Summary of scientific achievements

1. Name and surname:

Irmina Czarna

2. Scientific degrees:

2006 M.Sc. in Mathematics,
Mathematical Institute, Faculty of Mathematics and Computer Science, University of Wrocław,
Master's dissertation: *Stochastic differential equations modeling financial phenomena*,
supervisor: Prof. dr hab. Ewa Damek.

2011 Ph.D. in Mathematics,
Mathematical Institute, Faculty of Mathematics and Computer Science, University of Wrocław,
doctoral dissertation: *Ruin probabilities and dividends with Parisian delay for one-dimensional and multivariate Lévy processes*,
supervisor: Prof. dr hab. Zbigniew Palmowski.

3. Information on previous employment in scientific institutions:

2011 – 2012	Assistant in Mathematical Institute, University of Wrocław
2012 – 2017	Assistant Professor in Mathematical Institute, University of Wrocław
2017 – present	Assistant Professor in Faculty of Pure and Applied Mathematics, Wroclaw University of Science and Technology

4. The indication of the scientific achievement:

(a) The title of the scientific achievement:

Fluctuation theory for level-dependent Lévy processes and the problem of Parisian delay

(b) The list of papers constituting the scientific achievement:

[H1] R. Loeffen, I. Czarna, Z. Palmowski, *Parisian ruin probability for spectrally negative Lévy processes*, Bernoulli. 2013, vol. 19, no. 2, p. 599–609.

[H2] I. Czarna, *Parisian ruin probability with a lower ultimate bankrupt barrier*, Scandinavian Actuarial Journal. 2016, vol. 2016, no. 4, p. 319–337.

[H3] M. A. Lkabous, I. Czarna, J-F. Renaud, *Parisian ruin for a refracted Lévy process*, Insurance Mathematics and Economics. 2017, vol. 74, p. 153–163.

[H4] I. Czarna, J.-L. Pérez, T. Rolski, K. Yamazaki, *Fluctuation theory for level-dependent Lévy risk processes*, Stochastic Processes and their Applications. DOI:10.1016/j.spa.2019.03.006

(c) A discussion of the above-mentioned papers and the obtained results, together with a discussion of their possible use

I. Introduction

Lévy processes can be seen as a natural generalization of random walks to processes with a continuous time parameter and thus constitute an important subclass of Markov processes. The most known examples of these processes are the Wiener process, the compound Poisson process or the Cauchy process. In addition, some of the results obtained for Lévy processes can be used to make hypotheses and check properties for more general classes of processes. This means that Lévy processes can be seen as a kind of "experimental" class for Feller processes or even more generally for Markov processes. Moreover, Lévy processes can be used to model, among others physical, insurance or economic phenomena, thanks to which the literature devoted to these processes is currently very rich [1, 3, 4, 12, 28, 39].

The main goal of the scientific achievement was to create a general fluctuation theory for spectrally negative level-dependent Lévy processes, which are defined as a strong solution to some stochastic differential equation and generalize Lévy processes to Feller processes. Examples of such processes are Lévy processes, refracted Lévy processes considered in the literature by [24, 25, 33, 48] and generalized Ornstein-Uhlenbeck process killed at the moment when it goes below a certain level, which (also without killing) was considered before in [27, 40, 46, 47].

The papers [H1]-[H3] examine the probability of ruin and fluctuation identities for spectrally negative Lévy processes and refracted Lévy processes, where the leitmotif was to consider the so-called Parisian delay. The Parisian delay means the predetermined, deterministic time $d \geq 0$ through which the considered process must be in a specific position: for example above/below a certain level (so-called barrier). Note that the case $d = 0$ means no delay, i.e. a classic theory that for Lévy processes is well known in the literature [7, 31, 36]. As it turns out, issues related to the Parisian delay are still very popular and the multitude of new problems has not yet exhausted the topic of Parisian delay in risk theory. This is evidenced by numerous papers created recently (see [6, 18, 23, 44, 45, 51]).

The presented scientific achievement concerns general class of spectrally negative level-dependent Lévy processes, which, depending on the considered parameters, can be Lévy processes, refracted and multi-refracted Lévy processes. Their properties were investigated using the techniques of fluctuation theory of Lévy processes, among others scale functions and techniques related to the theory of excursions for Lévy processes. In [H1] the formula for the Parisian ruin probability for any spectrally negative Lévy process was proved. The key idea was to use Kendall's formula and, thus, to invert the Laplace transform associated with the excursion below the level zero of the considered process. Note that the new formula obtained for the Parisian ruin probability is compact and does not depend on the type of path variation of the considered Lévy process. This resulted that the proof techniques used in [H1] were later used by other authors to obtain results related to the Parisian delay in various models [6, 45, 51]. These studies become the motivation to consider the Parisian delay model with an additional barrier at a certain fixed level $-a < 0$ (paper [H2]). We believe that the Parisian ruin probability with the lower ultimate bankrupt barrier could be a better and more realistic risk measure than the Parisian ruin probability, because in this new model the excursion of the risk process below zero, which is "too deep" i.e. is bigger than a fixed number a leads to ruin immediately. However, from a mathematical point of view, the model requires more advanced proof techniques and is more complicated computationally. Additionally, in [H2] we derive the asymptotics of the Parisian ruin probability with the lower barrier in two cases, that is when the claim size has light- and heavy-tailed distributions. Paper [H3] generalizes the result obtained in [H1] for the case of the so-called spectrally negative refracted Lévy process, which is defined as a strong solution to some stochastic differential equation. Moreover, in [H3] the formulas for the one-sided and two-sided exit problems were obtained for refracted process considered until the Parisian ruin moment. The proof techniques used in [H3] are based on Kendall's formula, properties of the scale functions and excursions of spectrally negative Lévy processes. The culmination of this research is paper [H4], which considers the general family of level-dependent Lévy processes that are Feller processes. In [H4] the one-sided, two-sided exit problems and the resolvents were obtained. The resolvents are the expected occupation measures of the associated process in a given Borel set. These formulas were expressed by the new scale functions that, as it was proved, fulfil integral equations of the Volterra type.

In summary, below we present the most important results obtained in the papers included in the scientific achievement:

- In [H1] the probability of the Parisian ruin for any spectrally negative Lévy process was studied. The obtained formula involves only the scale function of the spectrally negative Lévy process and the distribution of the process at time d.

- The paper [H2] concerns Parisian ruin probability with a lower ultimate bankrupt barrier, for which the formula for any spectrally negative Lévy process was found and additionally, the light-tailed and heavy-tailed asymptotics were obtained for this model.

- In [H3] for the spectrally negative refracted Lévy process the Parisian ruin probability was found. Moreover, the formulas for the one-sided and two-sided exit problems for this process with the Parisian delay were obtained.
- The paper [H4] concerns the fluctuation theory for the general class of the so-called level-dependent spectrally negative Lévy processes. The main result of [H4] is the proof of existence, characteristics and formulas for resolvents as well as for the one-sided and two-sided exit problems for these processes. In this paper fluctuation identities were obtained in the case when Parisian delay d equals zero.

Before we proceed to the detailed description of the above results, we will present the basic definitions, formulas and properties of the trajectories of Lévy processes that were used in the discussed above papers. We relate the reader to [7, 31] for a much more complete and detailed description.

Lévy processes and risk theory

Let $X = \{X_t, t \geq 0\}$ be a Lévy process in $\mathbb{R}$, defined on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t : t \geq 0\}, \mathbb{P})$, i.e. X is a stochastic process issued from the origin which has stationary and independent increments and càdlàg paths. As a strong Markov process we shall endow X with probabilities $\{\mathbb{P}_x : x \in \mathbb{R}\}$ such that under $\mathbb{P}_x$ we have $X_0 = x$ with probability one. Further $\mathbb{E}_x$ denotes expectation with respect to $\mathbb{P}_x$. Recall that $\mathbb{P} = \mathbb{P}_0$ and $\mathbb{E} = \mathbb{E}_0$.

The Lévy triple for process X is given by (γ, σ, Π), where $\gamma \in \mathbb{R}$, $\sigma \geq 0$ and Π is a measure concentrated on $\mathbb{R}\backslash\{0\}$ satisfying $\int_{\mathbb{R}}(1 \wedge x^2)\Pi(dx) < \infty$. The measure Π is called the Lévy (characteristic) measure.

The Lévy-Khintchine formula gives that

$$\Psi(\theta) = -\log \mathbb{E}[e^{i\theta X_1}] = -i\gamma\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}}(1 - e^{-i\theta x} - i\theta x\mathbf{1}_{\{|x|<1\}})\Pi(dx)$$

for every $\theta \in \mathbb{R}$ and Ψ is a characteristic exponent of X. If we assume additionally that X is a spectrally one-sided Lévy process, for example spectrally negative (i.e. the Lévy measure Π is supported only on the negative half-line) then its the Laplace exponent ψ is well defined by

$$\psi(\theta) = -\Psi(-i\theta) = \log \mathbb{E}[e^{\theta X_1}] = \gamma\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{(-\infty,0)}(e^{\theta x} - 1 - \theta x\mathbf{1}_{\{|x|<1\}})\Pi(dx)$$

for every $\theta \geq 0$. Moreover, $\Phi(q) = \sup\{\beta \geq 0 : \psi(\beta) = q\}$ is known as the right inverse of ψ.

Due to the trajectory variation, Lévy processes can be divided into two separate groups: bounded and unbounded variation. Spectrally negative bounded variation processes are such that $\int_{(-1,0)} x\Pi(dx) < \infty$ and $\sigma = 0$. They may be written in the form

$$X_t = x + pt - S_t,$$

where S_t is a pure jump subordinator (i.e. process which paths are are almost surely non-decreasing) and $p = \gamma + \int_{(-1,0)} x\Pi(dx) > 0$. In risk theory, the classic example of such a process of bounded variation is the so-called the Crámér–Lundberg process, which is given by the following formula

$$X_t = x + pt - \sum_{k=1}^{N_t} C_k, \tag{1}$$

where $\{N_t\}_{t\geq 0}$ is a Poisson process with rate λ, non-negative random variables C_k are i.i.d. and also independent of N.

Let us now define using the so-called Esscher's transform the following exponential change of measure:

$$\left.\frac{d\mathbb{P}_x^c}{d\mathbb{P}_x}\right|_{\mathcal{F}_t} = \exp\{c(X_t - X_0) - \psi(c)\, t\} \tag{2}$$

for any c such that $\mathbb{E}e^{cX_1} < \infty$. From the theory of Lévy processes (see [7, 31]) it is known that X with respect to $\mathbb{P}_x^c$ is still a spectrally negative Lévy process with the Laplace exponent

$$\psi_c(\theta) = \psi(\theta + c) - \psi(c) \quad \text{for } \theta \geq -c.$$

The above fact plays a key role in the analysis of insurance models, which are constructed based on spectrally negative Lévy processes. For example, using the exponential change of the measure (2) one can derive the expressions for the Laplace transforms of the first exit time of the process from a given set.

Most of the results of the scientific achievement concern issues related to the fluctuation theory for spectrally negative Lévy processes and spectrally negative level-dependent Lévy processes. In this theory, for the case of Lévy processes, the so-called scale functions play a key role.

Definition **1.** *For a given spectrally negative Lévy process X define a family of functions indexed by $q \geq 0$, $W^{(q)} : \mathbb{R} \to [0, \infty)$ as follows. For each given $q \geq 0$ we have $W^{(q)}(x) = 0$ when $x < 0$ and otherwise for all $x \geq 0$, $W^{(q)}$ is a strictly increasing and continuous function whose Laplace transform is given by*

$$\int_0^\infty e^{-\theta x} W^{(q)}(x)dx = (\psi(\theta) - q)^{-1}, \quad \theta > \Phi(q). \tag{3}$$

For convenience we shall always write $W^{(q)} = W$ when $q = 0$. Typically we shall refer the functions $W^{(q)}$ as q-scale functions, but we shall also refer to W as just the scale function.

