SUMMARY OF PROFESSIONAL ACCOMPLISHMENTS

1 Name and surname

Bartosz Frej

2 Diplomas and degrees held.

2003: PhD in Mathematics, Wrocław University of Science and Technology, Faculty of Fundamental Problems of Technology, Institute of Mathematics

title of the PhD thesis: Markov operators in topological setup
advisor: Tomasz Downarowicz

1998: MSc in Mathematics, Wrocław University of Science and Technology, Faculty of Fundamental Problems of Technology

3 Information on hitherto employment in scientific institutions.

- **1998–2003:** research assistant in Wrocław University of Science and Technology, Faculty of Fundamental Problems of Technology, Institute of Mathematics

  1998–1999: full-time

  1999–2003: reduced to 1/4-time because of doctoral studies

- **since 2004:** assistant professor in Wrocław University of Science and Technology, Institute of Mathematics; currently Faculty of Pure and Applied Mathematics

  in 2010–2014 – senior specialist for testing of e-exercises (1/5-time) in a project *Mathematics-Reactivation* in EU Human Capital Operational Programme, providing secondary schools with a complete e-course in mathematics (during the project the employment as an assistant professors was reduced to 4/5-time)

4 Scientific achievement, according to art. 16 sec. 2 of the bill from 14 March 2003 on scientific degrees and title and degrees and title in the domain of arts (J. of Laws 2016 item 882, as amended in J. of Laws 2016, item 1311)

Title of the achievement: Entropy theory for generalized dynamical systems induced by Markov operators

[A1] B. Frej, *Maličky-Riečan's entropy as a version of operator entropy.* Fund. Math. 189 (2) (2006), 185–193.

[A2] B. Frej, P. Frej, *An integral formula for entropy of doubly stochastic operators.* Fund. Math. 213 (3) (2011), 271–289.

[A3] B. Frej, P. Frej, *The Shannon-McMillan theorem for doubly stochastic operators.* Nonlinearity 25 (12) (2012), 3453–3467.

[A4] B. Frej, D. Huczek, *Doubly stochastic operators with zero entropy.* Ann. Funct. Anal. 10 (1) 2019, 144–156.

My research interests emerge mainly from two sources: ergodic theory and topological dynamics, with special emphasis put on the entropy theory. The entropy is both the measure of complexity of a dynamical system and an isomorphism invariant for dynamical systems. The subject of the scientific achievement is a consistent theory of entropy for Markov operators, which generalizes the classical entropy of a dynamical systems – one of the most important notions of ergodic theory. In particular, building on basic definitions and properties which I studied in my PhD dissertation, I generalize to operator case several crucial theorems concerning entropy of a dynamical system, e.g., the Shannon–McMillan's theorem in [A3], Kushnirenko's theorem and Rokhlin's theorem (on genericity of zero entropy) in [A4]. A special role is played by the first of these theorems. The main inspiration for undertaking the research in this case was the claim (attributed to B. Weiss) that a correct definition of entropy should imply a Shannon–McMillan–Breiman type theorem. The Shannon–McMillan–Breiman's theorem, the Orstein's isomorphism theorem and the Variational Principle are generally agreed to be the key three theorems in entropy theory. None of them translates easily to operator case. The problem of transferring the Variational Principle to operator setup was treated in my doctoral dissertation (one of the inequalities was proved—the other one is still an open question). The Ornstein theory is not meaningful for Markov operators as it is valid only for invertible maps. Moreover, the notion of a Bernoulli system requires a sequence of independent identically distributed random variables, which leads to an integral operator having zero entropy. So the only one left is the Shannon–McMillan–Breiman's theorem. For classical dynamical systems it states that assuming ergodicity one can determine the entropy investigating just one typical point. For appropriately defined information function, based on a partition of a phase space, we obtain the almost sure and L^1 convergence of average information, calculated along the trajectory of a fixed point, to a constant number, namely the entropy of a system with respect to the partition. According to this fact, in real situations, when the theoretical dynamical system models some physical experiment, we hope to be able to estimate complexity of the system observing a large but finite set of data. In [A2] and [A3] we obtained full analog of the Shannon-McMillan theorem for doubly stochastic operators (i.e., convergence in L^1). Clearly, we first had to define a generalized information function. This generalization covers the classical definition, i.e., used for classical dynamical systems (that is, Koopman operators of measure preserving maps) and collections of characteristic functions of partitions it restores the classical Shannon information function. Hence, it is proved that the operator entropy satisfies the "Weiss' correctness criterion" and so it solves one of the main questions of operator entropy theory. The remaining problems, solved in my papers, were also posed by top specialists from the field of dynamical systems: the problem of equality with Maličky–Riečan's entropy by W. Słomczyński, the problem of genericity of zero entropy by A. Vershik and the problem of studying null operators and transferring the Kushnirenko's theorem to the operator case by V. Bergelson.

Below I present my results with more details. In the first section I will recall basic knowledge necessary to understand the content of papers constituting the scientific achievement. In next four sections I will discuss the articles, which were listed above. Then I will describe in short other works which I have completed and activities which I have undertaken after receiving the PhD degree.

4.1 Preliminaries

In the classical case by a dynamical system we mean a quadruple (X, Σ, μ, S), where (X, Σ, μ) is a probability space (usually a Lebesgue space) and $S : X \to X$ is a measure preserving map. As a natural generalization one can consider a Markov operator, known also as doubly stochastic operators. By a *doubly stochastic operator* we mean a linear operator $T\colon L^1(\mu) \to L^1(\mu)$, which satisfies the following conditions:

(i) Tf is positive for every positive $f \in L^1(\mu)$,

(ii) $T\mathbb{1} = \mathbb{1}$ (where $\mathbb{1}(x) = 1$ for all $x \in X$),

(iii) $\int Tf\, d\mu = \int f\, d\mu$ for every $f \in L^1(\mu)$.

Every measure preserving transformation S on (X, Σ, μ) induces a doubly stochastic operator by the formula $Tf = f \circ S$ (the Koopman operator of S). More generally, from a stationary transition probability $P(\cdot, \cdot)$ we obtain a doubly stochastic operator letting

$$Tf(x) = \int f(y)P(x, dy). \tag{1}$$

On a standard Borel space every doubly stochastic operator is given by this formula. It is known that an operator defined on $L^p(\mu)$, $p > 1$, and satisfying (i)–(iii) uniquely extends to a doubly stochastic operator on the whole $L^1(\mu)$. Such an operator is called *ergodic* if every function invariant under the action of T is constant. In the operator context one uses the following definition of isomorphism:

Definition 4.1 ([EFHN]) *Let T be a Markov operator on $L^1(\mu)$.*

1. *T is a* Markov embedding *if it is a lattice homomorphism, i.e., $|Tf| = T|f|$.*
2. *T is a* Markov isomorphism *if it is a surjective Markov embedding.*

In fact, Markov embeddings correspond to homomorphisms of measure algebras and Markov isomorphisms—to conjugacies.

The relation between Markov operators and transition probabilities (also called stochastic kernels) allows one to interpret Markov operator dynamics as the dynamics in which additional randomness is present—future states of a system are not uniquely determined. Instead, the next state of the system is chosen according to some probability distribution. Consequently, one can think of using operator dynamics to model evolution of systems which are subject to random perturbations. Simultaneously, it is a base for generalizing notions known in classical dynamical systems. Among monographs which present this point of view are [EFHN], [F], [LM]. In the latter, there is even a chapter on entropy of a Markov operator. Yet this is not the entropy which extends the Kolmogorov–Sinai's invariant, based on Shannon's entropy of a probability vector (or a partition of a probability space), to operator case, but the Boltzmann's entropy.

Various attempts to transfer the Kolmogorov–Sinai's entropy to the world of doubly stochastic operators were made by many authors. One should mention here the proposition by Ghys, Langevin and Walczak published in [GLW], and developed later by Kamiński and de Sam Lazaro in [KS], definitions by Alicki, Andries, Fannes, Tuyls [AAFT] and Makarov [M], based on von Neumann's matrix entropy, the lattice-based definition by Palm [Pa] and, finally, the entropy defined by Malički and Riečan in [MR]. As stated in [DF], the majority of these attempts implements the following general scheme of construction:

(1) one specifies a T-invariant collection $\mathbb{F}$ of selected finite families $\mathcal{F}$ of measurable functions;

(2) one specifies an operation $\sqcup$ of *joining* these families, so that $\mathcal{F} \sqcup \mathcal{G} \in \mathbb{F}$ whenever $\mathcal{F} \in \mathbb{F}$ and $\mathcal{G} \in \mathbb{F}$; we will also assume that $\sqcup$ is associative and commutative and that the cardinality of $\mathcal{F} \sqcup \mathcal{G}$ depends only on the cardinalities of the components;

(3) one defines the *static* entropy $\mathrm{H}_\mu(\mathcal{F})$ of a family $\mathcal{F} \in \mathbb{F}$ with respect to μ;

(4) one then defines

$$\mathrm{h}_\mu(T, \mathcal{F}) = \limsup_{n\to\infty} \frac{1}{n} \mathrm{H}_\mu(\mathcal{F}^n) ,$$

where $\mathcal{F}^n = \bigsqcup_{k=0}^{n-1} T^k\mathcal{F}$ and $T^k\mathcal{F} = \{T^k f : f \in \mathcal{F}\}$;

(5) and eventually one defines the entropy of T as

$$\mathrm{h}_\mu(T) = \sup_{\mathcal{F}\in\mathbb{F}} \mathrm{h}_\mu(T, \mathcal{F}) .$$

For example, the classical Kolmogorov-Sinai entropy for measurable maps uses $\mathbb{F}$ defined as families of characteristic functions corresponding to measurable partitions and joining is obtained by pointwise multiplication (or equivalently by the application of pointwise infima). In [GLW] one uses for $\mathbb{F}$ measurable *partitions of unity*, i.e. collections $\mathcal{F} = \{f_i : 1 \leqslant i \leqslant r\}$ with each f_i nonnegative and with $\sum_i f_i = 1$. For [AAFT] the condition is changed to $\sum_i f_i^2 = 1$. In both cases joinings are done via pointwise multiplication.

In [DF], which was the main part of my doctoral dissertation, it was shown that the above scheme, enriched with some natural requirements concerning the entropy function, guarantee that the final quantity $\mathrm{h}_\mu(T)$ does not depend on the formula used for $\mathrm{H}_\mu(\mathcal{F})$. The set of articles presented here as the main achievement is a natural continuation of the research started in [DF]. Below I cite the axioms of entropy as presented in [D3]—these statements are mildly weaker than those proposed in [DF] (meaning that they are satisfied by potentially bigger class of definitions of entropy). The conditional entropy is given by the formula

$$\mathrm{H}_\mu(\mathcal{F}|\mathcal{G}) = \mathrm{H}_\mu(\mathcal{F} \sqcup \mathcal{G}) - \mathrm{H}_\mu(\mathcal{G}) .$$

(A) MONOTONICITY AXIOM

For $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ belonging to $\mathbb{F}$ it holds that

$$\mathrm{H}_\mu(\mathcal{F}|\mathcal{H}) \leqslant \mathrm{H}_\mu(\mathcal{F} \sqcup \mathcal{G}|\mathcal{H}) \qquad \text{and} \qquad \mathrm{H}_\mu(\mathcal{F}|\mathcal{G} \sqcup \mathcal{H}) \leqslant \mathrm{H}_\mu(\mathcal{F}|\mathcal{G}) ,$$

where we assume that $\mathrm{H}_\mu(\mathcal{F}|\mathcal{H}) = \mathrm{H}_\mu(\mathcal{F})$ if $\mathcal{H}$ is the empty collection.

