough to attract many researchers from all over the world to contribute to the author's model. We will shortly summarize the overall content of these results.

Papers [P1-P2,P4] have furnished the foundation of the mathematical model of human corneal topography and have been included in the author's PhD thesis. The modelling of sight associated phenomena becomes more and more important for the future of our civilisation. It helps to understand and treat many eye disorders and diseases. One of the most important constituent of the eye is its frontal part - the cornea. This in turn, is responsible for about 2/3 of the refractive power making it not only optical but, thanks to its protective role, a mechanical part of the eye.

The introduced model is based on a force balance for a membrane approximation of the cornea. In some sense, we can think of our model as being somewhat intermediate between the simplest ones built on the basis of conical curves, and those most complex - coming from the elasticity theory. In the notation, the model has the form

$$-T\,\nabla\cdot\left(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\right)+kh=\frac{P}{\sqrt{1+|\nabla h|^2}},\quad h|_{\Omega}=0, \tag{5.1.1}$$

where $h = h(x, y)$ describes the shape of cornea, Ω is the domain, T is the surface tension, k elasticity coefficient (associated with Young's modulus) and P is the intra-ocular pressure. This equation gives a relation between the corneal geometry and the pressure.

In [P1-P2] we have given the existence, uniqueness and estimates results for the axisymmetric solution of the above equation. We have been able to obtain several results stating that provided the pressure is not too large, the problem has a uniquely determined solution. This is not, however, the most general result that can be proven. Some other authors [19] showed that the same assertion can be shown without any strict assumption on the pressure [18, 110]. Our results have been improved in [P3,P9]. There, we show some stronger results and some existence results.

Works [P4-P5] concern an important topic of inverse problems associated with corneal topography. It boils down to being able to measure the intra-ocular pressure on the basis of the corneal topography. Because of the fact that the shape of cornea can be measured in a relatively straightforward and cheap way, while the measurement of the pressure is invasive and unpleasant for the patient, this inverse problem is well-motivated. As it appears, the problem is ill-posed and in the aforementioned works we present the regularization strategies used to overcome this difficulty. Moreover, we prove several results concerning the regularization inequalities.

The series of papers [P6-P8] is a result of a collaboration with researchers from Lehigh Univeristy and City Univeristy of London. They undertake the topic of numerical methods used in solving our model. Our results concern several different methods such as Method of Lines (MOL) the Meshless Method. Both of these approaches differ from the standard finite difference schemes and have certain advantages over them. Our papers are equipped with complete R implementation of the algorithms what is a step in the direction of practitioners without sufficient mathematical and coding experience (for ex. physicians).

5.2 Asymptotics of ordinary fractional equations (papers [P10-P12])

The next works included in this summary concern ordinary fractional differential equations. The main result has been obtained in [P10] and gives the asymptotic behaviour of the following problem

$${}^{C}D_0^{\alpha}y=\lambda q(x)y,\quad 0<\alpha<2,\quad \lambda\in\mathbb{R}. \tag{5.2.1}$$

The solution of the above is strongly dependent on the sign of λ. The most interesting happens for $\lambda < 0$ and $1 < \alpha < 2$, when the so-called fractional oscillations occur. As a result, the solution oscillates but has only a finite number of zeros which number is related to α. In this paper we also give an approximate formula for finding the number of zeros.

These results have been then used for investigation of the boundary value problem with

$$y(0) = 0, \quad y(1) = 0. \tag{5.2.2}$$

Using the generalization of the results concerning asymptotic behaviour of the solutions to fractional differential equations we show in [P11], that the number of different eigenvalues for the boundary value problem is finite. This result differs completely from the classical Sturm-Liouville theory.

The last paper [P12] presents similar results but concerning the fractional Bessel equation.

5.3 Analytical model of a optical vortex microscope (papers [P13-P15])

The papers [P13-P15] were a result of author's collaboration with physicists from the Faculty of Fundamental Problems of Technology from the Wroclaw University of Science and Technology. This work has been supported by the National Science Centre. The objective of author's work was to construct the analytical model of the optical vortex microscope. This observation technique is a novel and innovative method of magnifying objects.

The analytical model is based on a Fresnel type integral which contains different factors responsible for all of the constituents of the experimental set-up. The main difficulty is the fact that introduction of the vortex destroys the symmetry of the integral and in that case, all the calculations become severely complicated. In [P13-P15] we have presented a series of results with an increasing level of generality which lead to the full expression of the light amplitude after passing through the whole route.

5.4 The analysis of the inflection point method in sensor technology (paper [P16])

These results have been a result of author's collaboration with researchers from the Faculty of Environmental Engineering from the Wroclaw University of Science and Technology. They concern the analysis of a sensor used in detecting various volatile gases such as benzene, toluene or xylene. A typical measurement is based on waiting for a sufficiently long time required for the sensor to approach its saturation state. This steady-state can then be related to the compounds concentration.

As became apparent in [P16], the necessity of a long wait can be overridden. The investigated signal has an inflection point which, quite remarkably, depend linearly on the compound's concentration. This inflection point appears much earlier than the saturation level (about 20 times faster) which gives us a quick method of finding concentration.

The mathematical problem in this work was to regularize the signal and provide some appropriate estimates on the concentration. First, in order to find the inflection point we have to calculate the maximum of signal's derivative which can be done by the Brent's method (we do not want to compute any derivatives unless necessary). The signal is always given with some additive noise so that the determination of the inflection point is an unstable operation. We introduce a number of different algorithms which are used in order to smooth the signal and to determine the inflection point in a stable way. Moreover, we prove several *a-priori* estimates on the error between the exact inflection point and its version calculated from the noisy data. The theoretical results are then verified by some simulations and real data.

5.5 Analysis of an equation modelling capillary rise (paper [P17])

An equation describing capillary rise under the influence of surface tension, gravity and viscosity can be stated in the nondimensional form as follows

$$HH' + H + \omega\,(HH')' = 1, \tag{5.5.1}$$

where $\omega > 0$ and $H(0) = 0$, $H'(0) = \frac{1}{\sqrt{\omega}}$. It is a nonlinear, singular second order ordinary equation which was the main object of our work. The most important result, apart from global existence and uniqueness, is the classification of the solution's behaviour. It appeared that there exists a nondimensional constant ω which acts as a switch between two regimes: for $\omega < 0.25$ the solution is monotone while for $\omega > 0.25$ it oscillates. This result was verified with the real experimental data and the agreement was perfect.

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