The scale function $W^{(q)}$ appears in classical fluctuation theory as a solution of the so-called *two-sided exit problem* from interval $[0, a]$, which has a long history (see [7, 30, 49]). Define $\tau_0^- = \inf\{t > 0 : X_t < 0\}$ and $\tau_a^+ = \inf\{t > 0 : X_t > a\}$. It is common in insurance and actuarial applications that the stopping time τ_0^- is called the moment of ruin. Then for any $q \geq 0$ and $0 \leq x \leq a$ we have that

$$\mathbb{E}_x\left[e^{-q\tau_a^+}\mathbf{1}_{\{\tau_a^+<\tau_0^-\}}\right] = \frac{W^{(q)}(x)}{W^{(q)}(a)}.$$

In fact, it is through this identity that the *scale function* gets its name. We also mention here that for some particular examples of Lévy processes (like for example Brownian motion with drift or Cramér-Lundberg process with exponential jumps) the form of the function $W^{(q)}$ can be obtained explicitly (see [36, H1]). Another function appearing in fluctuation identities for Lévy processes is the so-called second scale function $Z^{(q)}$.

Definition **2.** *For any $q \geq 0$ and $x \in \mathbb{R}$ define*

$$Z^{(q)}(x) = 1 + q \int_0^x W^{(q)}(y)dy.$$

Since in fluctuation theory most of the identities are expressed by the above defined scale functions, their properties have been extensively studied in literature [10, 31, 32, 34, 36]. For example, the solution of the one-sided exit problem for spectrally negative Lévy process, for $q \geq 0$ and $x \geq 0$ is given by

$$\mathbb{E}_x[e^{-q\tau_0^-}\mathbf{1}_{\{\tau_0^- < \infty\}}] = Z^{(q)}(x) - \frac{q}{\Phi(q)} W^{(q)}(x).$$

When $q = 0$ we get a formula for ruin probability, where we understand $\frac{q}{\Phi(q)}$ in the limiting sense when $q \downarrow 0$. Then

$$\mathbb{P}_x(\tau_0^- < \infty) = \begin{cases} 1 - \psi'(0+)W(x) & \text{if } \psi'(0+) > 0 \\ 1 & \text{if } \psi'(0+) \leq 0. \end{cases} \tag{4}$$

In risk theory, one of the basic characteristic is the above ruin probability $\mathbb{P}_x(\tau_0^- < \infty)$, which is treated as a certain measure of risk and it is also an important component taken into account when calculating the insurance premium. There exist already a vast number of papers considering this subject. Regarding the references, it is enough to mention the books of Rolski et al. [50] and Asmussen [2, 3]. In the papers [13] and [H1] the concept of ruin was generalized to the so-called Parisian ruin, which occurs when the process stays below zero for a period of time longer than fixed $d > 0$ (see Fig. 1). Formally define Parisian stopping time as follows:

$$\tau^d = \inf\{t > 0 : t - \sup\{s < t : X_s \geq 0\} \geq d, X_t < 0\}$$

and the ruin probability is given by

$$\mathbb{P}(\tau^d < \infty | X_0 = x) = \mathbb{P}_x(\tau^d < \infty).$$

Note that the case $d = 0$ corresponds to the classical ruin.
The name for this problem is borrowed from the Parisian option. Depending on the type of such option the prices are activated or cancelled if underlying asset stays above or below barrier long enough in a row (see [11, 15, 16, 17]). We believe that Parisian ruin probability could be a better measure of risk in many situations giving the possibility for to an insurance company to get solvency. Therefore, the above-mentioned problem turned out to be both interesting from an economic and mathematical point of view.

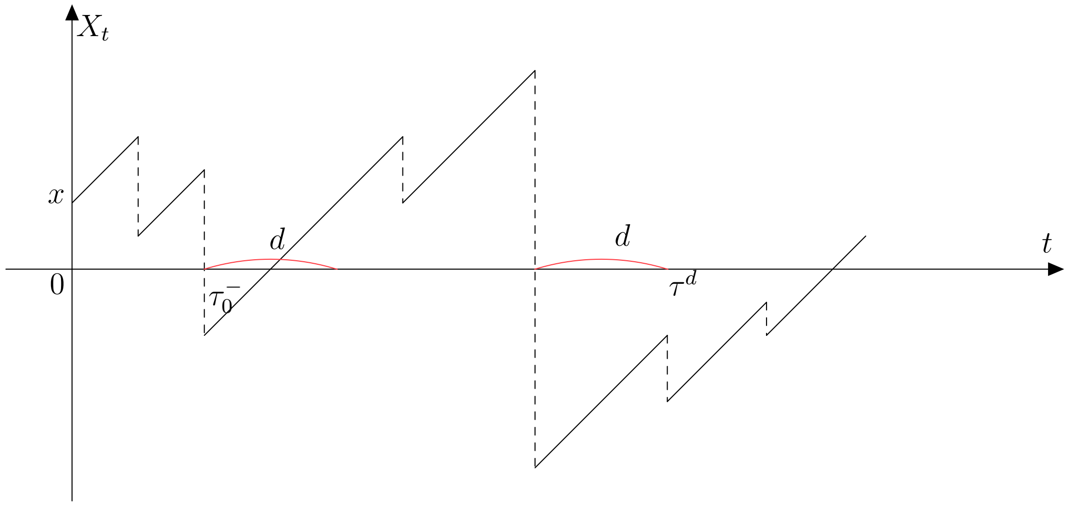

Fig. 1. A sample path of process X and moments of classical and Parisian ruin.

For further developments on this subject, in [H2] the Parisian ruin moment with lower ultimate bankrupt barrier $-a$ was defined:

$$\kappa^{d,a} = \min(\tau^d, \tau_{-a}^-),$$

where $\tau_{-a}^- = \inf\{t \geq 0 : X_t < -a\}$. Then the ruin probability is given by:

$$\mathbb{P}(\kappa^{d,a} < \infty | X_0 = x) = \mathbb{P}_x(\kappa^{d,a} < \infty).$$

The Parisian ruin probability with the lower ultimate bankrupt barrier occurs if the process X stays below zero for a longer period than a fixed $d > 0$ or goes below the level $-a$ (lower barrier). We believe that the Parisian ruin probability with the lower ultimate bankrupt barrier could be better and more realistic risk measure than the Parisian ruin probability, because in this new model the excursion of the risk process below zero, which is "too deep" i.e. is bigger than a fixed number $-a < 0$ leads to ruin immediately.

Moreover, in [H2] to obtain asymptotics of the above ruin probability when $x \to \infty$ a dual process $\widehat{X} = -X$ was considered. In this case the process $\widehat{X}$ is a spectrally positive Lévy process with the Lévy measure $\Pi_{\widehat{X}}(dy) = \Pi_X(-dy)$.

The characteristics of $\widehat{X}$ will be indicated by using a "hat" over the existing notation for the characteristics of X.

For the process X we define the ascending ladder process $(L^{-1}, H) = \{(L_t^{-1}, H_t)\}_{t \geq 0}$:

$$L_t^{-1} := \begin{cases} \inf\{s > 0 : L_s > t\} & \text{if } t < L_\infty \\ \infty & \text{otherwise} \end{cases}$$

and

$$H_t := \begin{cases} X_{L_t^{-1}} & \text{if } t < L_\infty \\ \infty & \text{otherwise,} \end{cases}$$

where $L = \{L_t\}_{t\geq 0}$ is a local time at the maximum (see [31, p. 140]). Recall that (L^{-1}, H) is a bivariate subordinator with the Laplace exponent $\kappa(\theta,\beta) = -\log \mathbb{E}\left(e^{-\theta L_1^{-1} - \beta H_1}\mathbf{1}_{\{1\leq L_\infty\}}\right)$ and with the Lévy measure denoted by Π_H. We define the descending ladder process $(\widehat{L}^{-1}, \widehat{H}) = \{(\widehat{L}_t^{-1}, \widehat{H}_t)\}_{t\geq 0}$ with the Laplace exponent $\widehat{\kappa}(\theta,\beta)$ constructed from the dual process $\widehat{X}$. Recall that $\widehat{L}_\infty$ has exponential distribution with parameter $\widehat{\kappa}(0,0)$ when $\widehat{\kappa}(0,0) > 0$ (see [31, p. 152]). Moreover, from the Wiener-Hopf factorization we have:

$$\kappa(\theta,\beta) = \Phi(\theta) + \beta, \qquad \widehat{\kappa}(\theta,\beta) = \frac{\theta - \psi(\beta)}{\Phi(\theta) - \beta};$$

see [31, p. 169-170]. Hence $\widehat{\kappa}(0,0) = \psi'(0+)$.

We introduce the potential measure $\mathcal{U}$ defined by

$$\mathcal{U}(dx, ds) = \int_0^\infty \mathbb{P}(L_t^{-1} \in ds, H_t \in dx)dt$$

supported on $[0,\infty)^2$, with the Laplace transform $\int_{[0,\infty)^2} e^{-\theta s - \beta x}\mathcal{U}(dx, ds) = 1/\kappa(\theta,\beta)$ and the renewal measure:

$$U(dx) = \int_{[0,\infty)} \mathcal{U}(dx, ds) = \mathbb{E}\left(\int_0^\infty \mathbf{1}_{\{H_t \in dx\}}dt\right).$$

It is konwn (see, e.g. [31]) that for a spectrally negative Lévy process the renewal measure equals: $U(dx) = dx$. Moreover the renewal measure $\widehat{U}(dz)$ for $(\widehat{L}^{-1}, \widehat{H})$ is characterized by:

$$\int_0^\infty e^{-\theta z}\widehat{U}(dz) = \frac{\theta}{\psi(\theta)};$$

see [31, p. 195].

Parisian ruin probability for spectrally negative Lévy processes (paper [H1])

Studies on the Parisian ruin for any spectrally negative Lévy process began with [13], where for the Parisian ruin probability two separate formulas were obtained, depending on the path variation of the considered process.

Theorem 1 (Theorem 1 in [13]). *The Parisian ruin probability for any spectrally negative Lévy processes equals:*

$$\begin{aligned}\mathbb{P}_x(\tau^d < \infty) &= \mathbb{P}_x(\tau_0^- < \infty)\mathbb{P}(\tau^d < \infty)\\ &\quad + \left(1 - \mathbb{P}(\tau^d < \infty)\right)\int_0^\infty \mathbb{P}(\tau_z^+ > d)\mathbb{P}_x(\tau_0^- < \infty, -X_{\tau_0^-} \in dz)\end{aligned} \tag{5}$$

and

$$\begin{aligned}&\int_0^\infty e^{-\theta s}\int_0^\infty \mathbb{P}(\tau_z^+ > s)\mathbb{P}_x(\tau_0^- < \infty, -X_{\tau_0^-} \in dz)\, ds\\ &= \frac{1 - \psi'(0+)W(x)}{\theta} - \frac{1}{\theta}e^{\Phi(\theta)x}\left(Z_{\Phi(\theta)}^{(-\theta)}(x) + \frac{\theta}{\Phi(-\theta)}W_{\Phi(\theta)}^{(-\theta)}(x)\right),\end{aligned} \tag{6}$$

where $W^{(-\theta)}_{\Phi(\theta)} = e^{-\Phi(\theta)x}W(x)$ *and* $Z^{(-\theta)}_{\Phi(\theta)} = 1 - \theta\int_0^x e^{-\Phi(\theta)y}W(y)\,dy$ *are the scale functions defined with respect to* $\mathbb{P}^{\Phi(q)}$ *given by* (2).