In particular, the axiom implies that

$$\mathrm{H}_\mu(\mathcal{F} \sqcup \mathcal{G}|\mathcal{H}) \leqslant \mathrm{H}_\mu(\mathcal{F}|\mathcal{H}) + \mathrm{H}_\mu(\mathcal{G}|\mathcal{H})$$

and

$$\mathrm{H}_\mu\left(\bigsqcup_{k=1}^{n} \mathcal{F}_k \middle| \bigsqcup_{k=1}^{n} \mathcal{G}_k\right) \leqslant \sum_{k=1}^{n} \mathrm{H}_\mu(\mathcal{F}_k|\mathcal{G}_k) .$$

Moreover, for any $n \geqslant 1$,

$$\mathrm{h}_\mu(T, T^n\mathcal{F}) = \mathrm{h}_\mu(T, \mathcal{F}) .$$

We define the L^1-distance of two collections $\mathcal{F} = \{f_i : 1 \leqslant i \leqslant r\}$ and $\mathcal{G} = \{g_i : 1 \leqslant i \leqslant r'\}$, $r' \leqslant r$ by the formula

$$\operatorname{dist}(\mathcal{F}, \mathcal{G}) = \min_{\pi} \left\{ \max_{1 \leqslant i \leqslant r} \int |f_i - g_{\pi(i)}| \, d\mu \right\},$$

where the minimum ranges over all permutations π of the set $\{1, 2, \ldots r\}$, and where $\mathcal{G}$ is considered an r-element family by setting $g_i \equiv 0$ for $r' < i \leqslant r$.

(B) CONTINUITY AXIOM

For any $r \in \mathbb{N}$ and $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that if cardinalities of $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ are bounded by r and $\operatorname{dist}(\mathcal{F}, \mathcal{G}) < \delta_\varepsilon$ then

$$|\mathrm{H}_\mu(\mathcal{F}|\mathcal{H}) - \mathrm{H}_\mu(\mathcal{G}|\mathcal{H})| < \varepsilon \qquad \text{and} \qquad |\mathrm{H}_\mu(\mathcal{H}|\mathcal{F}) - \mathrm{H}_\mu(\mathcal{H}|\mathcal{G})| < \varepsilon.$$

For a finite partition $\mathcal{A}$ of the space X we denote by $\mathbb{1}_\mathcal{A}$ the collection $\{\mathbb{1}_A : A \in \mathcal{A}\}$ of characteristic functions of elements of $\mathcal{A}$.

(C) PARTITIONS AXIOM

For any measurable partition $\mathcal{A}$ the collection $\mathbb{1}_\mathcal{A}$ belongs to $\mathbb{F}$ and the entropy H_μ of such collection is defined as the Shannon entropy of the corresponding partition:

$$\mathrm{H}_\mu(\mathbb{1}_\mathcal{A}) = -\sum_{A \in \mathcal{A}} \mu(A) \log \mu(A).$$

Moreover, for a collection of partitions $\mathcal{A}_1, \ldots, \mathcal{A}_n$ it holds that

$$\mathrm{H}_\mu\left(\bigsqcup_{k=1}^{n} \mathbb{1}_{\mathcal{A}_k}\right) = \mathrm{H}_\mu\left(\bigvee_{k=1}^{n} \mathcal{A}_k\right)$$

(D) DOMINATION AXIOM

For any $r \in \mathbb{N}$ and $\varepsilon > 0$ there exists $\gamma > 0$ such that for any collection $\mathcal{F} = \{f_i : 1 \leqslant i \leqslant r\}$ and any partition α of the unit interval $[0,1]$ into finitely many subintervals of lengths not exceeding γ it holds that

$$\mathrm{H}_\mu\left(\mathcal{F} | \mathbb{1}_{\bigvee_i f_i^{-1}(\alpha)} \sqcup \overline{\alpha}\right) < \varepsilon,$$

where $\overline{\alpha}$ is some collection depending only on α and satisfying

$$\lim_n \frac{1}{n} \mathrm{H}_\mu\left(\bigsqcup_{k=1}^{n} \overline{\alpha}\right) = 0$$

(usually, $\overline{\alpha}$ is the empty collection or it consists only of constant functions).

Theorem 4.2 ([DF]) *If T is a doubly stochastic operator on $L^1(\mu)$ then the axioms (A)-(D) (along with the construction steps (1)-(5)) completely determine the value of $\mathrm{h}_\mu(T)$.*

In order to prove the theorem, in the paper we developed the theory of asymptotic lattice stability of Markov operators. The following lemma was its starting point:

Lemma 4.3 ([DF]) *Let f and g be measurable bounded functions. For any $\delta > 0$ there exists $N \in \mathbb{N}$ such that for every $k \in \mathbb{N}$ and $l \geqslant N$ the following inequalities are satisfied*

$$\int |T^k(T^l f \vee T^l g) - (T^{k+l} f \vee T^{k+l} g)| \, d\mu < \delta$$

and

$$\int |T^k(T^l f \wedge T^l g) - (T^{k+l} f \wedge T^{k+l} g)| \, d\mu < \delta.$$

As a consequence we obtain convenient properties of operators concerning the action of their far iterates on certain characteristic functions or on lattice polynomials on a fixed set of variables.

Below I present the results of the papers [A1]–[A4].

4.2 Item [A1]

The definition of entropy proposed by Maličky and Riečan in [MR] does not implement the scheme described above and the first paper of the achievement is devoted to studying its properties. The study was motivated by the belief that equality between Maličky–Riečan's entropy and the one given axiomatically would strongly support the claim that the operator entropy is worth investigating.

The [MR] definition is based on the notion of a partition of unity, i.e., a finite collection of nonnegative measurable functions, which sum to one. For instance, if $\mathcal{A}$ is a finite partition of X into measurable subsets, then the set $\mathbb{1}_{\mathcal{A}} = \{\mathbb{1}_A : A \in \mathcal{A}\}$ of their characteristic functions is a partition of unity. Contrary to our definition, Maličky and Riečan do not use a joining operation, but they introduce an order relation on the set of all partitions of unity in the following way:

$$\Psi \succeq \Phi \text{ if } \Psi = \bigcup_{\varphi \in \Phi} \Psi_\varphi, \text{ where } \Psi_\varphi \text{ are pairwise disjoint and } \sum_{\psi \in \Psi_\varphi} \psi = \varphi$$

(to ensure that the relation is antisymmetric one disregards these partitions of unity which contain functions constantly equal to zero). We distinguish between the elements which are equal, i.e., a partition of unity is a finite sequence of functions (or a multiset) rather than a set, e.g., $\{\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\}$ consists of n identical elements.

The entropy of a partition of unity Φ is defined in [MR] by the formula

$$\mathrm{H}^{MR}(\Phi) = -\sum_{\varphi \in \Phi} \int \varphi \, d\mu \cdot \log \int \varphi \, d\mu.$$

For n partitions of unity $\Phi_1, \ldots, \Phi_n$ they define

$$\mathrm{H}^{MR}(\Phi_1, \ldots, \Phi_n) = \inf\{\mathrm{H}^{MR}(\Gamma) : \Gamma \succeq \Phi_1, \Gamma \succeq \Phi_2, \ldots, \Gamma \succeq \Phi_n\}.$$

Denoting $\mathrm{H}^{MR}(\Phi, T\Phi, \ldots, T^{n-1}\Phi)$ by $\mathrm{H}^{MR}(\Phi, n)$ one obtains a subadditive sequence, so it is possible to put

$$\mathrm{h}^{MR}(T, \Phi) = \lim_{n \to \infty} \frac{1}{n} \mathrm{H}^{MR}(\Phi, n).$$

Finally, for an arbitrary set $\mathcal{R}$ of partitions of unity one defines

$$\mathrm{h}^{\mathrm{MR}}_{\mathcal{R}}(T) = \sup_{\Phi \in \mathcal{R}} \mathrm{h}^{MR}(T, \Phi).$$

The main result in the first item of the achievement is the following theorem:

Theorem 4.4 ([A1, Theorem 3.7])

$$\mathrm{h}^{\mathrm{MR}}_{\mathcal{R}}(T) = \mathrm{h}_{\mu}(T)\,,$$

where $\mathcal{R}$ is the set of all partitions of unity on X and $\mathrm{h}_\mu(T)$ is the operator entropy compliant with the axioms given in the preceding section.

To prove the theorem it suffices to compare $\mathrm{h}^{\mathrm{MR}}_{\mathcal{R}}(T)$ with any explicit definition of operator entropy. In the current paper it is compared with the one defined in [DF] in the following way.

We use the same notations as in the general scheme of the construction of entropy. The [DF]-definition accepts the set of all measurable functions with range in $[0,1]$ as $\mathbb{F}$ and uses the set-theoretic union (or rather the operation of concatenating if we interpret collections as finite sequences) as the joining operation $\sqcup$. For $f\colon X \to [0,1]$ we define

$$A_f = \{(x,t) \in X \times [0,1] : t \leqslant f(x)\},$$

and by $\mathcal{A}_f$ we denote a partition of the product $X \times [0,1]$ consisting of A_f and its complement. For a collection $\mathcal{F}$ of measurable functions let $\mathcal{A}_{\mathcal{F}} = \bigvee_{f\in\mathcal{F}} \mathcal{A}_f$. It is easy to verify that $\mathcal{A}_{\mathcal{F}\sqcup\mathcal{G}} = \mathcal{A}_{\mathcal{F}} \vee \mathcal{A}_{\mathcal{G}}$. Let λ be the Lebesgue measure on the unit interval. We define

$$\mathrm{H}^{DF}(\mathcal{F}) = \mathrm{H}_{\mu\times\lambda}(\mathcal{A}_{\mathcal{F}}) = -\sum_{A\in\mathcal{A}_{\mathcal{F}}} (\mu\times\lambda)(A)\cdot\log(\mu\times\lambda)(A), \tag{2}$$

$$\mathrm{h}^{DF}(T,\mathcal{F}) = \lim_{n\to\infty}\frac{1}{n}\mathrm{H}^{DF}(\mathcal{F}^n)\,,$$

$$\mathrm{h}^{DF}(T) = \sup_{\mathcal{F}} \mathrm{h}^{DF}(T,\mathcal{F})\,,$$

where the infimum ranges over the set of all finite collections of measurable functions from X into $[0,1]$. Though the sequence $\mathrm{H}_\mu(\mathcal{F}^n)$ is not subadditive, the limit in the formula defining $\mathrm{h}_\mu(T,\mathcal{F})$ exists, but the proof of the fact is quite laborious.