To prove formula (5) we use the Strong Markov property. Then on the event $\{\tau^d = \infty\}$ we decompose the possible trajectory that goes below zero into two parts. The first one starts at the undershoot of 0 of size, say, $-z < 0$ visiting zero in a continuous way because of the spectral negativity of X in a shorter period than d. The second part starts at 0 after this excursion below 0. Then the right-hand side of the above equality is partially characterized by objects that are already known in the classical ruin theory for spectrally negative Lévy processes. These objects are the classical ruin probability and the Laplace transform of the first passage time above a given level. The only unknown term is the Parisian ruin probability $\mathbb{P}(\tau^d < \infty)$, when the process starts at $x = 0$. This probability is characterized below in Theorem 2.

First, for $\epsilon > 0$ denote by $p^+(s) = \mathbb{P}_\epsilon(\tau_0^- < s)$ the probability that the excursion of X above 0 is shorter than s. Let

$$\begin{aligned} p(s,t) &= \int_0^\infty \mathbb{P}(\tau^+_{z+\epsilon} \le t)\,\mathbb{P}_\epsilon(\tau_0^- < s, -X_{\tau_0^-} \in dz) \\ &\quad + \mathbb{P}(\tau^+_\epsilon \le t)\mathbb{P}_\epsilon(\tau_0^- < s, X_{\tau_0^-} = 0) \end{aligned}$$

be the probability that the upper excursion above 0 is shorter than s and that the first consecutive excursion below 0, which is shifted downward by $-\epsilon$, is shorter then than t. Note that $p^+(s) = p(s,\infty)$. Then we can write the following theorem.

Theorem 2 (Theorem 2 in [13]). *(i) If X is a process of bounded variation, then*

$$\mathbb{P}(\tau^d < \infty) = \frac{\int_0^\infty \mathbb{P}(\tau_z^+ > d)\mathbb{P}(\tau_0^- < \infty, -X_{\tau_0^-} \in dz)}{1 - \rho + \int_0^\infty \mathbb{P}(\tau_z^+ > d)\mathbb{P}(\tau_0^- < \infty, -X_{\tau_0^-} \in dz)},$$

where $\rho = \mathbb{P}(\tau_0^- < \infty)$ *and*

$$\int_0^\infty e^{-\theta s}\int_0^\infty \mathbb{P}(\tau_z^+ > s)\mathbb{P}(\tau_0^- < \infty, -X_{\tau_0^-} \in dz)\,ds = \frac{1}{p}\left(\frac{1}{\Phi(\theta)} - \frac{\psi'(0+)}{\theta}\right).$$

(ii) If X is a process of unbounded variation, then

$$\mathbb{P}(\tau^d < \infty) = \lim_{b\to\infty}\lim_{\epsilon\downarrow 0}\frac{p^+(b) - p(b,d)}{1 - p(b,d)}. \tag{7}$$

The above result was improved in [H1], because the Laplace transform (6) was inverted, whereby a compact formula for Parisian ruin probability for any spectrally negative Lévy process was obtained.

Theorem 3 (Theorem 1 in [H1]). *Let $d > 0$. Assume that $\mathbb{E}[X_1] = \psi'(0+) > 0$. Then for any $x \in \mathbb{R}$,*

$$\mathbb{P}_x(\tau^d < \infty) = 1 - \mathbb{E}[X_1]\frac{\int_0^\infty W(x+z)z\mathbb{P}(X_d \in \mathrm{d}z)}{\int_0^\infty z\mathbb{P}(X_d \in \mathrm{d}z)}. \tag{8}$$

The formula (8) for Parisian ruin probability involves only the distribution of the process X at time d and its scale function, what can be calculated explicitly for certain classes of Lévy processes.

The proof of the above theorem is multi-staged. First of all, the key is following lemma, which can be proved using Kendall's identity (see [7] Corollary VII.3) i.e. the measures $r\mathbb{P}(\tau_z^+ \in dr)$ and $z\mathbb{P}(X_r \in dz)$ coincide on $[0,\infty)^2$.

Lemma 1 (Lemma 2 in [H1]). *For $\theta > 0$,*

$$\begin{aligned}\mathbb{E}_x\left[\mathbf{1}_{\{\tau_0^- < \infty\}} \mathrm{e}^{\Phi(\theta) X_{\tau_0^-}}\right] =& \frac{\theta}{\Phi(\theta)} \int_0^\infty \mathrm{e}^{-\Phi(\theta)y} W'(x+y)\mathrm{d}y,\\ \int_0^\infty \mathrm{e}^{-\theta r} \int_y^\infty \frac{z}{r}\mathbb{P}(X_r \in \mathrm{d}z)\mathrm{d}r =& \frac{1}{\Phi(\theta)}\mathrm{e}^{-\Phi(\theta)y}, \quad y \geq 0,\\ \int_0^\infty W(z)\frac{z}{r}\mathbb{P}(X_r \in \mathrm{d}z) =& 1.\end{aligned}$$

Then, the proof for bounded variation processes is provided and next for unbounded variation processes. For the bounded variation case the proof is simpler and is mostly based on the strong Markov property and spatial homogeneity of Lévy processes. In this case, at first, the formula for the Parisian ruin probability is obtained when $x = 0$ and then for any $x \in \mathbb{R}$. However, for the case of unbounded processes, the proof of the main theorem of [H1] is more complicated and requires the use of a limiting argument. It results from the fact that in this case 0 is regular for $(-\infty, 0)$ (i.e. $\mathbb{P}(\tau_0^- = 0) = 1$), which together with (4) gives that $W(0) = 0$, and hence the formula for the Parisian ruin probability when $x = 0$ requires an approximation approach. Therefore in this case, for $\epsilon \geq 0$ define the stopping time τ_ϵ^d by

$$\tau_\epsilon^d = \inf\{t > d : t - g_t^\epsilon > d, X_{t-d} < 0\}, \quad \text{where } g_t^\epsilon = \sup\{0 \leq s \leq t : X_s \geq \epsilon\}.$$

The stopping time τ_ϵ^d is the first time that an excursion starting when X gets below zero, ending before X gets back up to ϵ and of length greater than d, has occurred. Next, for the stopping moment τ_ϵ^d, using a similar reasoning as in the bounded variation case we get the formula for $\mathbb{P}_\epsilon(\tau_\epsilon^d < \infty)$ (in [H1] formula (13)). Finally, taking $\epsilon \downarrow 0$ we get the statement of Theorem 3.

Parisian ruin probability with a lower ultimate bankrupt barrier (paper [H2])

Another natural question in the context of ruin theory concerns the probability of Parisian ruin with a lower ultimate bankrupt barrier, set at a certain level $-a < 0$. As it was written in the introduction, this model extends Parisian ruin concept and it is interesting both for applications and theory, because it is more realistic model and, moreover, it is a more complex problem from a mathematical point of view.

Let's define the following stopping moment:

$$\kappa^{d,a} = \min(\tau^d, \tau_{-a}^-),$$

where $\tau_{-a}^- = \inf\{t \geq 0 : X(t) < -a\}$. The goal of [H2] was to express the formula for Parisian ruin probability with a lower ultimate bankrupt barrier $\mathbb{P}_x(\kappa^{d,a} < \infty)$ using already known objects from the theory of Lévy processes, such as scale functions. Thus, in [H2] the following formulas were obtained for the Parisian ruin probability with a lower ultimate bankrupt barrier:

Theorem 4 (Theorem 1 in [H2]). *For $x > 0$:*

$$\mathbb{P}_x(\kappa^{d,a} < \infty)$$
$$= \mathbb{P}_x(\tau_0^- < \infty) - \mathbb{P}(\kappa^{d,a} = \infty)\left(\int_0^a \mathbb{P}_{a-z}(\tau_a^+ < \min(d, \tau_0^-))\mathbb{P}_x(\tau_0^- < \infty, -X_{\tau_0^-} \in dz) + \frac{\sigma^2}{2}W'(x)\right)$$

and

$$\mathbb{P}_x(\tau_0^- < \infty) = 1 - \psi'(0+)W(x),$$

$$\int_0^\infty e^{-\theta s}\int_0^a \mathbb{P}_{a-z}(\tau_a^+ < \min(s, \tau_0^-))\mathbb{P}_x(\tau_0^- < \infty, -X_{\tau_0^-} \in dz)\, ds$$
$$= \frac{1}{\theta W^{(\theta)}(a)}\int_0^a W^{(\theta)}(a-z)\int_0^x \widehat{U}(x-dy)\int_0^\infty \Pi_{\widehat{X}}(dz+v+y)dv$$
$$= \frac{1}{\theta W^{(\theta)}(a)}\int_0^a W^{(\theta)}(a-z)\int_0^x \Pi_X(-dz-y)\left(W(x)-W(x-y)\right)dy.$$

Theorem 5 (Theorem 2 in [H2]). *(i) If process X has bounded variation paths, then*

$$\mathbb{P}(\kappa^{d,a} < \infty) = \frac{\mathbb{P}(\tau_0^- < \infty) - \int_0^a \mathbb{P}_{a-z}(\tau_a^+ < \min(d, \tau_0^-))\mathbb{P}(\tau_0^- < \infty, -X_{\tau_0^-} \in dz)}{1 - \int_0^a \mathbb{P}_{a-z}(\tau_a^+ < \min(d, \tau_0^-))\mathbb{P}(\tau_0^- < \infty, -X_{\tau_0^-} \in dz)},$$

where

$$\int_0^\infty e^{-\theta s}\int_0^a \mathbb{P}_{a-z}(\tau_a^+ < \min(s, \tau_0^-))\mathbb{P}(\tau_0^- < \infty, -X_{\tau_0^-} \in dz)\, ds$$
$$= \frac{1}{p\theta W^{(\theta)}(a)}\int_0^a W^{(\theta)}(a-z)\Pi_X(z,\infty)dz.$$

(ii) If process X has unbounded variation paths, then

$$\mathbb{P}(\kappa^{d,a} < \infty) = \lim_{\epsilon\downarrow 0}\frac{p^+(\epsilon) - p(d,\epsilon)}{1 - p(d,\epsilon)},$$

where $p^+(\epsilon) = \mathbb{P}_\epsilon(\tau_0^- < \infty)$ and

$$p(t,\epsilon) = \int_0^a \mathbb{P}_{a-z}(\tau_{a+\epsilon}^+ < \min(t, \tau_0^-))\,\mathbb{P}_\epsilon(\tau_0^- < \infty, -X_{\tau_0^-} \in dz)$$
$$+\mathbb{P}_a(\tau_{a+\epsilon}^+ < \min(t, \tau_0^-))\frac{\sigma^2}{2}W'(\epsilon)$$

is the probability that the first excursion starting when X gets below zero (or to zero), but above $-a$, ending before X gets back up to ϵ and of length shorter than t, has occurred.

The above two theorems were proved by the strong Markov property, properties of scale functions and spatial homogeneity of Lévy processes. In addition, for Brownian motion with drift $X_t = x + pt + \sigma B_t$ the above probability was obtained explicitly (Example 6.2 in [H2]). Moreover, the Cramér–Lundberg process with exponential jumps (1) and the Cramér-Lundberg process with exponential jumps perturbed by a Brownian motion were also considered (Examples 6.1 and 6.3 in [H2]). For this particular examples the numerical analysis was provided using *fast Fourier transformation* method (FFT) followed by the so-called "ϵ-algorithm" to speed up the convergence of infinite complex Fourier series. Detailed analysis with conclusions is available in [H2] in section Examples.