The definitions that we consider exploit different tools— the Maličky–Riečan's entropy uses partitions of unity, while our explicit formula for entropy makes use of partitions of the product $X\times[0,1]$, generated by collections of functions. The first step in the proof is to describe some operations which interchange between these objects. These operations depend on enumeration of elements of a collection of functions or a partition of unity. For a partition of unity Φ the collection $\Sigma(\Phi)$ consists of partial sums of Φ, i.e., if $\Phi = \{\varphi_1,\ldots,\varphi_r\}$ then $\Sigma(\Phi) = \{\sum_{i=1}^{j}\varphi_i : j = 1,\ldots,r\}$. Conversely, given a collection $f_1 \leqslant f_2 \leqslant \ldots \leqslant f_r = 1$, called *an increasing collection*, we obtain a partition of unity by taking differences $f_{i+1} - f_i$ of subsequent functions. Moreover, each collection $\mathcal{F}$ may be transformed into an increasing collection $\Theta(\mathcal{F})$ by the multiple use of lattice operation, so that the following equality holds:

$$\mathcal{A}_{\mathcal{F}} = \mathcal{A}_{\Theta(\mathcal{F})}.$$

The partition of unity obtained from a collection $\mathcal{F}$ by forming an increasing collection $\Theta(\mathcal{F})$ and then taking differences of subsequent elements will be denoted by $\mathcal{PU}(\mathcal{F})$. Unfortunately, in general $\Theta(T\mathcal{F}) \neq T(\Theta(\mathcal{F}))$ and, consequently, $\mathcal{PU}(T\mathcal{F}) \neq T(\mathcal{PU}(\mathcal{F}))$. On the other hand, it holds that

$$T\Sigma(\Phi) = \Sigma(T\Phi)\,, \qquad \Sigma(\mathcal{PU}(\mathcal{F})) = \Theta(\mathcal{F}), \qquad \mathcal{PU}(\Sigma(\Phi)) = \Phi,$$

$$\mathrm{H}^{MR}(\mathcal{PU}(\mathcal{F})) = \mathrm{H}^{DF}(\Theta(\mathcal{F})) = \mathrm{H}^{DF}(\mathcal{F})$$

Furthermore, if $\mathcal{F} = \{f_0, ..., f_r\}, \mathcal{G} = \{g_0, ..., g_s\}$, where $f_0 = 0 \leqslant f_1 \leqslant ... \leqslant f_r = 1$ and $g_0 = 0 \leqslant g_1 \leqslant ... \leqslant g_s = 1$, then $\mathcal{A}_\mathcal{G} \succcurlyeq \mathcal{A}_\mathcal{F}$ implies $\mathcal{PU}(\mathcal{G}) \succeq \mathcal{PU}(\mathcal{F})$.

For any partition of unity Φ we denote by Φ_Σ^n the collection $\bigcup_{k=0}^{n-1} T^k(\Sigma(\Phi))$. The above observations indicate that for an arbitrary partition of unity it holds that $\mathcal{A}_{\Theta(\Phi_\Sigma^n)} \succcurlyeq \mathcal{A}_{T^k(\Sigma(\Phi))}$, so $\mathcal{PU}(\Phi_\Sigma^n) \succeq T^k\Phi$ for every $n \in \mathbb{N}$ and $k < n$. This, in turn, yields the following inequality:

$$\mathrm{H}^{MR}(\Phi, n) \leqslant \mathrm{H}^{MR}(\mathcal{PU}(\Phi_\Sigma^n)) = \mathrm{H}^{DF}(\Phi_\Sigma^n),$$

implying that $\mathrm{h}^{MR}(T) \leqslant \mathrm{h}^{DF}(T)$.

Proving the converse inequality is a much more difficult task. The goal is to estimate from above the quantity $\mathrm{h}^{DF}(T, \mathcal{F})$ for an arbitrary finite collection $\mathcal{F}$. We know that for any natural number l the equality $\mathrm{h}^{DF}(T, \mathcal{F}) = \mathrm{h}^{DF}\left(T, T^l\mathcal{F}\right)$ holds. Using the theory of asymptotic lattice stability of doubly stochastic operators we can replace a given collection $\mathcal{F}$ by its far image $T^l\mathcal{F}$, so that for a previously fixed $\varepsilon > 0$ and $\delta > 0$ the following inequalities are simultaneously satisfied:

$$\mathrm{H}^{DF}\left(T^k\mathcal{F} \,\middle|\, \mathbb{1}_{\bigvee_{f \in T^k\mathcal{F}} f^{-1}(\alpha)} \cup \overline{\alpha}\right) < \epsilon,$$

where $\overline{\alpha}$ is a collection of constants (as aforementioned in the domination axiom), and

$$\operatorname{dist}\left(\mathbb{1}_{\bigvee_{f \in T^k\mathcal{F}} f^{-1}(\alpha)},\ T^k(\mathbb{1}_{\bigvee_{f \in \mathcal{F}} f^{-1}(\alpha)})\right) < \delta.$$

At the same time, we can adjust the number $\delta = \delta(\varepsilon)$ so that the condition $\operatorname{dist}(\Phi_i, \Psi_i) < \delta$ for $i = 1, ..., n$ (with cardinalities of partitions of unity Φ_i and Ψ_i being fixed) ensures that

$$|\mathrm{H}^{MR}(\Phi_1, ..., \Phi_n) - \mathrm{H}^{MR}(\Psi_1, ..., \Psi_n)| < (n+1)\epsilon$$

(see lemmas 3.3, 3.4 and 3.5 in [A1]). Thus, we obtain the following bound on the level of static entropies:

$$\mathrm{H}^{DF}(\mathcal{F}^n) \leqslant \mathrm{H}^{MR}\left(\mathbb{1}_{\bigvee_{f \in \mathcal{F}} f^{-1}(\alpha)}, n\right) + \mathrm{H}^{DF}(\overline{\alpha}) + (2n+1)\epsilon.$$

Finally, we get $\mathrm{h}^{DF}(T, \mathcal{F}) \leqslant \mathrm{h}^{MR}\left(T, \mathbb{1}_{\bigvee_{f \in \mathcal{F}} f^{-1}(\alpha)}\right) + 2\epsilon$, which, in fact, ends the proof.

The current paper contains also a simpler proof of equality between $\mathrm{h}_{\mathfrak{R}}^{\mathrm{MR}}(T)$ and the Kolmogorov–Sinai entropy of a measure preserving map S in case if T is a Koopman operator of S (in [MR] there was no proof and even no statement of this fact). Since the presented result is much more general, I will refrain from commenting on the proof of the weaker one.

I presented results of the paper on one of the conferences of the Czech–Slovak Workshop on Discrete Dynamical Systems series, while one of the authors of [MR] (Petr Maličký) was present in the audience. The second author asked me to send him a copy of my paper.

4.3 Item [A2]

In ergodic theory the classical notion of entropy has the following interpretation. The space X is understood as as a phase space of some physical system, while the r-element partition models the experiment performed on that system, having r possible outcomes. The apparatus used to measure results of the experiment is assumed to be faultless, i.e., in each state of the system (point of a phase space) it yields an outcome unambiguously assigned to this state. Doubly stochastic operators may be used to deal with situations in which the measurement of an experiment is disturbed or unclear; in each state the machinery gives outcomes according to some probability distribution. In this way one obtains a vector-valued function on X assigning to each $x \in X$ a probability vector or, in other words, values of a r-element partition of unity on X. After taking partial sums we are led to an increasing collection. (However, from the mathematical point of view it seems more elegant and more convenient to formulate the definition of entropy for arbitrary collections of functions with range in $[0,1]$, not only for increasing ones, and so we do.) The action of an operator models a change in settings of the measuring tool or a flow of time. The entropy of $\mathcal{F}$ is understood as the information content in the experiment. One would thus expect that the entropy of the family $\mathcal{F}$ satisfies the following conditions:

(i) if $\mathcal{F}$ consists solely of constant functions then its entropy is equal to zero, because an experiment modeled by such family yields the same results regardless of the state of the system, providing no information about the actual state;

(ii) for every family $\mathcal{F}$ the conditional entropy $H_\mu(\mathcal{F}|\mathcal{F}) = 0$, because copying the results of a once performed experiment does not give any new information.

The aim of [A2] was to come up with the formula for static entropy $\mathrm{H}_\mu(\mathcal{F})$, which would fulfil the above postulates. The axioms do not guarantee that these properties are satisfied. In fact, none of the already mentioned versions of entropy has both these properties at the same time: definitions based on partitions of unity, where the joining operation is implemented by pointwise multiplication, essentially increase the size of the joining $\mathcal{F} \sqcup \mathcal{F}$, which results in increasing $\mathrm{H}_\mu(\mathcal{F} \sqcup \mathcal{F})$, and, consequently, they increase the conditional entropy; definitions from [DF] and [MR] give positive values of entropy to collections consisting of constant functions.

We assume that the set $\mathbb{F}$ from the general scheme is the set of all finite sequences of measurable functions $X \longrightarrow [0,1]$. The joining $\sqcup$ is done by concatenating these sequences. We also assume that $\mathbb{F}$ contains the empty sequence $\mathcal{O}$ and that $\mathcal{A}_{\mathcal{O}} = \{X\}$. The partitions $\mathcal{A}_f$ and $\mathcal{A}_{\mathcal{F}}$ of the product $X \times [0,1]$ are defined in the same way as in the previous section. By A^t we denote the t-section of the set $A \subset X \times [0,1]$, i.e. $A^t = \{x \in X : (x,t) \in A\}$. By $\mathcal{A}^t_{\mathcal{F}}$ we mean the partition of X consisting of appropriate t-sections A^t, where $A \in \mathcal{A}_{\mathcal{F}}$. Clearly, it holds that $\mathcal{A}_{\mathcal{F}\sqcup\mathcal{G}} = \mathcal{A}_{\mathcal{F}} \vee \mathcal{A}_{\mathcal{G}}$ and $(\mathcal{A}_{\mathcal{F}\sqcup\mathcal{G}})^t = (\mathcal{A}_{\mathcal{F}} \vee \mathcal{A}_{\mathcal{G}})^t = \mathcal{A}^t_{\mathcal{F}} \vee \mathcal{A}^t_{\mathcal{G}}$.

Definition 4.5 *The entropy of $\mathcal{F}$ is defined by*

$$\mathrm{H}_\mu(\mathcal{F}) = \int_0^1 H_\mu(\mathcal{A}^t_{\mathcal{F}})\, d\lambda(t),$$

where $\mathrm{H}_\mu(\alpha)$ is the Shannon entropy of the partition α of X.

The idea which stands behind this definition is to treat the product $X \times [0,1]$ as a set of copies of the space X and to study evolution of the partitions induced on these copies, which is governed by the dynamics generated by a given operator T.

It is obvious that the definition satisfies the above postulates. In [A2], in a series of lemmas it is verified that the formula satisfies axioms of entropy. Much of the paper is devoted to verification of the existence of the limit in:

$$\mathrm{h}_\mu(T, \mathcal{F}) = \lim_{n\to\infty} \mathrm{H}_\mu(\mathcal{F}^n)\,.$$

Similarly to the explicit definition from [DF] and contrary to the classical case, existence of the limit is not automatic, because the sequence $\mathrm{H}_\mu(\mathcal{F}^n)$ is not subadditive; indeed, the invariance of H_μ under the action of T is missing, i.e., in general, the entropy $\mathrm{H}_\mu(\mathcal{F})$ is not equal to $\mathrm{H}_\mu(T\mathcal{F})$. In the proof of existence of the limit we use the Iwanik's theorem on integral representation of stochastic operators:

Theorem 4.6 ([I]) *If T is an operator on the set of bounded measurable functions of a standard Borel space and T is induced by a transition probability (see the formula (1)) then*

$$Tf(x) = \int_{[0,1]} f(\varphi_\omega(x))d\lambda(\omega),$$

where λ denotes the Lebesgue measure on the unit interval and $(\omega, x) \mapsto \varphi_\omega(x)$ is a jointly measurable map from $[0,1] \times X$ into X.

The map, whose existence is granted by the above theorem, allows one to introduce a measurable transformation $\phi : [0,1] \times X \to [0,1] \times X$ by $\phi(\omega, x) = (\omega, \varphi_\omega(x))$. Unfortunately, this map does not preserve the product measure $\lambda \times \mu$. Since the iterates T^k are also induced by transition probabilities, in the same way one can obtain maps ϕ_k corresponding to T^k. Let us denote by Φ_k the operator pointwise generated by the map ϕ_k (that is, $\Phi_k f = f \circ \phi_k$). For a function $f : X \to [0,1]$ let $\bar{f} : X \times [0,1] \to [0,1]$ be given by $\bar{f}(\omega, x) = f(x)$ and let $\overline{\mathcal{F}}$ denote the collection $\{\bar{f} : f \in \mathcal{F}\}$. The following lemma is crucial in the proof of existence of the limit.