An important part of this paper are the sections about Cramér-type (light-tailed) and convolution-equivalent (heavy-tailed) asymptotics of the Parisian ruin probability with a lower ultimate bankrupt barrier when the reserves $X_0 = x$ tends to infinity.

In the light-tailed case (Section 4 in [H2]) assume Cramér's conditions i.e. that there exists $\gamma > 0$ satisfying

$$\widehat{\psi}(\gamma) = \psi(-\gamma) = 0 \tag{9}$$

and that $\widehat{\psi}(\gamma)$ is finite in the neighborhood of γ. Then $\mathbb{E}e^{-\gamma X_1} < \infty$ and we can define the measure $\mathbb{P}^{-\gamma}$ via (2) (see [8]). Define $\widehat{U}_\gamma(dx) = \widehat{U}_\gamma^{(0)}(dx) := e^{\gamma x}\widehat{U}(dx)$ and

$$\mu = \int_0^\infty x\widehat{U}_\gamma^{(1)}(dx),$$

where $\widehat{U}_\gamma^{(q)}(dx) = \int_0^\infty e^{-(qt+\gamma x)}\mathbb{P}(\widehat{H}_t \in dx)dt$ for $q \geq 0$. Note from [8] that $\widehat{U}_\gamma$ is a renewal function of the ladder height process calculated on $\mathbb{P}^{-\gamma}$. Moreover, from [8] we have:

$$\mu = \frac{\partial \widehat{\kappa}_{-\gamma}(0,\beta)}{\partial \beta}|_{\beta=0} = \frac{\partial \widehat{\kappa}(0,\beta)}{\partial \beta}|_{\beta=-\gamma}.$$

Now, we can present the main result of Chapter 4 of [H2], which derives the exponential asymptotics of the Parisian ruin probability with a lower barrier.

Theorem 6 (Theorem 3 in [H2])**.** *Assume Cramér's conditions* (9) *and that the support of* $\widehat{\Pi}$ *is not lattice when* $\widehat{\Pi}(\mathbb{R}) < \infty$*. We have*

$$\lim_{x\uparrow\infty} e^{\gamma x}\mathbb{P}_x(\kappa^{d,a} < \infty) \;=\; \frac{\widehat{\kappa}(0,0)}{\gamma\mu} - \big(1 - \mathbb{P}(\kappa^{d,a} < \infty)\big)\, f^{(c)}(d,a), \tag{10}$$

where

$$\int_0^\infty e^{-\theta s} f^{(c)}(s,a)\, ds = \frac{1}{\mu\theta W^{(\theta)}(a)} \int_0^a W^{(\theta)}(a-z) \int_0^\infty e^{\gamma y} \int_0^\infty \Pi_{\widehat{X}}(dz+\upsilon+y)d\upsilon dy$$

and $\mathbb{P}(\kappa^{d,a} < \infty)$ *is given in Theorem 5. If* $\mu = \infty$*, then left-hand side of (10) is understood to be* 0.

To prove the above theorem we use the Key Renewal theorem saying that $\widehat{U}_\gamma(dx)$ on $(0,\infty)$ converges weakly as a measure to $\mu^{-1}dx$ (see [31, p. 188] and [8]). Moreover, to obtain asymptotics of $e^{\gamma x}\mathbb{P}(\widehat{X}_{\tau_x^+} - x \in dz)$, when $x \uparrow \infty$ we apply Lebesgue dominated convergence.

The results given in Section 5 of [H2] deal with spectrally negative Lévy process X for which the tail of Lévy measure of its dual process belongs to the class $\mathcal{S}^{(\alpha)}$. It means that we allow that the process $\widehat{X}$ may have heavy-tailed jumps. Formally, the class $\mathcal{S}^{(\alpha)}$ is defined as follows (see [29]):

Definition **3.** *(Class $\mathcal{L}^{(\alpha)}$) For a parameter $\alpha \geq 0$ we say that measure G on $[0, \infty)$ with tail $\overline{G} = G([0, \infty))$ belongs to class $\mathcal{L}^{(\alpha)}$ if*

(i) $\overline{G}(x) > 0$ *for each* $x \geq 0$,

(ii) $\lim_{u \to \infty} \frac{\overline{G}(u-x)}{\overline{G}(u)} = e^{\alpha x}$ *for each* $x \in \mathbb{R}$, *and* G *is non-lattice,*

(iii) $\lim_{n \to \infty} \frac{\overline{G}(n-1)}{\overline{G}(n)} = e^{\alpha}$ *if* G *is lattice (then assumed of span* 1*).*

Definition **4.** *(Class $\mathcal{S}^{(\alpha)}$) We say that G belongs to class $\mathcal{S}^{(\alpha)}$ if*

(i) $G \in \mathcal{L}^{(\alpha)}$,

(ii) for some $M_0 < \infty$, *we have*

$$\lim_{u \to \infty} \frac{\overline{G * G}(u)}{\overline{G}(u)} = 2M_0, \tag{11}$$

where $\overline{G * G}(u) = 1 - G * G(u)$ *and* $*$ *denotes convolution.*

For all $a \in \mathbb{R}$ such that the following integral is finite we define the moment generating function δ such that

$$\delta_a(G) = \int_0^\infty e^{au} G(du), \tag{12}$$

where G is a distribution function of a finite measure.

Recall that $\widehat{X} = -X$ is a spectrally positive Lévy process. Throughout this section we assume that for $\widehat{X}$ and for some fixed $\alpha \geq 0$ we have:

(i)

$$\overline{\Pi}_{\widehat{X}} \in \mathcal{S}^{(\alpha)} \quad \text{if} \quad \alpha > 0 \tag{13}$$

and

$$\int_0^x \overline{\Pi}_{\widehat{X}}(y)\, dy \in \mathcal{S}^{(0)} \tag{14}$$

(ii)

$$\widehat{\varphi}(\alpha) < 0 \quad \text{if} \quad \alpha > 0 \tag{15}$$

(iii)

$$e^{-q}\delta_\alpha(\widehat{H}) < 1, \quad \text{where} \quad q = \lim_{\beta \downarrow 0} \frac{-\widehat{\varphi}(-\beta)}{\kappa(0, -\beta)}. \tag{16}$$

Above $\delta_\alpha(\widehat{H})$ denotes the moment generating function (12) of the distribution function of $\widehat{H}_1$. The first condition gives $\overline{\Pi}_{\widehat{H}} \in \mathcal{S}^{(\alpha)}$. The latter condition (iii) is necessary when $\alpha > 0$; by the drift assumption for $\alpha = 0$ this condition is automatically satisfied. An example of a process that fulfills the conditions (13)-(16) is the dual process to Cramér–Lundberg process (1) with Pareto distributed jumps.

We denote: $f(x) \sim g(x)$ iff $\lim_{x\to\infty} f(x)/g(x) = 1$. Then the main result of Section 5 in [H2] is as follows

Theorem 7 (Theorem 4 in [H2]). *Under assumptions (13)-(16) the asymptotic Parisian ruin probability equals:*

$$\mathbb{P}_x(\kappa^{d,a} < \infty) \sim \mathbb{E}X_1 \left(\frac{\alpha}{\widehat{\psi}(\alpha)}\right)^2 \big(1 - (1 - \mathbb{P}(\kappa^{d,a} < \infty))f^{(e)}(d,a)\big) \int_x^\infty \overline{\Pi}_{\widehat{X}}(y)\, dy,$$

where

$$\int_0^\infty e^{-\theta s} f^{(e)}(s,a)\, ds = \frac{1}{\theta W^{(\theta)}(a)} \int_0^a W^{(\theta)}(a-z)B(z)dz \tag{17}$$

and

$$B(z) = \frac{e^{-\alpha z}}{EX_1}\left(-\widehat{\psi}(\alpha) + \alpha \int_z^\infty e^{\alpha y}\overline{\Pi}_{\widehat{X}}(y)\, dy\right).$$

Above for $\alpha = 0$ the quantity $-\widehat{\psi}(\alpha)/\alpha$ is understood in limiting sense and equals $-\widehat{\psi}'(0+) = \mathbb{E}X_1$.

Parisian ruin for a refracted Lévy process (paper [H3])

The next paper included in the scientific achievement is [H3], which gives formula for Parisian ruin probability and fluctuation identities for refracted Lévy process considered with Parisian delay. In [33] refracted Lévy process was described in detail for the first time. The authors proved its existence, the strong Markov property, found formulas for the one-sided and the two-sided exit problems and gave formulas for resolvents. Formally, a refracted Lévy process is defined as a strong unique solution of the following stochastic differential equation (SDE):

$$\mathrm{d}U_t = \mathrm{d}X_t - \delta \mathbf{1}_{\{U_t > b\}}\mathrm{d}t, \quad t \geq 0, \tag{18}$$

where $\delta, b \geq 0$ and X is a spectrally Lévy process, which models the dynamics of the process U under level b. In addition, from the above SDE, it is easy to see that above the level b process U evolves as $Y = \{Y_t = X_t - \delta t, t \geq 0\}$. The process Y is also a Lévy process and has the same Lévy measure as X, but different drift. Let's denote by $\mathbb{W}^{(q)}$ and φ the scale function and the Laplace exponent of the process Y, respectively. In the case when $\delta = 0$ we get that $U = X$, which gives that all fluctuation identities previously unknown in the literature obtained for the process U considered until the Parisian ruin moment are automatically solved for any spectrally negative Lévy process X. The results presented in [33] required the introduction of a new scale function $w^{(q)}$ associated with the process U and defined as follows

Definition 5. *For $q \geq 0$ and $x, z \in \mathbb{R}$, $b \geq 0$, define*

$$w^{(q)}(x;z) := W^{(q)}(x-z) + \delta \mathbf{1}_{\{x>b\}} \int_b^x \mathbb{W}^{(q)}(x-y) W^{(q)'}(y-z) dy, \tag{19}$$

where $\mathbb{W}^{(q)}$ is the scale function of Y.

Note that when $x \leq b$, we have

$$w^{(q)}(x;z) = W^{(q)}(x-z).$$

For $q = 0$, we will write $w^{(0)}(x;z) = w(x;z)$.

In [H3] is assumed that $b = 0$. Note that we could interpret this change in the premium rate as a way to invest (for R&D, modernization, etc.) or to pay an income tax: if the surplus of the company is in a good financial situation, i.e. above the *critical level* 0, then it invests or pays at rate $\delta > 0$; otherwise it does not.

The following results were obtained in [H3].

Let

$$\tau_U^d = \inf\{t > 0\colon t - \sup\{0 \leq s \leq t\colon U_s \geq 0\} > d\}$$

be the Parisian ruin moment of U. Then the Parisian ruin probability is given by

Theorem 8 (Theorem 2 in [H3])**.** *For $x \in \mathbb{R}$,*

$$\mathbb{P}_x\left(\tau_U^d < \infty\right) = 1 - (\mathbb{E}[X_1] - \delta)_+ \frac{\int_0^\infty w(x;-z) z \mathbb{P}(X_d \in dz)}{\int_0^\infty z \mathbb{P}(X_d \in dz) - \delta d}. \tag{20}$$

One can easily see that for $\delta = 0$ the above formula matches with the result obtained in Theorem 1 in [H1]. The above theorem is proved by the Kendall identity and the strong Markov property, which gives that the decomposition of the trajectory of the process U leads us to get fluctuation identities associated with the first passage times of the processes X and Y to given levels. These formulas are presented in the following lemma. Note that the case when U has unbounded variation paths must be considered separately. We define the stopping time $\tau_U^{d,\epsilon} = \inf\{t > 0\colon t - \sup\{0 \leq s \leq t\colon U_s \geq \epsilon\} > d\}$, and then the Parisian ruin probability is approximated by $\mathbb{P}_\epsilon(\tau_U^{d,\epsilon} < \infty)$, when $\epsilon \downarrow 0$.

Lemma 2 (Lemma 8 in [H3])**.** *Let $\nu_0^- = \inf\{t > 0 : Y_t < 0\}$ and $\nu_a^+ = \inf\{t > 0 : Y_t > a\}$. Then for $x \in \mathbb{R}$, $a, q \geq 0$, we get*

$$\mathbb{E}_x\left[\mathbb{P}_{Y_{\nu_0^-}}\left(\tau_0^+ \leq d\right) \mathbf{1}_{\{\nu_0^- < \infty\}}\right] = \int_0^\infty \left(w(x;-z) - \mathbb{W}(x)\right) \frac{z}{d} \mathbb{P}(X_d \in dz) + \delta \mathbb{W}(x),$$

$$\mathbb{E}_x\left[e^{-q\nu_0^-} \mathbb{E}_{Y_{\nu_0^-}}\left[e^{-q\tau_0^+} \mathbf{1}_{\{\tau_0^+ \leq d\}}\right] \mathbf{1}_{\{\nu_0^- < \nu_a^+\}}\right]$$
$$= \int_0^\infty e^{-qd} \left(w^{(q)}(x;-z) - \frac{\mathbb{W}^{(q)}(x)}{\mathbb{W}^{(q)}(a)} w^{(q)}(a;-z)\right) \frac{z}{d} \mathbb{P}(X_d \in dz)$$

and

$$\mathbb{E}_x\left[e^{-q\nu_0^-}\mathbb{P}_{Y_{\nu_0^-}}(\tau_0^+\leq d)\mathbf{1}_{\{\nu_0^-<\nu_a^+\}}\right]$$
$$=\int_0^\infty\left(\mathcal{W}_{x,\delta}^{(q,-q)}(x+z)-\frac{\mathbb{W}^{(q)}(x)}{\mathbb{W}^{(q)}(a)}\mathcal{W}_{a,\delta}^{(q,-q)}(a+z)\right)\frac{z}{d}\mathbb{P}\left(X_d\in dz\right),$$

where for any $x,c\in\mathbb{R}$

$$\mathcal{W}_{c,\delta}^{(q,-q)}(x)=W(x)+\int_0^c\left(qW(x-y)-\delta W'(x-y)\right)\mathbb{W}^{(q)}(y)\,dy. \tag{21}$$

The proof techniques used in [H3] are based on the fact that depending on its position the process U behaves like the process X or the process Y. Therefore, the above lemma was proved most of all using of the strong Markov property and the spatial homogeneity of Lévy processes X and Y. Moreover, the important result of [H3] is the following theorem, which gives formulas for the two-sided and the one-sided exit problems for the process U considered until the Parisian ruin. The following results are expressed using the scale functions associated with X, Y, U, and the distribution of X at time d.

Theorem 9 (Theorem 4 in [H3])**.** *Let $\kappa_a^+=\inf\{t>0:U_t>a\}$ and $d>0$, then the Parisian exit problems for refracted Lévy process U are given by*

(i) For $q>0$, $x\leq a$,

$$\mathbb{E}_x\left[\mathrm{e}^{-q(\tau_U^d-d)}\mathbf{1}_{\{\tau_U^d<\kappa_a^+\}}\right]$$
$$=\mathbb{Z}^{(q)}(x)+\int_0^\infty\left(w^{(q)}(x;-z)\,\mathbb{E}\left[e^{-q\tau_U^d}\mathbf{1}_{\{\kappa_d^U<\kappa_a^+\}}\right]-\mathcal{W}_{x,\delta}^{(q,-q)}(x+z)\right)\frac{z}{d}\mathbb{P}\left(X_d\in dz\right),$$

where

$$\mathbb{E}\left[e^{-q\tau_U^d}\mathbf{1}_{(\tau_U^d<\kappa_a^+)}\right]=1-\frac{\mathbb{Z}^{(q)}(a)+\int_0^\infty\left(w^{(q)}(a;-z)-\mathcal{W}_{a,\delta}^{(q,-q)}(a+z)\right)\frac{z}{d}\mathbb{P}\left(X_d\in dz\right)}{\int_0^\infty w^{(q)}(a;-z)\,\frac{z}{d}\mathbb{P}\left(X_d\in dz\right)}$$
$$=\frac{\int_0^\infty\mathcal{W}_{a,\delta}^{(q,-q)}(a+z)\,\frac{z}{d}\mathbb{P}\left(X_d\in dz\right)-\mathbb{Z}^{(q)}(a)}{\int_0^\infty w^{(q)}(a;-z)\,\frac{z}{d}\mathbb{P}\left(X_d\in dz\right)}.$$

(ii) For $q>0$ and $x\in\mathbb{R}$,

$$\mathbb{E}_x\left[\mathrm{e}^{-q(\tau_U^d-d)}\mathbf{1}_{\{\tau_U^d<\infty\}}\right]$$
$$=\mathbb{Z}^{(q)}(x)+\int_0^\infty\left(w^{(q)}(x;-z)\,\mathbb{E}\left[e^{-q\tau_U^d}\mathbf{1}_{\{\tau_U^d<\infty\}}\right]-\mathcal{W}_{x,\delta}^{(q,-q)}(x+z)\right)\frac{z}{d}\mathbb{P}\left(X_d\in dz\right),$$

where

$$\mathbb{E}\left[\mathrm{e}^{-q(\tau_U^d-d)}\mathbf{1}_{\{\tau_U^d<\infty\}}\right]=\frac{\int_0^\infty\mathcal{H}_\delta^{(q,-q)}(z)\frac{z}{d}\mathbb{P}\left(X_d\in dz\right)-\frac{q}{\psi(q)}-\delta}{\int_0^\infty\mathcal{H}_\delta^{(q,0)}(z)\frac{z}{d}\mathbb{P}\left(X_d\in dz\right)-\delta e^{qd}}$$

and for $p, p+q \geq 0$

$$\mathcal{H}_\delta^{(p,q)}(x) = e^{\varphi(p)x}\left(1 + (q - \delta\varphi(p))\int_0^x e^{-\varphi(p)y} W^{(p+q)}(y)dy\right).$$

(iii) For $q \geq 0$ *and* $x \leq a$*,*

$$\mathbb{E}_x\left[e^{-q\kappa_a^+}\mathbf{1}_{\{\kappa_a^+ < \tau_U^d\}}\right] = \frac{\int_0^\infty w^{(q)}(x;-z)\frac{z}{d}\mathbb{P}(X_d \in dz)}{\int_0^\infty w^{(q)}(a;-z)\frac{z}{d}\mathbb{P}(X_d \in dz)}.$$

The proofs of each sub-items of the above theorem are technical and quite complicated. They use the identities given in Lemma 2 and the strong Markov property. For the unbounded variation case we use an approximation argument.

Fluctuation theory for level-dependent Lévy processes (paper [H4])

The last part of the description of the scientific achievement is devoted to level-dependent Lévy processes, which are Feller processes defined as the strong unique solution of the following stochastic differential equation (SDE):

$$dU(t) = dX(t) - \phi(U(t))\, dt, \tag{22}$$

where $X(t)$ is a spectrally negative Lévy process. Additionally, we assume that the function ϕ is non-decreasing and for fixed $r^* \in \mathbb{R}$, $\phi(x) = 0$ for $x \leq r^*$. In the bounded variation case, we assume that $\phi(x) \leq p = \gamma + \int_{(-1,0)} x\Pi(dx) > 0$ for all $x \in \mathbb{R}$. This is a technical assumption that has been used to construct the solution of (22). Furthermore ϕ is either locally Lipschitz continuous or ϕ is of the form

$$\phi(x) = \phi_k(x) = \sum_{j=1}^{k} \delta_j \mathbf{1}_{\{x \geq b_j\}},$$

where $k \geq 1$, $0 < \delta_1, \ldots, \delta_k$, and $-\infty < b_1 < \ldots < b_k < \infty$. The above conditions were denoted in [H4] as assumption **[A]**.

For $\phi = \phi_k$ the process $U = U_k$ is a so-called multi-refracted Lévy process and it is a direct, but nontrivial, extension of the case $k = 1$, considered in [33]. Observe that for any $0 \leq j \leq k$ in each level interval $(b_j, b_{j+1}]$ (where $b_0 := -\infty$ and $b_{k+1} := \infty$) the process U_k evolves as $X_j := \{X(t) - \sum_{i=1}^{j}\delta_i t : t \geq 0\}$, which is a spectrally negative Lévy process that is not the negative of a subordinator, because we assumed that that $\phi_k(x) \leq p$ for all $x \in \mathbb{R}$. The Laplace exponent of X_j is given by

$$\psi_j(\theta) := \psi(\theta) - (\delta_1 + \ldots + \delta_j)\theta \quad \text{for} \quad \theta \geq 0$$

with right-inverse

$$\varphi_j(q) = \sup\{\theta \geq 0 \mid \psi_j(\theta) = q\}.$$

The paper [H4] consists of two parts. The first one concerns multi-refracted Lévy processes. In Theorem 1 of [H4] it has been proved that when ϕ is a step function, then there exists strong a unique solution to (22), which possess the strong Markov property.

Theorem 10 (Theorem 1 in [H4]). *Let $k \geq 1$, $\phi(x) = \phi_k(x) = \sum_{j=1}^{k} \delta_j \mathbf{1}_{\{x \geq b_j\}}$, $0 < \delta_1, \dots, \delta_k$, and $-\infty < b_1 < \dots < b_k < \infty$. Then there exists a strong solution U_k to the SDE* (22).

At first we provide a pathwise solution to (22) when the driving Lévy process X is of bounded variation. It means that we construct the process U_k satisfying the equation (22) at time intervals during which this process is above or below the highest level b_k. We prove the result by induction. First, the base case ($k = 1$) holds by [33]. Next we assume that process U_{k-1} it is well defined at time intervals where it is between levels $-\infty < b_1 < \dots < b_{k-1} < \infty$ and then we show that U_k is also well defined at all levels $-\infty < b_1 < \dots < b_k < \infty$. The above construction holds only for bounded variation processes X, because in this case b_k is irregular for $(-\infty, b_k)$. Moreover, it is important that process U_k goes upward continuously and down only with jumps (here we use the assumption that $\phi_k(x) \leq p = \gamma + \int_{(-1,0)} x\Pi(dx) > 0$ for all $x \in \mathbb{R}$).

The case when the process X and hence when the process U_k has unbounded variation is proved using the approximating sequences X^n and U_k^n, where the processes X^n, U_k^n have bounded variation paths. This is due to the fact that any spectrally negative Lévy process, which has unbounded variation paths, can be approximated by a sequence of bounded variation spectrally negative Lévy processes (see [7, p. 210]).

Moreover, in Section 2 of [H4] fluctuation identities and resolvents were obtained for process U_k.