Lemma 4.7 ([A2, Lemma 3.2]) *For an increasing collection of functions $\mathcal{G}$ it holds that*

$$\mathrm{H}_{\mu\times\lambda}\left(\Phi_k\bar{\mathcal{G}}\right) = \mathrm{H}_\mu(\mathcal{G}).$$

Every collection of functions can be made into an increasing one by means of the operation Θ, which was mentioned in the previous section. Moreover, for any $\delta > 0$ there exists $N \in \mathbb{N}$ such that for any $k, l \in \mathbb{N}$, where $l \geqslant N$, it holds that $\left\|\overline{T^{k+l}f} - \Phi_k\overline{T^l f}\right\|_1$, so we obtain the asymptotic invariance of our static entropy:

Lemma 4.8 ([A2, Lemma 3.3]) *For every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for any k, $m \in \mathbb{N}$ it holds that*

$$|\mathrm{H}_\mu\left(T^{k+N}\mathcal{F}^m\right) - \mathrm{H}_\mu\left(T^N\mathcal{F}^m\right)| < m\varepsilon.$$

Finally, the asymptotic subadditivity follows:

Lemma 4.9 ([A2, Lemma 3.4]) *For every $\varepsilon > 0$ there exist $N \in \mathbb{N}$ and $c > 0$ such that for any $k \in \mathbb{N}$ and $m \geqslant N$ it holds that*

$$\mathrm{H}_\mu\left(\mathcal{F}^{k+m}\right) \leqslant \mathrm{H}_\mu\left(\mathcal{F}^k\right) + \mathrm{H}_\mu(\mathcal{F}^m) + c + m\varepsilon.$$

The existence of the limit from the fourth step of the general scheme is now proved similarly to the classical case, when full subadditivity of entropy is present ([A2, Theorem 3.5]).

New definition of entropy allowed us to prove the product rule—a classical theorem, which was hitherto not known to be true in the operator case. If T is a doubly stochastic operator on $L^1(X, \mu)$ and S is a doubly stochastic operator on $L^1(Y, \nu)$ then the product $T \times S$ of the operators is defined first for the functions of the form $f(x) \cdot g(y)$, where $f \in L^1(X, \mu)$, $g \in L^1(Y, \nu)$, by

$$(T \times S)(fg)(x, y) = Tf(x) \cdot Sg(y).$$

Then one extends it linearly to the set of all linear combinations of such functions. This set is dense in $L^1(\mu \times \nu)$, so there is a unique extension to a continuous (and doubly stochastic) operator on the whole $L^1(\mu \times \nu)$. Alternatively, if the operators T and S are induced according to the formula (1) by transition probabilities P_T and P_S, respectively, then the product $T \times S$ is given by the same formula by the corresponding product measure.

Theorem 4.10 ([A2, Theorem 4.5])

$$\mathrm{h}_{\mu\times\nu}(T \times S) = \mathrm{h}_\mu(T) + \mathrm{h}_\nu(S)$$

The proof contains a number of unpleasant "technicalities", connected with the extensive use of the domination axiom (axiom (D)) and lemmas from asymptotic lattice stability of doubly stochastic operators. But the central idea relies on a simple observation that for partitions α of the space X and β of the space Y the following equality holds:

$$\mathrm{H}_\mu(\mathbb{1}_{\alpha\times Y} \sqcup \mathbb{1}_{X\times\beta}) = \mathrm{H}_\mu(\mathbb{1}_{\alpha\times\beta}) = \mathrm{H}_\mu(\mathbb{1}_\alpha) + \mathrm{H}_\mu(\mathbb{1}_\beta),$$

where partitions $\alpha \times Y$, $X \times \beta$ of the product $X \times Y$ are defined as

$$\alpha \times Y = \{A \times Y : A \in \alpha\}, \qquad X \times \beta = \{X \times B : B \in \beta\}.$$

Without going deep into details, the sketch of the proof is as follows. To show that the sum of entropies of single operators is the upper bound for the entropy of their product, one fixes a collection $\mathcal{E}$ of measurable functions $X \times Y \to [0,1]$ and approximates each of its elements by a function of the form $\sum f_i g_i$, where f_i and g_i are functions on X and Y, respectively. Then one chooses a number $L \in \mathbb{N}$ and sequences of partitions α_n of X and β_n of Y $(n \in \mathbb{N})$ so that the following conditions hold:

1. all functions $T^L f_i$ are well approximated by simple functions, which combine characteristic functions of elements of the partition α_L, and all functions $S^L g_i$ are well approximated by simple functions, which combine characteristic functions of elements of β_L,
2. for $n \in \mathbb{N}$ distances $\mathrm{dist}(T^n \mathbb{1}_{\alpha_L}, \mathbb{1}_{\alpha_{L+n}})$ and $\mathrm{dist}(S^n \mathbb{1}_{\beta_L}, \mathbb{1}_{\beta_{L+n}})$ are small.

Then we derive that the functions $(T \times S)^{L+n}(\sum_i f_i g_i)$ are well approximated by simple functions combining characteristic functions of elements of the partition $\alpha_{L+n} \times \beta_{L+n}$. It follows that

$$\begin{aligned}
\mathrm{H}_{\mu\times\nu}\left(\mathcal{E}^{L+N}\right) &\leqslant \mathrm{H}_{\mu\times\nu}\left(\mathbb{1}_{\bigvee_{n<N}\alpha_{L+n}\times\bigvee_{n<N}\beta_{L+n}}\right) + \varepsilon O(N) \\
&= \mathrm{H}_\mu\left(\mathbb{1}_{\bigvee_{n<N}\alpha_{L+n}}\right) + \mathrm{H}_\nu\left(\mathbb{1}_{\bigvee_{n<N}\beta_{L+n}}\right) + \varepsilon O(N) \\
&\leqslant \mathrm{H}_\mu\left((\mathbb{1}_{\alpha_L})^N\right) + \mathrm{H}_\nu\left((\mathbb{1}_{\beta_L})^N\right) + \varepsilon O(N),
\end{aligned}$$

which implies

$$\mathrm{h}_{\mu\times\nu}(T \times S, \mathcal{E}) \leqslant \mathrm{h}_\mu(T, \mathbb{1}_{\alpha_L}) + \mathrm{h}_\nu(S, \mathbb{1}_{\beta_L}) + \varepsilon \leqslant \mathrm{h}_\mu(T) + \mathrm{h}_\nu(S) + \varepsilon.$$

Conversely, starting from collections $\mathcal{F}$ of functions on X and $\mathcal{G}$ of functions on Y, we choose partitions of the underlying spaces with similar properties as in the first part, and bound the entropies of collections $\mathcal{F}^n$ and $\mathcal{G}^n$ (for large n) by the entropies of characteristic functions of these partitions. This allows us to pass to the entropy of a collection of characteristic functions of the adequate partition of the product.

The result was later reproved by T. Austin [Au] in a totally different way. Austin proved and used the equality between the entropy of an operator and the Kolmogorov-Sinai entropy of its *backward tail boundary*—a dynamical system induced by the operator.

The current paper contains also a result concerning continuity of our new formula, treated as a function of measure:

Theorem 4.11 ([A2, Theorem 5.1]) *If X is a compact metric space and $\mathcal{F}$ is a collection of continuous functions then the map assigning to each probability measure μ the entropy $\mathrm{H}_\mu(\mathcal{F})$ is continuous if the space of all probability measures on X is endowed with the weak* topology.*

4.4 Item [A3]

The paper continues the investigations from [A2] and [Fr] in this sense that it uses the same explict formula for static entropy. But the present study is much deeper and more detailed, because we introduce an additional definition of *entropy on level t*. To make the notation clearer let us denote $\mathcal{A}_i = \mathcal{A}_{T^i\mathcal{F}}$.

Definition 4.12 ([A3, Definition 1.1])

1. The entropy of a collection $\mathcal{F}$ on level $t \in [0, 1]$ *is defined by*

$$\mathrm{H}_\mu(\mathcal{F}, t) = \mathrm{H}_\mu\left(\mathcal{A}^t_{\mathcal{F}}\right).$$

2. The entropy of an operator T with respect to a collection $\mathcal{F}$ on level $t \in [0, 1]$ *is defined by*

$$\mathrm{h}_\mu(T, \mathcal{F}, t) = \lim_{n\to\infty} \frac{1}{n} \mathrm{H}_\mu\left(\bigvee_{i=0}^{n-1} \mathcal{A}^t_i\right).$$

The existence of the above limit was proved in [Fr].

In classical dynamical systems, given a map S, which preserves measure μ, and a partition $\alpha = \{A_1, ..., A_r\}$ of the space X, to each point $x \in X$ we can associate its α-name, that is a sequence $(i_n)_{n=0}^{\infty}$ such that $S^n x \in A_{i_n}$. The knowledge about first n elements of the sequence is equivalent to knowing which element of the partition $\bigvee_{i=0}^{n-1} S^{-i}\alpha$ contains x. According to Shannon-McMillan-Breiman's theorem, this knowledge allows one to localize almost each point of the space up to a set of measure close to $e^{-n\mathrm{h}_\mu(S)}$. In case of operator dynamics induced by a transition probability, it is not possible to study the evolution of partitions. Nevertheless, a collection of images of characteristic functions $\{T^n\mathbb{1}_A : A \in \alpha\}$ is a partition of unity for every n. The value of the function $T^n\mathbb{1}_A(x)$ could be interpreted as the probability that after time n the trajectory starting at x belongs to the set A. By solving the inequality $T^n\mathbb{1}_A(x) \geqslant t$ one finds out which elements of α are visited by nth element of the orbit of x with probability greater than or equal to t. Taking in our predictions t close to 1, we decide to accept only the most likely future states. For t close to 0 we rule out only the states which are the least probable. In the interpretation given in the description of the former paper, if T is assumed to model measurement of an experiment with some uncertainty, the level t is the sensitivity parameter. It allows to control the errors, known in hypothesis testing theory as type I and type II errors - for

t close to 0 we minimize the risk of disregarding a state which is the actual state of the system, while for t close to 1 we decrease the risk of accepting an improper state.

In ergodic theory the information function with respect to a partition α of X is defined as

$$I_\alpha(x) = -\sum_{A\in\alpha} \log \mu(A) \cdot \mathbb{1}_A(x).$$

In operator case, the variety of definitions of static entropy $\mathrm{H}_\mu(\mathcal{F})$ makes it necesary to adjust the formula for information function to the choice of entropy formula. For entropy defined in the previous section we propose the following

$$\mathbb{I}_{\mathcal{F}}(x) = \int_0^1 I_{\mathcal{A}^t_{\mathcal{F}}}(x)\lambda(dt).$$

It is the average information gained for partitions $\mathcal{A}^t_{\mathcal{F}}$, which harmonizes with the interpretation of the product $X \times [0,1]$ as a set of copies of the phase space X. Let us remark that for collections $\mathcal{F} = \mathbb{1}_\alpha$, where α is a partition of X, it holds that $I_{\mathcal{A}^t_{\mathcal{F}}}(x) = I_\alpha(x)$ for every x and $t \neq 0$, which implies $\mathbb{I}_{\mathcal{F}} = I_\alpha$. As it was already mentioned, our definition is thus a generalization of the classical one. Moreover, for an operator T, which is pointwise generated by a map S, we have $\mathbb{I}_{TF} = I_{\mathcal{A}^t_{T\mathcal{F}}}(x) = I_\alpha(Sx)$, so results obtained for the operator case cover the analogous classical theorems of ergodic theory.