Definition **6.** *For any $0 \leq k$, $-\infty =: b_0 < b_1 < \cdots < b_k < b_{k+1} := \infty$, and $y \in \mathbb{R}$, define*

$$\Xi_{\phi_k}(y) := 1 - W^{(q)}(0)\phi_k(y).$$

For the unbounded variation case, we note that the fact that $W^{(q)}(0) = 0$ implies that $\Xi_{\phi_k}(y) = 1$ for all $y \in \mathbb{R}$. On the other hand, in the bounded variation case with $y \in (b_i, b_{i+1}]$ and $i \leq k - 1$, we have that

$$\Xi_{\phi_k}(y) = 1 - W^{(q)}(0) \sum_{j=1}^{i} \delta_j = \prod_{j=1}^{i} \left(1 - \delta_j W_{j-1}^{(q)}(0)\right),$$

and similarly for $y > b_k$, we obtain $\Xi_{\phi_k}(y) = \prod_{j=1}^{k} \left(1 - \delta_j W_{j-1}^{(q)}(0)\right)$.

For $a \in \mathbb{R}$ and $k \geq 1$, define the following first-passage stopping times

$$\kappa_k^{a,-} := \inf\{t > 0 \colon U_k(t) < a\} \quad \text{oraz} \quad \kappa_k^{a,+} := \inf\{t > 0 \colon U_k(t) \geq a\},$$

with the convention that $\inf \emptyset = \infty$.

In the following theorem, we present formulas for two-sided exit problems for multi-refracted process U_k. Right-hand sides of these identities are represented by new scale functions given by some recursive integral equations. The recursion can be solved and an explicit formula can be obtained for these scale functions listed below (see Proposition 13 in [H4]).

Theorem 11 (Theorem 5 in [H4]). *Fix $k \geq 1$ and $q \geq 0$.*

(i) For $r < b_1 < \cdots < b_k \leq a$ and $r \leq x \leq a$, we have

$$\mathbb{E}_x \left[\mathrm{e}^{-q\kappa_k^{a,+}} \mathbf{1}_{\{\kappa_k^{a,+} < \kappa_k^{r,-}\}} \right] = \frac{w_k^{(q)}(x; r)}{w_k^{(q)}(a; r)}, \tag{23}$$

where $w_k^{(q)}$ *is defined by the recursion*

$$w_k^{(q)}(x;r) := w_{k-1}^{(q)}(x;r) + \delta_k \int_{b_k}^{x} W_k^{(q)}(x-y) w_{k-1}^{(q)\prime}(y;r)dy. \tag{24}$$

with initial function $w_0^{(q)}(x;r) = W^{(q)}(x-r)$. *The function* $w_k^{(q)}(x;r)$ *is the scale function associated with the process* U_k.

(ii) For $0 < b_1 < \cdots < b_k \le a$ *and* $0 \le x \le a$,

$$\mathbb{E}_x\left[\mathrm{e}^{-q\kappa_k^{0,-}} \mathbf{1}_{\{\kappa_k^{0,-}<\kappa_k^{a,+}\}}\right] = z_k^{(q)}(x) - \frac{z_k^{(q)}(a)}{w_k^{(q)}(a)} w_k^{(q)}(x), \tag{25}$$

where $z_k^{(q)}$ *is defined by the recursion*

$$z_k^{(q)}(x) := z_{k-1}^{(q)}(x) + \delta_k \int_{b_k}^{x} W_k^{(q)}(x-y) z_{k-1}^{(q)\prime}(y)dy.$$

with initial function $z_0^{(q)}(x) = Z^{(q)}(x)$. *The function* $z_k^{(q)}(x)$ *is the scale function associated with the process* U_k.

The proof of the above theorem consists of two parts. At first, the formula (23) was proved in case the process X has bounded variation paths. It can be showed by using the strong Markov property, and then determine (23) when $x = b_k$. Such reasoning is appropriate only for bounded variation process, because in this case $W^{(q)}(x - b_k) = W^{(q)}(0) > 0$. The proof for the unbounded variation case is more complicated and requires the use of approximation sequences. The remaining fluctuation identities that were obtained in this section for the multi-refracted process U_k are the one-sided exit problems and resolvents. One-sided exit problems give the Laplace transform of the first-exit time of the process U_k from the half line $[0, \infty)$ (point *(i)* in the following corollary) and from $(-\infty, a)$, for some fixed $a \in \mathbb{R}$ (point *(ii)* in the following corollary). These identities are expressed by scale functions related to the process U_k and specified recursively.

Corollary **7** (Corollary 6 in [H4]). *Fix* $k \ge 1$.

(i) For $x \ge 0$, $b_1 > 0$ *and* $q > 0$, *we have*

$$\mathbb{E}_x\left[\mathrm{e}^{-q\kappa_k^{0,-}} \mathbf{1}_{\{\kappa_k^{0,-}<\infty\}}\right] = z_k^{(q)}(x) - \frac{\int_{b_k}^{\infty} e^{-\varphi_k(q)z} z_{k-1}^{(q)\prime}(z)dz}{\int_{b_k}^{\infty} e^{-\varphi_k(q)z} w_{k-1}^{(q)\prime}(z)dz} w_k^{(q)}(x). \tag{26}$$

(ii) For $b_1 < \cdots < b_k \le a$, $x \le a$ *and* $q \ge 0$,

$$\mathbb{E}_x\left[\mathrm{e}^{-q\kappa_k^{a,+}} \mathbf{1}_{\{\kappa_k^{a,+}<\infty\}}\right] = \frac{u_k^{(q)}(x)}{u_k^{(q)}(a)}. \tag{27}$$

Here, $u_k^{(q)}$ is defined by the recursion

$$u_k^{(q)}(x) := u_{k-1}^{(q)}(x) + \delta_k \int_{b_k}^{x} W_k^{(q)}(x-y) u_{k-1}^{(q)\prime}(y) dy.$$

with initial function $u_0^{(q)}(x) = e^{\Phi(q)x}$. The function $u_k^{(q)}(x)$ is the scale function associated with the process U_k.

Resolvents or q-potential measures give expressions for the expected occupation measure of process in a given Borel set over its entire lifetime as well as when time is restricted up to the first passage times. Such expected occupation measures play an important role in fluctuation theory and in applications, for example to determine some characteristics in optimal dividend models (see e.g. [14, 35]).

Theorem 12 (Theorem 7 in [H4]). *Fix a Borel set $\mathcal{B} \subseteq \mathbb{R}$ and $k \geq 1$.*

(i) For $r < b_1 < \cdots < b_k \leq a$, $r \leq x \leq a$ and $q \geq 0$,

$$\mathbb{E}_x \left[\int_0^{\kappa_k^{a,+} \wedge \kappa_k^{r,-}} e^{-qt} \mathbf{1}_{\{U_k(t) \in \mathcal{B}\}} dt \right] = \int_{\mathcal{B} \cap (r,a)} \frac{\frac{w_k^{(q)}(x;r)}{w_k^{(q)}(a;r)} w_k^{(q)}(a;y) - w_k^{(q)}(x;y)}{\Xi_{\phi_k}(y)} dy, \tag{28}$$

where the scale function $w_k^{(q)}(x;r)$ is defined in Theorem 11 (i).

(ii) For $x \geq 0$, $b_1 > 0$ and $q > 0$,

$$\mathbb{E}_x \left[\int_0^{\kappa_k^{0,-}} e^{-qt} \mathbf{1}_{\{U_k(t) \in \mathcal{B}\}} dt \right] = \int_{\mathcal{B} \cap (0,\infty)} \frac{\frac{w_k^{(q)}(x)}{v_k^{(q)}(0)} v_k^{(q)}(y) - w_k^{(q)}(x;y)}{\Xi_{\phi_k}(y)} dy, \tag{29}$$

where $v_k^{(q)}(y) := \delta_k \int_{b_k}^{\infty} e^{-\varphi_k(q)z} w_{k-1}^{(q)\prime}(z;y) dz$, and the scale function $w_k^{(q)}(x;z)$ is defined in Theorem 11 (i).

(iii) For $x, b_k \leq a$ and $q \geq 0$,

$$\mathbb{E}_x \left[\int_0^{\kappa_k^{a,+}} e^{-qt} \mathbf{1}_{\{U_k(t) \in \mathcal{B}\}} dt \right] = \int_{\mathcal{B} \cap (-\infty,a)} \frac{\frac{u_k^{(q)}(x)}{u_k^{(q)}(a)} w_k^{(q)}(a;y) - w_k^{(q)}(x;y)}{\Xi_{\phi_k}(y)} dy, \tag{30}$$

where the functions $w_k^{(q)}(x;y)$ and $u_k^{(q)}(x)$ are defined in Theorems 11 and 7, respectively.

(iv) For $x \in \mathbb{R}$ and $q > 0$,

$$\mathbb{E}_x \left[\int_0^{\infty} e^{-qt} \mathbf{1}_{\{U_k(t) \in \mathcal{B}\}} dt \right] = \int_{\mathcal{B}} \frac{\frac{u_k^{(q)}(x) \int_{b_k}^{\infty} e^{-\varphi_k(q)z} w_{k-1}^{(q)\prime}(z;y) dz}{\int_{b_k}^{\infty} e^{-\varphi_k(q)z} u_{k-1}^{(q)\prime}(z) dz} - w_k^{(q)}(x;y)}{\Xi_{\phi_k}(y)} dy, \tag{31}$$

Proof techniques for the above results are similar to those used in the Theorem 11. Moreover, we get Corollary 7 directly from Theorem 11 by taking the appropriate limits: $a \to \infty$ in (25) and $r \to -\infty$ in (23).

In Subsection 2.3 of [H4], we summarized the properties of the multi-refracted scale function $w_k^{(q)}(x; z)$ such as its behaviour at zero and at infinity, monotonicity and smoothness. Moreover, it was crucial for the second part of [H4] that the scale functions $w_k^{(q)}(x; r)$, $z_k^{(q)}(x)$ and $u_k^{(q)}(x)$ fulfil the following integral equations:

Proposition 8 (Section 2.4 in [H4])**.** *Suppose that $\phi_k(x) = \sum_{i=1}^{k} \delta_i \mathbf{1}_{\{x>b_i\}}$ and $q \geq 0$, $k \geq 1$. Then, for $x, r \in \mathbb{R}$, $z < b_1$ functions $w_k^{(q)}(x; r)$, $z_k^{(q)}(x)$ and $u_k^{(q)}(x)$ are unique solutions to the following equations:*

$$w_k^{(q)}(x; r) = W^{(q)}(x - r) + \int_d^x W^{(q)}(x - y)\phi_k(y) w_k^{(q)\prime}(y; r)\, dy,$$

$$z_k^{(q)}(x) = Z^{(q)}(x) + \int_0^x W^{(q)}(x - y)\phi_k(y) z_k^{(q)\prime}(y) dy,$$

$$u_k^{(q)}(x) = e^{\Phi(q)x} + \int_0^x W^{(q)}(x - y)\phi_k(y) u_k^{(q)\prime}(y) dy.$$

Indeed, the above proposition is important, because thanks to the above observation the analogous equations were obtained for scale functions for a general class of functions ϕ (that fulfil the condition **[A]**).

In Section 3 of [H4], the theory of multi-refracted processes was extended to the solutions of (22) with a general premium rate function ϕ. The existence and uniqueness of these solutions were proved using the already developed theory for the multi-refracted case. To this end, we approximated a general rate function ϕ by a sequence of rate functions $(\phi_n)_{n\geq 1}$ that satisfy the following conditions

(a) $\lim_{n\to\infty} \phi_n = \phi$ uniformly on compact sets.

(b) For $x \in \mathbb{R}$,

$$\phi_1(x) \leq \phi_2(x) \leq \ldots \leq \phi(x).$$

(c) For each $n \geq 1$ and $x \in \mathbb{R}$, we have that $\phi_n(x) = \sum_{j=1}^{m_n} \delta_j^n \mathbf{1}_{\{x>b_j^n\}}$, for some $m_n \in \mathbb{N}$, $0 < b_1^n < \ldots < b_{m_n}^n$, and $\delta_j^n > 0$ with $j = 1, \ldots, m_n$.

For each $n \geq 1$ we denote the solution of (22) with the rate function ϕ_n by U_n. We now show how to construct a specific sequence $(\phi_n)_{n\geq 1}$ that satisfies the conditions mentioned above. For each $n \geq 1$ we choose a grid $\Pi^n = \{b_l^n = l2^{-n} : l = 1, \ldots, m_n = n2^n\}$ and set $\delta_j^n = \phi(b_j^n) - \phi(b_{j-1}^n)$, with $b_0^n = 0$. Furthermore we define the *approximating sequence* of the rate function ϕ as follows:

$$\phi_n(x) = \sum_{j=1}^{m_n} \delta_j^n \mathbf{1}_{\{x>b_j^n\}} \qquad \text{dla } n \geq 1 \text{ oraz } x \in \mathbb{R}.$$

For any $n \geq 1$, we have from Theorem 1 in [H4] that there exists a unique solution U_n to (22) with the rate function ϕ_n. Moreover, the following lemma implies that the sequence $(U_n(t))_{n\geq 1}$ is non-decreasing for any $t \geq 0$.

Lemma 3 (Lemma 20 in [H4]). *Suppose that for each $n \geq 1$, $\phi_n(x) \leq \phi_{n+1}(x)$ for all $x \in \mathbb{R}$. Then $U_{n+1}(t) \leq U_n(t)$ for all $t \geq 0$.*

Then using the above mentioned lemma it was proved that

Proposition 9 (Proposition 21 in [H4]). *Suppose that the rate function ϕ satisfies condition **[A]**. Then, there exists a unique solution U to the SDE (22) with rate function ϕ. Furthermore, the sequence $(U_n)_{n\geq 1}$ converges uniformly to U a.s. on compact time intervals.*

The next step was to show that the scale function $w^{(q)}$ associated with the general process U satisfies the following integral equation:

$$w^{(q)}(x) = W^{(q)}(x) + \int_0^x W^{(q)}(x-y)\phi(y)w^{(q)\prime}(y)dy, \tag{32}$$

where $W^{(q)}$ is the scale function of the driving Lévy process X.

Proof techniques used in this part of [H4] are related to the general theory of Volterra integral equations:

$$u(x) = g(x) + \int_0^x K(x,y)u(y)\, dy, \tag{33}$$

where the continuity of g and K is not required and the presented theory is in the spirit of L^1 kernels as in Chapter 9.2 of [26]. It was showed in Lemma 22 of [H4] that if we assume that a measurable function u is differentiable almost everywhere, then u is the solution to

$$u(x;r) = W^{(q)}(x-r) + \int_r^x W^{(q)}(x-y)\phi(y)u'(y;r)\, dy$$

if and only if u' is the solution to the following Volterra integral equation

$$u'(x;r) = \Xi_\phi(x)^{-1}W^{(q)\prime}((x-r)+) + \int_r^x \Xi_\phi(x)^{-1}\phi(y)W^{(q)\prime}(x-y)u'(y;r)\, dy, \tag{34}$$

with boundary condition $u(r;r) = W^{(q)}(0)$. Next, using Picard's iterations we construct the solution to (34) and then we determine a majorant for the Neumann series, which in our case is an infinite series of convolution of the kernels

$$K(x,y) = \Xi_\phi(x)^{-1}\phi(y)W^{(q)\prime}(x-y).$$

The next step was to show that the scale functions of general level-dependent process can be approximated by the ones for the multi-refracted case. Then Theorem 27 in [H4] gives that for each $x \geq r$, we have

$$\lim_{n\to\infty} w_n^{(q)\prime}(x;r) = w^{(q)\prime}(x;r) \qquad \text{and} \qquad \lim_{n\to\infty} z_n^{(q)\prime}(x) = z^{(q)\prime}(x).$$

Thanks to the above approximation in the last subsection of [H4] we obtained fluctuaction identities for the general level-dependent process.

Let $a \in \mathbb{R}$ and define the following first-passage stopping times for the level-dependent process:

$$\kappa^{a,-} := \inf\{t > 0\colon U(t) < a\} \quad \text{and} \quad \kappa^{a,+} := \inf\{t > 0\colon U(t) \geq a\}.$$

In the following theorem we derive formulas for resolvents. Notice that (36) for $\phi = \phi_k$ is consistent with (29), however we do not have a more explicit formula for $c^{(q)}(y;r)$ in a general case.

Theorem 13 (Theorem 30 in [H4]). *Fix a Borel set $\mathcal{B} \subseteq \mathbb{R}$, then*

(i) For $q \geq 0$ and $r \leq x \leq a$,

$$\mathbb{E}_x\left[\int_0^{\kappa^{a,+}\wedge\kappa^{r,-}} e^{-qt}\mathbf{1}_{\{U(t)\in\mathcal{B}\}}\,dt\right] = \int_{\mathcal{B}\cap(r,a)} \Xi_\phi(y)^{-1}\left(\frac{w^{(q)}(x;r)}{w^{(q)}(a;r)}w^{(q)}(a;y) - w^{(q)}(x;y)\right)\,dy. \tag{35}$$

(ii) For $q > 0$ and $x \geq 0$, there exists $c^{(q)}(y;r) > 0$ such that $w^{(q)}(x;r)c^{(q)}(y;r) - w^{(q)}(x;y) \geq 0$ and

$$\mathbb{E}_x\left[\int_0^{\kappa^{r,-}} e^{-qt}\mathbf{1}_{\{U(t)\in\mathcal{B}\}}dt\right] = \int_{\mathcal{B}\cap(0,\infty)} \Xi_\phi(y)^{-1}\left(c^{(q)}(y;r)w^{(q)}(x;r) - w^{(q)}(x;y)\right)dy. \tag{36}$$

(iii) For $q \geq 0$ and $x \leq a$,

$$\mathbb{E}_x\left[\int_0^{\kappa^{a,+}} e^{-qt}\mathbf{1}_{\{U(t)\in\mathcal{B}\}}dt\right] = \int_{\mathcal{B}\cap(-\infty,a)} \Xi_\phi(y)^{-1}\left(\frac{u^{(q)}(x)}{u^{(q)}(a)}w^{(q)}(a;y) - w^{(q)}(x;y)\right)dy, \tag{37}$$

where $u^{(q)}(x) = e^{\Phi(q)x} + \int_0^x \phi(y)W^{(q)}(x-y)u^{(q)\prime}(y)\,dy$.

In addition, in [H4] the two-sided exit problems were found for the process U with a general function ϕ fulfilling the condition **[A]**. The right-hand sides of the following identities are represented by scale functions that fulfil Volterra type integral equations.

Theorem 14 (Theorem 32 in [H4]). *(i) For $r \leq x \leq a$ and $q \geq 0$,*

$$\mathbb{E}_x\left[\mathrm{e}^{-q\kappa^{a,+}}\mathbf{1}_{\{\kappa^{a,+}<\kappa^{r,-}\}}\right] = \frac{w^{(q)}(x;r)}{w^{(q)}(a;r)}. \tag{38}$$

(ii) For $0 \leq x \leq a$ and $q \geq 0$,

$$\mathbb{E}_x\left[\mathrm{e}^{-q\kappa^{0,-}}\mathbf{1}_{\{\kappa^{0,-}<\kappa^{a,+}\}}\right] = z^{(q)}(x) - \frac{z^{(q)}(a)}{w^{(q)}(a)}w^{(q)}(x), \tag{39}$$

where $z^{(q)}(x) = Z^{(q)}(x) + \int_0^x \phi(y)W^{(q)}(x-y)z^{(q)\prime}(y)\,dy$.

Finally, the ruin probability for the general process U was also computed in [H4]. Notice that from Theorem 14 (ii) for $q = 0$, we get

$$\rho(x) := \mathbb{P}_x\left(\kappa^{0,-} < \infty\right) = 1 - \lim_{a\to\infty}\frac{w(x)}{w(a)}.$$

Then one can prove the following lemma.

Lemma 4 (Proposition 34 in [H4]). *Assume that $x \geq 0$.*

(i) If $\mathbb{E}[X(1)] \leq 0$*, then* $\rho(x) = 1$ *for all* $x \geq 0$*.*

(ii) If $\mathbb{E}[X(1)] > 0$ *and* $\int_0^\infty \phi(x)w'(x)\,dx$ *exists, then the ruin function*

$$\rho(x) = 1 - A^{-1}w(x),$$

where

$$A = \frac{1 + \int_0^\infty \phi(x)w'(x)\,dx}{\mathbb{E}[X(1)]}$$

and ρ *satisfies the following integral equation:*

$$\rho(x) = 1 - A^{-1}W(x) + \int_0^x W(x-y)\phi(y)\rho'(y)dy.$$

Moreover, when $\int_0^\infty \phi(x)w'(x)\,dx = \infty$ *it follows that* $A = \infty$*, and hence* $\rho(x) = 1$ *for all* $x \geq 0$*.*

Above we presented the most important results of [H1]-[H4], that form the scientific achievement. Summarizing, the main results of [H1]-[H3] give fluctuation identities for Lévy processes and refracted Lévy processes considered with Parisian delay, the paper [H4] shows the existence and fluctuations of the level-dependent processes, including the multi-refracted case.

5. Description of other scientific achievements

Besides the four papers, which constitute mono-thematic series of publications, after Ph.D., I published seven articles, one was accepted and now is waiting for publication since August 2018. The total number of my publications (together with the articles before PhD) is 15. My papers were published with co-authors from various research centres around the world. The number of citations, according to the Web of Science database ('Sum of the Times Cited' on 2019-04-12), is 105 (93 without self-citations), and the h-index (Hirsh index) is 4. Total *impact factor* of the journals for five publications included in the *scientific achievement*, according to the Journal Citation Reports, is 4,959; total *impact factor* of the journals for all publications is 13,573, see Table 1.

Table 1: Impact factor of the journals according to Journal Citation Report from the publication year (or 2018 for publication from 2019).

article	journal	publication year	impact factor
[H1]	Bernoulli	2013	1,296
[H2]	Scandinavian Actuarial Journal	2016	1,347
[H3]	Insurance: Mathematics and Economics	2017	1,265
[H4]	Stochastic Processes and their Applications	2019	1,051 (2018)
[P1]	Journal of Optimization Theory and Applications	2014	1,509
[P2]	Statistics and Probability Letters	2016	0,540
[P3]	Scandinavian Actuarial Journal	2017	1,550
[P4]	Statistics and Probability Letters	2017	0,533
[P5]	Computational and Applied Mathematics	2017	1,632
[P6]	Probability and Mathematical Statistics	2018*	0,286 (2018)
[P7]	Insurance: Mathematics and Economics.	2018	1,265
[D1]	Stochastic Models	2011	0,667
[D2]	Journal of Applied Probability	2011	0,632
[D3]	Research Papers of Wrocław University of Economics	2011	–
[D4]	Research Papers of Wrocław University of Economics	2011	–
		Sum:	13,573

* - article [P6] was accepted for publication in August 2018.

[P1] I. Czarna, Z. Palmowski, *Dividend problem with Parisian delay for a spectrally negative Lévy risk process*, Journal of Optimization Theory and Applications. 2014, vol. 161, no. 1, p. 239–256.