The Fubini theorem easily implies the basic requirement concerning the information function:

$$\mathrm{H}_\mu(\mathcal{F}) = \int_X \mathbb{I}_{\mathcal{F}}\, d\mu.$$

Furthermore, for a collection $\mathcal{F}$ consisting of constant functions, the information function is constantly equal to zero. On the other hand, the functions $\mathbb{I}_{T\mathcal{F}}$ and $T\mathbb{I}_{\mathcal{F}}$ need not be equal—for instance, for $Tf = \int f d\mu$ the first one is equal to zero while the second one is a positive constant.

Theorem 4.13 ([A3, Main theorem A]) *Let $(X, \mathcal{B}, \mu)$ be a probability space and let $\mathcal{F}$ be a finite collection of measurable functions $X \longrightarrow [0,1]$. Let T be an ergodic doubly stochastic operator on $L^1(\mu)$.*

Then, for almost every $t \in [0,1]$ the sequence $\frac{1}{n} I_{\bigvee_{i=0}^{n-1} \mathcal{A}^t_i}$ converges to entropy $\mathrm{h}_\mu(T, \mathcal{F}, t)$ in L^1 norm.

Theorem 4.14 ([A3, Main theorem B]) *With the assumptions of the preceding theorem the sequence $\frac{1}{n}\mathbb{I}_{\mathcal{F}^n}$ converges to $\mathrm{h}_\mu(T, \mathcal{F})$ in L^1 norm.*

The main part of reasoning is performed for operators induced by transition probabilities and then the theorem is extended to the general case. So let us first assume that T is given by formula (1). An important role in proving the above theorems is played by a trajectory space of the operator T, as defined and studied in [Fr]. It is the space $X^{\mathbb{N}}$ endowed with the product σ-algebra and the probability measure ν given by the equality

$$\nu\left(A_0 \times A_1 \times \cdots \times A_n \times X^{\mathbb{N}}\right) = \int_{A_0}\int_{A_1} \cdots \int_{A_n} P(x_{n-1}, dx_n) \dots P(x_0, dx_1)\mu(dx_0)$$

for all measurable sets $A_0, ..., A_n \subset X$. By the Ionescu-Tulcea theorem [IT], the measure is well-defined on the whole product σ-algebra. The space $X^{\mathbb{N}}$ is the domain of the shift map σ given by

$$(\sigma x)_n = x_{n+1} \qquad \text{for } x = (x_n)_{n\in\mathbb{N}}.$$

Lemma 4.15 ([Fr]) *For almost every $t \in [0,1]$ it holds that*

$$h_\mu(T, \mathcal{F}, t) = \lim_{l \to \infty} h_\nu\Big(\sigma, \mathcal{A}_l^t \times X^{\mathbb{N}}\Big).$$

In the present paper we show the following relation:

Theorem 4.16 ([A3, Theorem 3.3]) *If T is an ergodic doubly stochastic operator induced on $L^1(\mu)$ by a transition probability P, then the system $(X^{\mathbb{N}}, \nu, \sigma)$ is also ergodic.*

The proof relies on verifying one of the equivalent conditions of ergodicity of a map, namely that $\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \nu(A \cap \sigma^{-k}(B)) = \nu(A)\nu(B)$ for any measurable A and B, with the additional use of the Chacon–Ornstein's ergodic theorem [CO].

The first step in the proof of theorem 4.13 is the following fact concerning the size of elements of the partition $\bigvee_{i=0}^{n-1} \mathcal{A}_{l_0+i}^t$.

Theorem 4.17 ([A3, Theorem 4.1]) *For any collection $\mathcal{F}$ and $\epsilon > 0$ there exist: $l_0 \in \mathbb{N}$ and a set $\tau \subset [0,1]$ of Lebesgue measure $\lambda(\tau) < \epsilon$, such that for every $t \in [0,1] \setminus \tau$ there is a number $n_t \in \mathbb{N}$ such that the inequality*

$$\mu(A) \leqslant 2^{-n(h_\mu(T,\mathcal{F},t)-\varepsilon)}$$

holds for all $n \geqslant n_t$ and all $A \in \bigvee_{i=0}^{n-1} \mathcal{A}_{l_0+i}^t$ except some number of sets with aggregate measure not greater than ε.

The minimal n_t given by the above theorem may not be arbitrarily large on a big set of t, which allows to make the choice of n_t independent of t and obtain:

Theorem 4.18 ([A3, Theorem 4.2]) *For every collection $\mathcal{F}$ and every $\epsilon > 0$ there exist: $l_0 \in \mathbb{N}$, $N \in \mathbb{N}$ and a set $\tau \subset [0,1]$ with $\lambda(\tau) < \epsilon$, such that for every $t \in [0,1] \setminus \tau$ and $n \geqslant N$ the inequality*

$$\mu(A) \leqslant 2^{-n(h_\mu(T,\mathcal{F},t)-\varepsilon)}$$

holds for each $A \in \bigvee_{i=0}^{n-1} \mathcal{A}_{l_0+i}^t$ except some sets of aggregate measure not greater than ε.

The inequality in the above theorem may be rewritten in the form

$$\frac{1}{n} I_{\bigvee_{i=0}^{n-1} \mathcal{A}_{l_0+i}^t}(x) \geqslant h_\mu(T, \mathcal{F}, t) - \epsilon,$$

where x belongs to the complement of some set of measure μ smaller than or equal to ε.

For almost all $t \in [0,1]$ it also holds that

$$\lim_{n\to\infty} \int \frac{1}{n} I_{\bigvee_{i=0}^{n-1} \mathcal{A}_{l_0+i}^t} \, d\mu = \lim_{n\to\infty} \frac{1}{n} H_\mu\left(\bigvee_{i=0}^{n-1} \mathcal{A}_i^t\right) = h_\mu(T, \mathcal{F}, t)$$

Now, the theorem 4.13 for operators induced by transition probabilities follows, after some derivation, by analogy to the following simple fact:

if for a constant h it holds that $f \geqslant h - \varepsilon$ and $\int f d\mu = h$ then $\int |f - h| \leqslant 2\varepsilon$.

Theorem 4.14 is a corollary from Theorem 4.13 and the following equality proved in [Fr]:

$$\mathrm{h}_\mu(T,\mathcal{F}) = \int_0^1 \mathrm{h}_\mu(T,\mathcal{F},t)\,\lambda(dt)$$

by integrating along the variable t (with respect to the Lebesgue measure on the unit interval).

It remains to clarify the idea of the proof of Theorem 4.17. Because of Lemma 4.15 we plan to transfer the argument to the space of trajectories, replacing the quantity $\mathrm{h}_\mu(T,\mathcal{F},t)$ by $h_\nu(\sigma,\mathcal{A}^t_l\times X^{\mathbb{N}})$ for a large l. By another result of [Fr], for any collection $\mathcal{F}$ and any $\varepsilon>0$ there exist $l_0\in\mathbb{N}$ and a measurable set $\tau\subset[0,1]$ of Lebesgue measure not greater than ε, such that

$$\mathrm{dist}(T^n\mathbb{1}_{\mathcal{A}^t_l},\mathbb{1}_{\mathcal{A}^t_{l+n}})<\varepsilon$$

if only $l\geqslant l_0$, $n\in\mathbb{N}$ and $t\in[0,1]\setminus\tau$. In the following, we assume that the parameter t is always taken from outside of τ. For a fixed i it allows to set a one-to-one correspondence between sets $A\in\mathcal{A}^t_{l+i}$ and $B=\pi(A)\in\mathcal{A}^t_l$, for which the distance $\|\mathbb{1}_A-T^i\mathbb{1}_B\|_1$ is small. Passing to the space of trajectories we obtain that for pairs (A,B) of corresponding sets the measure $\nu\big((A\times X^{\mathbb{N}})\triangle(X^i\times B\times X^{\mathbb{N}})\big)$ is also small. Clearly, $\nu(A\times X^{\mathbb{N}})$ is for any measurable set $A\subset X$ equal to $\mu(A)$.

Let $A=\bigcap_{i=0}^{n-1}A_i\in\bigvee_{i=0}^{n-1}\mathcal{A}^t_{l_0+i}$. To any such set we assign, by the above correlation, a cylinder set $B_0\times B_1\times\ldots\times B_{n-1}\times X^{\mathbb{N}}$. Each set $A\times X^{\mathbb{N}}$ may also intersect many other cylinder sets $C_0\times C_1\times\ldots\times C^{n-1}\times X^{\mathbb{N}}$. However, one derives that except for a small subset of the space X the condition

$$x\in(A\times X^{\mathbb{N}})\cap(C_0\times C_1\times\ldots\times C_{n-1}\times X^{\mathbb{N}})$$

implies $C_i=B_i$ for all $i=0,\ldots,n-1$ except a small fraction of indexes. Since on the space of trajectories we consider a pointwise map, and not an operator, we can make use of the corollary from the classical Shannon–McMillan–Breiman's theorem, so called equipartition rule (see [P]), which in our case states that for any number $\delta>0$ for sufficiently large n we have lower and upper bounds:

$$2^{-n\left(\mathrm{h}_\nu\left(\sigma,\mathcal{A}^t_{l_0}\times X^{\mathbb{N}}\right)+\delta\right)}\leqslant\nu(C)\leqslant 2^{-n\left(\mathrm{h}_\nu\left(\sigma,\mathcal{A}^t_{l_0}\times X^{\mathbb{N}}\right)-\delta\right)},\tag{3}$$

for C being an arbitrary element of the partition $\bigvee_{i=0}^{n-1}\sigma^{-i}(\mathcal{A}^t_{l_0}\times X^{\mathbb{N}})$, apart from a set being a union of elements of the partition having aggregate measure less than δ. We prove that except for sets A of small total measure μ, a part of each set $A\times X^{\mathbb{N}}$ of measure exceeding $\frac{1}{2}\mu(A)$ is covered by cylinder sets satisfying condition (3) and differing from $B_0\times B_1\times\ldots\times B_{n-1}\times X^{\mathbb{N}}$ on a small number of coordinates. After calculating the amount of such cylinders we obtain the hypothesis of the theorem.

In the general case, for T not necessarily given by a transition probability, we fix a collection of functions $\mathcal{F}$ and consider a complex subalgebra $\mathcal{L}$ of $L^\infty(\mu)$, generated by all functions of the form T^nf, where $f\in\mathcal{F}$, $n\in\mathbb{N}$, and by lattice polynomials over variables of the form T^nf. By lattice polynomials we mean all expressions $w_1\vee w_2\vee\ldots\vee w_k$, where $w_i=x_1\wedge x_2\wedge\ldots\wedge x_{l_i}$ and where $\vee$ and $\wedge$ are lattice operations, in this case pointwise minima and maxima of functions. According to Gelfand's theorem there is an isomorphism between $\mathcal{L}$ and the algebra of all continuous (complex) functions on a compact Hausdorff space Δ. The isomorphism preserves the involution, so it sends real functions to real functions and positive functions to positive functions. In particular, the collection $\mathcal{F}$ is transformed into a collection $\widehat{\mathcal{F}}$ of functions $\Delta \longrightarrow [0,1]$. It also allows one to transport the measure μ to a Borel probability measure $\hat{\mu}$ on Δ and to define a Markov operator $\widehat{T}$ acting on the space of continuous functions $C(\Delta)$ by $\widehat{T}\hat{f} = \widehat{Tf}$, $f \in \mathcal{L}$. It is known that such an operator is always induced by a transition probability, so Theorem 4.13 holds for $\widehat{T}$, and, consequently, the theorem holds also for T.

Theorem 4.13 implies also the following equipartition rule ([A3, Remark 4.6]):

> for any collection $\mathcal{F}$ and any $\epsilon > 0$ there exist: $l_0 \in \mathbb{N}$, $N \in \mathbb{N}$ and a set $\tau \subset [0,1]$ of Lebesgue measure $\lambda(\tau) < \epsilon$, such that for every $t \in [0,1] \setminus \tau$ and $n \geqslant N$ the inequality
> $$2^{-n(\mathrm{h}_\mu(T,\mathcal{F},t)+\varepsilon)} \leqslant \mu(A) \leqslant 2^{-n(\mathrm{h}_\mu(T,\mathcal{F},t)-\varepsilon)}$$
> holds for every $A \in \bigvee_{i=0}^{n-1} \mathcal{A}^t_{l_0+i}$ except some sets of aggregate measure not greater than ε.

The final section of the paper contains the discussion on possible information functions for entropies defined in [DF] and [GLW], indicating their advantages and disadvantages. The function $\mathbb{I}^{DF}_{\mathcal{F}}$ proposed for [DF]-entropy allows one to prove a convenient asymptotic property:

$$\lim_{n\to\infty} \|\mathbb{I}^{DF}_{T^{l+n}\mathcal{F}} - T^n \mathbb{I}^{DF}_{T^l\mathcal{F}}\|_1 = 0.$$

Unfortunately, for collections of constant functions this function may be strictly positive. The definition $\mathbb{I}^{GLW}_{\mathcal{F}}$ proposed for [GLW]-entropy does not have the asymptotic property, but it is equal to zero for constant functions—thus it seems to be closer to the information function considered in [A3].

4.5 Item [A4]

The last paper is devoted to translating three famous classical theorems concerning zero entropy to the operator case. It is divided into two separate parts, which essentially differ as to the techniques used. The first one deals with genericity of entropy zero.

In the set of all automorphisms of a given probability space (X,μ) the weak topology is introduced by demanding that the sequence of automorphisms S_n converges to an automorphism S if for any measurable set A we have $\lim_{n\to\infty} \mu(S_n^{-1}A \triangle S^{-1}A) = 0$. The Rokhlin's theorem [R] says that automorphisms with zero entropy constitute a residual subset of the set of all automorphisms in the weak topology. For group actions, genericity of zero Rokhlin entropy was proved by Rudolph in case of amenable groups (see *subclaim* on page 288 of [FW]) and by L. Bowen for actions of arbitrary countable groups (see [B]). In the topological setup, Glasner and Weiss proved in [GW] that in the set of all homeomorphisms of a compact metric space, endowed with the topology of uniform convergence, the set of homeomorphisms with zero entropy is a G_δ set, and if the underlying space is a Cantor set, then the set of zero entropy homeomorphisms is also dense.

In [V], A. Vershik considers typical properties of doubly stochastic operators, which he also calls polymorphisms, and poses a question about genericity of zero entropy. In this context he mentions, among others, the entropy defined in [DF]. This was the direct motivation to begin the research of the current paper. Answering the question of Vershik, we prove the following theorem:

Theorem 4.19 ([A4, Theorem 3.5]) *The set of doubly stochastic operators with zero entropy is residual in the set of all doubly stochastic operators on $L^1(\mu)$ in strong operator topology and in the norm topology.*

The set of zero entropy operator is dense because in this class we find operators of the form $(1-\frac{1}{n})T+\frac{1}{n}S$, where $Sf=\int f d\mu$ and T is an arbitrary doubly stochastic operator. To verify that the set of zero entropy operators is a G_δ, we show that for any $\varepsilon>0$, $n\in\mathbb{N}$ and a collection $\mathcal{F}$ of measurable functions the set

$$U(\varepsilon,n,\mathcal{F})=\left\{T : T \text{ is doubly stochastic and } \frac{1}{n}\,\mathrm{H}_\mu(\mathcal{F}^n)<\varepsilon\right\}$$

is open in both considered topologies. Then we prove that the set of zero entropy operators is a countable intersection of sets of the form $U(\varepsilon,n,\mathcal{F})$, where one can restrict ε to rational numbers and $\mathcal{F}$ to collections consisting only of functions from a dense subset of $L^1(\mu)$.

In the second part of the paper we generalize the celebrated theorems by Kushnirenko (see [K]) and by Halmos–von Neumann (see [HvN]). Originally, these theorems concern dynamical systems with disrete spectrum, that is, systems for which the set of eigenvectors of the induced Koopman operator spans the whole $L^2(\mu)$. *The sequence entropy of a measure preserving map S with respect to a partition ξ along a sequence $A=(i_n)_{n\in\mathbb{N}}$* of positive integers is defined by

$$\mathrm{h}_A(S,\xi)=\limsup_{n\to\infty}\frac{1}{n}\,\mathrm{H}_\mu\left(\bigvee_{k=1}^{n}S^{-i_k}\xi\right).$$

The sequence entropy of S along a sequence A is given by

$$\mathrm{h}_A(S)=\sup_\xi \mathrm{h}_A(S,\xi)\,,$$

where the supremum ranges over all finite measurable partitions of the space X. The Kushnirenko's theorem states that a map S has discrete spectrum if and only if the sequence entropy $\mathrm{h}_A(S)$ is equal to zero along every sequence A.

It is fairly easy to translate the notion of the sequence entropy to Markov operators, using, for instance, the definition given in [DF]. For a sequence $A=(i_n)$ we let

$$\mathrm{h}_A(T,\mathcal{F})=\limsup_{n\to\infty}\frac{1}{n}\mathrm{H}_\mu\left(\bigvee_{k=1}^{n}T^{-i_k}\mathcal{F}\right),$$

where $\mathrm{H}_\mu(\mathcal{F})=\mathrm{H}^{DF}(\mathcal{F})$ is as defined in (2), and finally we take the supremum over all finite collections $\mathcal{F}$. However, the equivalence ensured in the classical case by the Kushnirenko's theorem does not hold in operator case.

The main tool used in this part of the paper is the Jacobs–de Leeuw–Glicksberg decomposition. In case of a Markov operator it is the decomposition of the domain of an operator into a direct sum of two invariant subspaces: the *reversible part* E_{rev}, spanned by eigenvectors corresponding to eigenvalues of modulus 1, and the *almost weakly stable part* E_{aws}, which is characterized by the following property:

> for every function $f\in E_{\mathrm{aws}}$ either the orbit $\{T^n f : n\in\mathbb{N}\}$ is not precompact in the norm topology or $\inf_{n\in\mathbb{N}}||T^n f||=0$.

This allows to obtain the following generalization of the Kushnirenko's theorem:

Theorem 4.20 ([A4, Theorem 5.4]) *The following conditions are equivalent for a doubly stochastic operator $T : L^p(\mu)\to L^p(\mu)$:*

1. *T has zero sequence entropy for every sequence,*

2. *$L^p(\mu) = V \oplus W$, where T has discrete spectrum on V and $\lim_{n\to\infty} \|T^n f\|_p = 0$ for every $f \in W$,*

3. *if $E_{\mathrm{rev}} \oplus E_{\mathrm{aws}}$ is a Jacobs de Leeuw–Glicksberg decomposition of $L^p(\mu)$ associated with T, then $\lim_{n\to\infty} \|T^n f\|_p = 0$ for every $f \in E_{\mathrm{aws}}$.*

For $p = 2$ we also obtained the adequate formulation in terms of the Nagy-Foiaş decomposition into the unitary and the completely non-unitary part of an operator (the decomposition is constructed in [NF]).

In the proof it is shown that a doubly stochastic operator has zero sequence entropy along every sequence if and only if the orbit of each function $f \in L^p(\mu)$ is precompact ([A4, Corollary 5.8]). This part of the proof is similar to the one given in the original Kushnirenko's paper, but it seems to be more natural in the operator case. For an operator with zero sequence entropy we obtain that, according to the above characterization, all orbits of functions from the almost weakly stable part must converge to zero (since there are no orbits with noncompact closure). Thus it is proved that the second condition in our theorem follows from the first one. To prove the converse implication, one shows that orbits of functions from the reversible part are precompact. Since the orbits of functions in almost weakly stable part are also precompact (in fact they converge to zero), we get compactness of all closed orbits, that is, zero sequence entropy. The third condition is just the reformulation of the second one.

According to Theorem 5.12 of [A4], the class of quasi-compact operators is an example of the class of operators with zero sequence entropy. These are the linear operators on $L^2(\mu)$, such that there exist two invariant subspaces F, H and a number $r < 1$, for which the following holds

1. $L^2(\mu) = F \oplus H$,

2. $\dim(F) < \infty$ and all eigenvalues of the restriction $T|_F$ have modulus greater than r,

3. the spectral radius of the restriction $T|_H$ is smaller than r.

In a similar way one can prove that if the unitary part of T in the Nagy–Foiaş decomposition has discrete spectrum, and the completely nonunitary part has spectral radius less than 1, then T has zero sequence entropy ([A4, Remark 5.13]). Nevertheless, the spectral radius equal to 1 (outside the unitary part) does not exclude the possibility of sequence entropy being equal to zero—Example 5.14 in the paper contains a construction of a completely nonunitary operator, having zero sequence entropy and spectral radius equal to one on the orthogonal complement of constants.

The Halmos–von Neumann theorem proved in 1942 in [HvN] states that an ergodic dynamical system with discrete spectrum is isomorphic to a Kronecker system, that is, a rotation of a compact abelian group. Since in the operator case, having zero sequence entropy seems to be a more natural condition, one can expect that the analog of the Halmos–von Neumann theorem will concern the representation of operators satisfying the entropy condition. Indeed, the following theorem is true:

Theorem 4.21 ([A4, Theorem 6.3]) *Let T be an ergodic doubly stochastic operator on $L^1(X, \Sigma, \mu)$. Let $\Sigma_{\mathrm{rev}} = \{A \in \Sigma : \mathbb{1}_A \in E_{\mathrm{rev}}\}$.*

The operator T has zero sequence entropy (for all sequences) if and only if the following two conditions are simultaneously satisfied:

1. *the action of* T *on* $E_{\mathrm{rev}} = L^1(X, \Sigma_{\mathrm{rev}}, \mu)$ *is Markov isomorphic to a rotation* R *of a compact abelian group* G *with Haar measure* λ,

2. $$\lim_{n\to\infty} \|T^n f - T^n E(f|\Sigma_{\mathrm{rev}}))\|_1 = 0$$

   for every $f \in L^1(\mu)$.

Furthermore, if X *is a Lebesgue space and* P_T *is a transition probability inducing* T, *then the Markov isomorphism becomes a point isomorphism of dynamical systems, i.e., there is a measure-preserving map* $\pi : X \to G$ *satisfying* $T(g \circ \pi) = g \circ R \circ \pi$ *for every* $g \in L^1(G, \lambda)$ *and* $P_T(x, \cdot)$ *is supported on* $\pi^{-1}R\pi(x)$.

After publishing the preliminary version on arxiv.org, I was invited to give a lecture on this research on a workshop *Operator Theoretic Aspects of Ergodic Theory* in Wuppertal. Also, results of the paper were presented on several international conferences and on a local seminar.

5 The description of other scientific achievements.

Below I present the results and activities which took place after receiving the PhD degree. Articles appear in chronological order, with one exception – a popular paper in Polish is presented at the end of already published papers.

5.1 B. Frej, A. Kwaśnicka. *Minimal models for Z^d-actions.* Colloq. Math. 2008, vol. 110, nr 2, 461–476.

The paper has its roots in the famous Jewett–Krieger theorem, proved in [J] for weakly mixing dynamical systems and generalized in [Kr] to all ergodic systems. It states that for every ergodic system there exists a measure-theoretically isomorphic strictly ergodic system, that is, a minimal topological dynamical system with single invariant measure. In particular, every topological dynamical system with any of its ergodic invariant measures can be modeled in an isomorphic minimal system. It is known that in general the set of all probability invariant measures of a topological dynamical system is a compact convex subset of the set of all probability measures on X, and it has the structure of a Choquet simplex (see [Ph]). A natural next question concerning possibility of modeling topological systems in minimal systems with the simplex of invariant measures remaining unchanged was considered in [D1] and in [KO]. The current paper extends the results obtained there to the case of $\mathbb{Z}^d$-actions in the following way. Let X be a compact zero-dimensional metrizable space and let $T = \{T_1, \dots, T_d\}$ be a set of commuting homeomorphisms $X \to X$. A pair (X,T) is called a *d-dimensional topological dynamical system* or a *$\mathbb{Z}^d$-action.* We say that a system (X,T) is *aperiodic*, if $T_1^{n_1}...T_d^{n_d}(x) = x$ only for $n_1 = ... = n_d = 0$. A system (X,T) is *minimal* if X does not contain nonempty proper subsets which are closed and invariant (by an invariant subset we mean $F \subset X$ such that $T_iF = F$ for $i = 1, ..., d$). The set of all T-invariant Borel probability measures on X will be denoted by $P_T(X)$. We say that a set X_0 is a full subset of X if $\mu(X_0) = 1$ for every $\mu \in P_T(X)$.

Definition 5.1 *We say that two d-dimensional dynamical systems (X,T) and (Y,S) are Borel* isomorphic if there exists an equivariant (i.e. satisfying $\Phi(T_ix) = S_i\Phi(x)$ for $i = 1, ..., d$) Borel-measurable bijection $\Phi : X_0 \to Y_0$ between full invariant subsets $X_0 \subset X$ and $Y_0 \subset Y$, such that the conjugate map $\Phi^* : \mathcal{P}_T(X) \to \mathcal{P}_S(Y)$ given by the formula $\Phi^*(\mu) = \mu \circ \Phi^{-1}$ is a (affine) homeomorphism with respect to weak* topologies.*

The main result in the paper is the following theorem:

Theorem 5.2 *If X is a compact zero-dimensional and metrizable space then every aperiodic $\mathbb{Z}^d$-action is Borel* isomorphic to a minimal one.*

The system which is constructed in the proof is in fact a symbolic system over an uncountable alphabet. The main idea in the construction is similar to the one used in [D1] and is based on replacing the original system by a symbolic system in which each point is represented by a d-dimensional infinite array. Then, each point is modified, so that every word from a "dense" subset of the set of all words occurring in the representation occurs syndetically, which guarantees that the resulting system is minimal. A $\mathbb{Z}^d$-version of the Krieger's marker lemma is used (its proof is given in our paper). It allows to build a sequence of block codes, uniformly convergent to the final isomorphism. The main difficulties, completely new when compared to the one-dimensional case, arise from the fact that orbits of points are no longer sequences. They are multidimensional infinite arrays, so the linear order on elements of orbits is lost. The question how to construct a nested sequence of shift invariant decompositions of orbits into blocks becomes much more difficult. In the paper we use a new method of constructing such decompositions, based on maximolexicographic order. It turns out that we additionally need to estimate the amount of points, for which the decompositions behave badly. We call such an improper behavior the existence of "eternal boundaries".

Actually, I have a feeling that the article might have appeared in a more prestigious journal, at least because of level of complication of the construction. It did not happen simply because of the authors' low experience in publishing papers. The natural continuation of the paper (see section 5.4) appeared recently in *Groups, Geometry, and Dynamics.*

5.2 T. Downarowicz, B. Frej, P.-P. Romagnoli. *Shearer's inequality and infimum rule for Shannon entropy and topological entropy.* Contemp. Math., ISSN 0271-4132; vol. 669 (2016), 63–75

The paper was planned as a survey build around the property of subadditivity of the Shannon entropy of a partition, but it also contains some new results. In accordance with the title, the emphasis is put on Shearer's inequality, rarely used in classical monographs in the field of ergodic theory. To formulate the Shearer's inequality we need the notion of a *k-cover* of a finite set F, that is a collection $\mathcal{K} = \{K_1, K_2, \ldots, K_r\}$ of finite sets (with possible repetitions), such that every element of F belongs to K_i at least for k indexes $i \in \{1, 2, \ldots, r\}$. We say that a nonnegative function H defined on finite sets satisfies the Shearer's inequality if for any finite set F and any k-cover $\mathcal{K}$ of F it holds that

$$H(F) \leqslant \frac{1}{k} \sum_{K \in \mathcal{K}} H(K).$$

By the amenable group we mean a group G, such that there exists a sequence (F_n) of finite subsets of G satisfying for every $g \in G$ the following condition:

$$\lim_{n\to\infty} \frac{|gF_n \triangle F_n|}{|F_n|} = 0,$$

where $gF = \{gf : f \in F\}$ and $|\cdot|$ denotes the cardinality of a set. Such sequence is called a Følner sequence. The action of G on X is given by a group homomorphism defined on G, taking values in the group of all automorphisms of X if we consider the measure case or all homeomorphisms from X onto itself in the topological case.

Let G be an amenable group and let (F_n) be a Følner sequence. Denote by $\mathcal{F}(G)$ the collection of all finite subsets of G.

Definition 5.3 *A nonnegative function H defined on $\mathcal{F}(G)$ satisfies the* infimum rule, *if*

$$\limsup_{n\to\infty} \frac{1}{|F_n|} H(F_n) = \inf_{F \in \mathcal{F}(G)} \frac{1}{|F|} H(F).$$

We say that H is *G-invariant* if for every $g \in G$ it holds that $H(Fg) = H(F)$. The importance of the Shearer's inequality is highlighted in the following theorem:

Theorem 5.4 *If H is a nonnegative and G-invariant function on $\mathcal{F}(G)$ which satisfies Shearer's inequality, then it also obeys the infimum rule.*

The function $H(F) = \mathrm{H}_\mu\big(\alpha^F\big)$, where α is a finite partition of a probability space (X, μ), $\alpha^F = \bigvee_{g \in F} g\alpha$ and $\mathrm{H}_\mu(\alpha)$ is the Shannon entropy of α, obeys the Shearer's inequality. Hence, the above theorem states, in particular, that for amenable groups the definition of the entropy of a G-action with respect to a partition does not depend on the choice of the Følner sequence. Moreover, a Følner sequence is not needed in the definition of this entropy; instead of the limit along such sequence one can consider the infimum over all finite subsets. Therefore, the formula

$$\mathrm{h}^*_\mu(G, \alpha) = \inf_{F \in \mathcal{F}(G)} \frac{1}{|F|} \mathrm{H}_\mu\big(\alpha^F\big)$$

may be used to define the entropy in more general groups. Letting

$$\mathrm{h}^*_\mu(G) = \sup_\alpha \mathrm{h}^*_\mu(G, \alpha),$$

where the supremum ranges over all finite partitions, we obtain the notion which is presently known in the literature as *naive entropy*. The notion was deeply investigated in the works of Burton, L. Bowen and Seward. Besides the striking simplicity, it has an important virtue that it cannot increase under taking factors (contrary to sofic entropy). Unfortunately, it was shown in [L] that for non-amenable groups the naive entropy takes only two values: zero or infinity. One of the questions posed in our paper, concerning relations between the entropy defined by $\mathrm{h}^{**}_\mu(G) = \inf\{\mathrm{h}^*_\mu(G, \alpha) : \alpha \text{ is a generator}\}$ and the Rokhlin entropy $\mathrm{h}^{***}_\mu(G) = \inf\{\mathrm{H}^*_\mu(\alpha) : \alpha \text{ is a generator}\}$ was answered (even in version for conditional entropies) by Seward in [S]. He showed that for a free action of G it holds that $\mathrm{h}^{***}_\mu(G) \leqslant \mathrm{h}^{**}_\mu(G)$, which yields the equality between these notions in view of a former result of the same author.

New results in our paper appear mostly in chapter 6, concerning topological entropy. In the paper the examples were constructed to show that the topological entropy obeys neither Shearer's inequality nor the infimum rule (though the entropy is subadditive). In the latter case the action of the finite group $\mathbb{Z}_3$ was used. We also managed to formulate positive results.

Theorem 5.5 *If an open cover $\mathcal{U}$ consists of pairwise disjoint sets then the corresponding function $\mathrm{H}_{\mathrm{top}}(\mathcal{F}) = \mathrm{H}_{\mathrm{top}}\left(\mathcal{U}^{\mathcal{F}}\right)$ on $\mathcal{F}(G)$ obeys Shearer's inequality.*

Elements of a symbolic system $X = \Lambda^G$ with action of a group G, where Λ is a finite alphabet, will be denoted by $(x_g)_{g \in G}$, where $x_g \in \Lambda$ for every $g \in G$. The action in this system is given by $(gx)_h = x_{hg}$ and the system is called a subshift. As an open cover one can take a partition into cylinder sets. In particular, by P_Λ we denote the so-called *time-zero partition*: $P_\Lambda = \{[a] : a \in \Lambda\}$, where $[a] = \{(x_g)_{g \in G} : x_e = a\}$.

Corollary 5.6 *If (X, G) is a subshift and $\mathcal{U} = P_\Lambda$ then the infimum rule holds for* $\mathrm{H}_{\mathrm{top}}$, *i.e.,*

$$\mathrm{h}_{\mathrm{top}}(G) = \mathrm{h}_{\mathrm{top}}(G, \mathcal{U}) = \inf_{F \in \mathcal{F}(G)} \frac{1}{|F|} \mathrm{H}_{\mathrm{top}}\left(\mathcal{U}^F\right).$$

Despite the lack of the Shearer's inequality, using the Variational Principle we prove the following fact.

Theorem 5.7 *Let X be a compact metric space and G a countable amenable group acting on X by continuous maps. Define*

$$\mathrm{h_{top}}^*(G,\mathcal{U}) = \inf_F \frac{1}{|F|}\mathrm{H_{top}}\left(\mathcal{U}^F\right)$$
$$\mathrm{h_{top}}^*(G) = \sup_{\mathcal{U}} \mathrm{h_{top}}^*(G,\mathcal{U})$$

Then

$$\mathrm{h_{top}}^*(G) = \mathrm{h_{top}}(G)$$

The question if the exact infimum rule holds for topological entropy in case of infinite amenable groups, i.e., if the equality $\mathrm{h_{top}}^*(G,\mathcal{U}) = \mathrm{h_{top}}(G,\mathcal{U})$ holds, remains open.

5.3 B. Frej, A. Kwaśnicka. *A map maintaining the orbits of a given Z^d-action.* Colloq. Math. 2016, vol. 143, nr 1, s. 1-15.

The paper deals with the notion of orbital equivalence of topological dynamical systems. In [GMPS1] it was shown that every minimal action of $\mathbb{Z}^2$ on a Cantor space, i.e., a compact zero-dimensional perfect metrizable space, is orbitally equivalent to a (minimal) action of the group $\mathbb{Z}$. The same authors generalized the result to the case of $\mathbb{Z}^d$-actions (for arbitrary d) in [GMPS2]. Before the second paper was published, we made an attempt to prove the fact, without the complicated machinery of K-theory used by Giordano, Matui, Putnam and Skau. The goal was to explicitly construct the appropriate orbit-preserving map, and the direct inspiration was the paper [F] by Forrest, which used as the main tool Bratteli–Vershik diagrams. Our efforts led only to a partial succes. Using similar techniques as in 5.1 (Marker lemma, cutting trajectories into blocks with help of maximolexicographic order) we proved the following theorem:

Theorem 5.8 *Let $T_1, ..., T_d$ be homeomorphisms of a Cantor space X. Let $\mathcal{O}_T(x)$ denote the orbit of x, that is, the set $\{T_1^{i_1}...T_d^{i_d}x : i_1, ..., i_d \in \mathbb{Z}\}$. For every free minimal $\mathbb{Z}^d$-action $T = \{T_1, ..., T_d\}$ and every $x_0 \in X$ there is a continuous injection $F\colon X \setminus \{x_0\} \to X$ such that*

$$\mathcal{O}_T(x) = \bigcup_{n\in\mathbb{Z}} F^n\{x\}$$

for every x from some residual subset of X.

Moreover, for all other points x it holds that

$$\mathcal{O}_T(x) = \bigcup_{j=1}^{J} \bigcup_{n\in\mathbb{Z}} F^n\{x_j\}$$

for a finite set $\{x_1, ..., x_J\}$.

After [GMPS2] was published, we gave up further investigations in this direction.

5.4 B. Frej, D. Huczek. *Minimal models for actions of amenable groups.* Groups Geom. Dyn. 2017, vol. 11, nr 2, s. 567-583.

It is a continuation of the research described in section 5.1. As a direct generalization of the case of $\mathbb{Z}^d$-actions we consider actions of countable amenable groups—a rapidly developing part od dynamical systems (see e.g. [KL]). We identify the acting group with the corresponding set of homeomorphisms and we write gx to denote the image of x via the homeomorphism assigned to g. We say that the action is *free* if $gx = x$, where $g \in G$, $x \in X$, implies that g is a neutral element of G. The action of G is *minimal* if for every $x \in X$ the closure of the orbit $\{gx : g \in G\}$ is equal to the whole space X or, equivalently, if X does not contain proper subsets which are closed and invariant (that is, $gF = F$ for all g).

The notion of the Borel* isomorphism (compare Def. 5.1) is translated to the group case in a natural way. Then the main theorem in the current paper has the following form:

Theorem 5.9 *If X is a compact zero-dimensional metrizable space and G is an amenable group acting freely on X then the system (X, G) is Borel* isomorphic to a minimal dynamical system (Y, G).*

Roughly speaking, the main difficulty comes from the fact that, compared to a $\mathbb{Z}^d$-action, we can no longer use shapes of subsets of G. Formally, we need to use deep theorems on tilings of amenable groups which were proved in [DH] and [DHZ]. Using them, we restore decompositions into blocks, which in $\mathbb{Z}^d$ case were done by means of the maximolexicographic order. Retaining the main plan from [D1] and the paper described in section 5.1, which relied on the construction of a sequence of block codes modifying trajectories (or symbolic representations of points of the original space), we perform reasonings of quantitative kind, using the notion of Banach density. According to Lindenstrauss' ergodic theorem (see [L]), these calculations are translated to the language of measures.

5.5 B. Frej, D. Huczek. *Faces of simplices of invariant measures for actions of amenable groups.* Monatsh. Math. 2018, vol. 185, nr 1, s. 61-80.

Equipped with the same toolbox as in the previous paper, we deal with the problem of representing faces of a given simplex of invariant measures, in case of an action of a countable amenable group, as full simplices of invariant measures. For classical dynamical systems the problem was solved in [D2].

Let K be an arbitrary metrizable Choquet simplex. We introduce the following definitions.

Definition 5.10

1. *An* assignment *on K is a function Φ defined on K such that for each $p \in K$, the value of $\Phi(p)$ is a measure-preserving group action $(X_p, \Sigma_p, \mu_p, G_p)$, where (X_p, Σ_p, μ_p) is a standard probability space and G_p acts on X via measure automorphisms.*

2. *Two assignments Φ on K and Φ' on K' are* equivalent *if there exists an affine homeomorphism $\pi : K \to K'$ such that $\Phi(p)$ and $\Phi'(\pi(p))$ are isomorphic for every $p \in K$.*

3. *Let (X, G) be a continuous group action on a compact metric space X. Let $\mathcal{B}_X$ be the Borel σ-algebra in X and let $P_G(X)$ denote the Choquet simplex of all G-invariant measures on X (with the weak* topology).*
   The assignment $\Phi(\mu) = (X, \mathcal{B}_X, \mu, G)$ is the natural assignment *of (X, G).*

4. *By a* face *of a simplex S we mean a compact convex subset of S which is a simplex itself and whose extreme points are also the extreme points of S.*

5. *If K is a face of a simplex $P_G(X)$ then by the* identity assignment *on K we mean the restriction of the natural assignment on $P_G(X)$ to K.*

The main result of the current paper is the following theorem:

Theorem 5.11 *Let X be a Cantor system with free action of an amenable group G and let K be a face in the simplex $P_G(X)$ of all G-invariant measures of X. There exists a Cantor system Y with free action of G, such that the natural assignment on Y is equivalent to the identity assignment on K.*

Combining it with the results of our former paper we obtain:

Theorem 5.12 *Let X be a Cantor system with free action of an amenable group G and let K be a face in the simplex $P_G(X)$. There exists a Cantor system Y with* minimal *free action of G, such that the natural assignment on Y is equivalent to the identity assignment on K.*

The main tool used in the proof of the one-dimensional case were block measures (that is, measures on subshifts supported by periodic orbits), which approximated arbitrary ergodic measures. To employ a similar strategy in case of group actions, we created analogs of such measures, which seems to be a standalone important result.

5.6 B. Frej, D. Huczek, *A comment on ergodic theorem for amenable groups*, to appear in Canad. Math. Bull., doi: 10.4153/S0008439519000110, arXiv:1901.01324 [math.DS]

In the paper one finds a proof of a version of ergodic theorem for actions of countable amenable groups, where a fixed Følner sequence needs not be tempered. Instead, it is assumed that a function, whose ergodic averages we study, satisfies the following mixing condition:

Definition 5.13 *We say that f is ε*-independent *from a* sub-σ-algebra Σ_0 *if for every $B \in \Sigma_0$ of positive measure it holds that*

$$\left|\int_B f d\mu_B - \int f d\mu\right| < \varepsilon,$$

where μ_B is the conditional measure on B.

Theorem 5.14 *Let $G = \{g_1, g_2, ...\}$ be an amenable group acting on a probability space (X, μ). Let $(F_n)_{n\in\mathbb{N}}$ be a Følner sequence in G, such that for every $\alpha \in [0,1)$ the series $\sum_{n=1}^{\infty} \alpha^{|F_n|}$ converges. Let $f \in L^\infty(\mu)$ be such that for every $\varepsilon > 0$ there exists a finite set $K \subset G$ such that f is ε-independent from $\sigma(\{f \circ g : g \notin K\})$. Then*

$$\lim_{n\to\infty} \frac{1}{|F_n|} \sum_{g\in F_n} f(gx) = \int f d\mu \qquad \mu\text{-a.e.}$$

The proof uses the Azuma–Hoeffding concentration inequality.

Results of the paper were already presented on an international conference and on a dynamical systems seminar held in AGH UST (on an invitation).

5.7 B. Frej, D. Huczek, *Extensions of full shifts with group actions.* to appear in Colloq. Math., doi: 10.4064/cm7843-3-2019, arXiv:1901.01145 [math.DS]

The paper was inspired by lectures, which were given by Aimee Johnson during Wrocław edition of Wandering Seminar workshop, about factorizing multidimensional subshifts with large entropy onto the full shift (see e.g., [JM] and [De]). We give the following sufficient condition for a symbolic topological dynamical system with action of a countable amenable group to be an extension of the full shift.

Definition 5.15 *We say that a symbolic dynamical system (X, G) is* strongly irreducible *if there exists a finite set D such that for any finite subsets T_1 and T_2 of G, satisfying $T_2 \cap DT_1 = \emptyset$, and any two blocks A and B, with domains respectively T_1 and T_2, there exists an $x \in X$ such that $x(T_1) = A$ and $x(T_2) = B$.*

Theorem 5.16 *If the symbolic dynamical system (X, G) with topological entropy greater than $\log k$ is strongly irreducible, then there exists a symbolic extension $(\tilde{X}, G)$ of X, having the same topological entropy as X, and such that $(\tilde{X}, G)$ factors onto the full shift over k symbols.*

5.8 B. Frej, *Exploding Markov operators*, submitted

The paper continues my research on operator dynamics. It contains a definition of a class of doubly stochastic operators, based on the notion of disintegration of measure. These operators come from pointwise maps, but they are not their Koopman operators. Let (X, Σ, μ) be a standard probability space and let $T : X \to X$ be a measure-preserving surjection. For a decreasing sequence (a_k), which sums to one, we define a measure m on $\mathbb{N}$ letting $m(\{k\}) = a_k$. For $k \in \mathbb{N}$ let ξ_k be a partition of X into sets $T^{-k}\{T^k x\}$, $x \in X$, and let $\xi_k(x)$ be an element of the partition ξ_k which contains x. Let $\{\mu_C : C \in \xi_k\}$ be the disintegration of measure μ with respect to X/ξ_k. Moreover, let (b_k) be given by $b_k = \frac{a_k - a_{k+1}}{a_1}$. Denote by $A|_k$ the section $\{x \in X : (x, k) \in A\}$. We define a doubly stochastic operator $L^1(X \times \mathbb{N}, \mu \times m)$ by the formula

$$\Upsilon_T f(y) = \int f(u) P_T(y, du),$$

where P_T is a probability kernel given by

$$\begin{aligned} P_T\big((x,1), A\big) &= \sum_{k=1}^{\infty} b_k \mu_{\xi_k(x)}(T^{-1}A|_k) \\ P_T\big((x,k), A\big) &= \delta_{(Tx,k-1)}(A) \qquad \text{for } k \geqslant 2. \end{aligned}$$

It is proved in the paper that such operator inherits some properties of a generating map, namely: it is ergodic if and only if T is ergodic and it has positive entropy if T has. Yet, it may have no pointwise factors.

5.9 B. Frej, *I jeszcze jeden, i jeszcze raz.* Matematyka, Społeczeństwo, Nauczanie 41 (VII 2008)

This short paper (in Polish) is a consequence of an invitation to one of the conferences in the series *Szkoła Matematyki Poglądowej*. It also emerges from my studies on multi-recurrence of Markov operators. The paper contains an exposition on the van der Waerden's theorem on monochromatic arithmetic progressions and the Birkhoff's theorem on multirecurrence in topological dynamical systems with d commuting homeomorphisms. In particular, I present the brilliant proof of the latter theorem, published in [BPT]. Unfortunately, an attempt to translate the proof to the language of dynamics induced by transition probabilities was not successful. The paper does not contain any new results.

5.10 Mathematics–Reactivation

In years 2010–2014 I was invited to take part in the educational project, mentioned at the beginning of the document (in the information about the employment). The project, known under the name *Mathematics–Reactivation*, was addressed to teachers and students of Polish secondary schools. The project was financed from the resources of the European Union as a part of Operational Programme III and the head of it was Jędrzej Wierzejewski, one of the forerunners of e-learning in Poland. The main goal was to create an interactive e-course in mathematics, covering the whole math curriculum in secondary schools. It was meant to attract students and help them in gaining necessary mathematical skills, but also to give teachers a convenient didactic tool. As a final outcome of the project, we created an electronic textbook with all the knowledge taught in secodary school, as well as a large set of e-exercises to let the students practice their skills. Apart from this, the e-platform allowed for carrying tests with use of computer terminal. My role was to take an active part in the creation of e-course, mainly e-exercises, since the prototype of an exercise. Every e-exercise needed careful testing to verify both the mathematical accuracy and reliability. My responsibility was to find all possible faults and propose improvements and modifications. I also used the experienced gained in the project, while serving as an e-learning coordinator in the Institute of Mathematics.

The e-course was introduced in several secondary schools in Wrocław.

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