[P2] I. Czarna, J-F. Renaud, *A note on Parisian ruin with an ultimate bankruptcy level for Lévy insurance risk processes,* Statistics and Probability Letters. 2016, vol. 113, p. 54–61.

[P3] I. Czarna, Z. Palmowski, P. Świątek, *Discrete time ruin probability with Parisian delay*, Scandinavian Actuarial Journal. 2017, vol. 2017, no. 10, p. 854–869.

[P4] I. Czarna, Z. Palmowski, *Parisian quasi-stationary distributions for asymmetric Lévy processes*, Statistics and Probability Letters. 2017, vol. 127, p. 75–84.

[P5] I. Czarna, Y. Li, Z. Palmowski, C. Zhao, *The joint distribution of the Parisian ruin time and the number of claims until Parisian ruin in the classical risk model*, Computational and Applied Mathematics. 2017, vol. 313, p. 499-514.

[P6] I. Czarna, Y. Li, Z. Palmowski, C. Zhao, *Optimal Parisian-type dividend payments penalized by the number of claims for the classical and perturbed classical risk process.*, Probability and Mathematical Statistics. 2018.

[P7] I. Czarna, J.-L. Pérez, K. Yamazaki, *Optimality of multi-refraction dividend strategies in the dual model*, Insurance: Mathematics and Economics. 2018, vol. 83, p. 148–160.

Before PhD I published the following four papers, which will not be discussed here.

[D1] I. Czarna, Z. Palmowski, *Ruin probability with Parisian delay for a spectrally negative Lévy risk process*, Journal of Applied Probability. 2011, vol. 48, no. 4, p. 984–1002.

[D2] I. Czarna, Z. Palmowski, *De Finetti's dividend problem and impulse control for a two-dimensional insurance risk process*, Stochastic Models. 2011, vol. 27, no. 2, p. 220–250.

[D3] I. Czarna, Z. Palmowski, *Porównanie prawdopodobieństw paryskiej i klasycznej ruiny dla procesu ryzyka typu Lévy'ego*, Research Papers of Wrocław University of Economics. 2011, nr 207, s. 9–21.

[D4] I. Czarna, Z. Palmowski, *Problem wyboru optymalnej paryskiej dywidendy dla procesu ryzyka typu Lévy'ego : numeryczna analiza*, Research Papers of Wrocław University of Economics. 2011, nr 207, s. 22–37.

I will now discuss the results obtained in the papers [P1]-[P7]. Papers [P1]-[P6] concern topics related to the Parisian delay used in various models. While the paper [P7] shows that the *multi-refracted* process (for $k = 2$) is the solution to the optimal dividend problem assuming that dividend strategies in this model are absolutely continuous with respect to the Lebesgue measure. For clarity, the description is divided into three sections relating to my main research interests.

5.1 Parisian dividends (Papers [P1], [P6])
Both papers are about optimal dividends which are paid until Parisian ruin. Additionally, in [P6] these dividends are discounted by a new factor $r \in (0; 1]$ related to the total number of claims which arrived in a compound Poisson process up to the Parisian ruin moment $\tau^{\pi,d}$ (i.e. $r^{N_{\tau^{\pi,d}}}$). While in [P1] we considered the case, when $r = 1$. Formally, the results obtained in [P1] and [P6] consist in maximizing the following integral functional

$$\sup_{\pi\in\Pi} \mathbb{E}_x \left[r^{N_{\tau^{\pi,d}}} \int_0^{\tau^{\pi,d}} e^{-qt} dL_t^\pi \right],$$

where $q > 0$ and Π represents a set of all admissible strategies such that $\pi = \{L_t^\pi : t \geq 0\}$ is a non-decreasing left-continuous adapted process which starts at zero. Thus L_t^π represents the cumulative dividends paid out by the company up till time t. Formally, we consider the risk process controlled by the dividend policy π given

$$U_t^\pi = X_t - L_t^\pi \tag{40}$$

In [P1], the process X is any spectrally negative Lévy process, but in [P6] the Cramér–Lundberg process perturbed by a Brownian motion $\{B_t\}_{t\geq 0}$ is considered. For fixed $d \geq 0$, for process U^π define the Parisian ruin moment:

$$\tau^{\pi,d} := \inf\{t > 0 : t - \sup\{s < t : U_s^\pi \geq 0\} > d, U_t^\pi < 0\}.$$

In both papers the dividends were paid according to the barrier strategy that corresponds to reducing the risk process $U^{\pi^{a,a}}$ to the level a if $U^{\pi^{a,d}}(0) = x > a$, by paying out the amount $x - a$, and subsequently paying out the minimal amount of dividends to keep the risk process below the level a (see Fig. 2). It is well known (see [5]) that for $0 < x \leq a$ the corresponding controlled risk process $U^{\pi^{a,d}}$ under $\mathbb{P}_x$ is equal in law to the process $\{a - Y_t : t \geq 0\}$ under $\mathbb{P}_x$ for

$$Y_t = \max(a, \overline{X}_t) - X_t$$

being the Lévy process X reflected at its past supremum: $\overline{X}_t = \sup_{0\leq s\leq t} X_s$. In [P1] we considered the case, when $r = 1$, so in this case for all $x \geq 0$,

$$v^{a,d}(x) := v^{\pi^{a,d}}(x) = \mathbb{E}_x\left(\int_0^{\tau^{\pi^{a,d}}} e^{-qt} dL_t^{\pi^{a,d}}\right)$$

and $L_t^{\pi^{a,d}} = \max(a, \overline{X}_t) - a$.

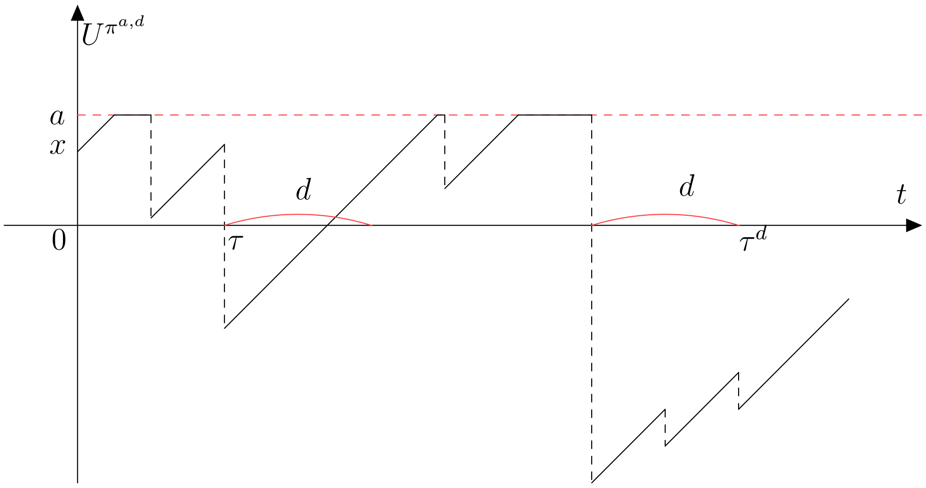

Fig. 2. A sample path of a regulated surplus process $U^{\pi^{a,d}}$.

Then, in [P1] it was proved that

$$v^{a,d}(x) = \begin{cases} \frac{V^{(q)}(x)}{V^{(q)\prime}(a)}, & x \leq a, \\ \\ x - a + \frac{V^{(q)}(a)}{V^{(q)\prime}(a)}, & x > a, \end{cases} \tag{41}$$

where $V^{(q)}(x) = \int_0^\infty W^{(q)}(x+z)z\mathbb{P}(X_d \in \mathrm{d}z)$. In particular,

$$(v^{a,d})'(a) = 1.$$

Hence we get the following theorem.

Theorem 15 (Theorem 4.1 in [P1]). *The value function $v^{a,d}$ corresponding to the barrier strategy $\pi^{a,d}$ is given by (41). Then from (41) it follows that the optimal barrier a^* satisfies:*

$$a^* = \inf\{a > 0 : V^{(q)\prime}(a) \leq V^{(q)\prime}(y) \quad \textit{for all } y \geq 0\}.$$

In particular, if $V^{(q)} \in \mathcal{C}^2(\mathbb{R})$ and there exists unique solution to the equation:

$$V^{(q)\prime\prime}(a^*) = 0,$$

then a^* is the optimal barrier.

Next step was to prove that the barrier strategy $\pi^{a^*,d}$ is optimal across all admissible strategies Π considered up till Parisian ruin moment $\tau^{\pi,d}$ (Theorem 5.2 in [P1]). To prove it, we used *verification lemma* (see, [41, 42, 43]), which assumptions require checking that the value function $v^{a^*,d}$ given by (41) is q-superharmonic and its first derivative is greater than one, i.e. it fulfils the following conditions:

$$\begin{aligned} (\Gamma - q)v^{a^*,d}(x) &\leq 0, \quad \text{if} \quad x \in \mathbb{R}, \\ (v^{a^*,d})'(x) &\geq 1, \quad \text{if} \quad x \in \mathbb{R}, \end{aligned}$$

where Γ is the infinitesimal generator of process X. Moreover, one can give another necessary condition for the barrier strategy to be optimal.

Corollary **10** (Corollary 5.1 in [P1]). *Suppose*

$$V^{(q)\prime}(a) \leq V^{(q)\prime}(b), \qquad \textit{for all } a^* \leq a \leq b.$$

Then the barrier strategy at a^* is an optimal strategy.

In [P6], similarly like in [P1], it was proved that the dividend strategy $\pi^{a^*,d}$ is optimal. The main difference is that, in [P6] the considered process is of the form

$$X_t = x + pt - \sum_{i=1}^{N_t} C_i + \sigma B_t,$$

and we maximize the following integral functional, for $0 < r < 1$ and $q > 0$:

$$v^{a,d,r}(x) := v^{\pi^{a,d}}(x) = \mathbb{E}_x \left[r^{N_{\tau^{\pi,d}}} \int_0^{\tau^{\pi^{a,d}}} e^{-qt} dL_t^{\pi^{a,d}} \right]. \tag{42}$$

In this case, the formula for the value function $v^{a,d,r}$ was also derived:

$$v^{a,d,r}(x) = \begin{cases} h^d(x) v^{a,d}(a) & \text{for } x \le a, \\ x - a + v^{a,d}(a) & \text{for } x > a, \end{cases} \tag{43}$$

where $h^d(x) := \mathbb{E}_x \left[r^{N_{\tau_a^+}} e^{-q\tau_a^+}, \tau_a^+ < \tau^d \right]$ is the expected value of the first moment of reaching level a before the Parisian ruin, discounted in time and additionally by the number of jumps of the process N.

We used two different methods to obtain an expression for the function h^d. The first one uses the Dickson-Hipp operator $(T_\rho f)$, which is a generalization of the Laplace transform (see [19]). Then we obtain the following theorem.

Theorem 16 (Theorem 3.8 in [P6]). *Let f be the density of the random variable C_i for all $i = 1, 2, \ldots$. Then the function h^d can be expressed as follows*

1. *For $\sigma = d = 0$ and $0 \le x \le a$,*

$$h^d(x) = h(x) = \frac{\sum_{n=0}^{\infty} (\frac{\lambda r}{p})^n (T_\rho f)^{*n} * \zeta(x)}{\sum_{n=0}^{\infty} (\frac{\lambda r}{p})^n (T_\rho f)^{*n} * \zeta(a)}. \tag{44}$$

2. *For $\sigma > 0$, $d = 0$ and $0 < x \le a$,*

$$h^d(x) = h(x) = \frac{\sum_{n=0}^{\infty} (\frac{2\lambda r}{\sigma^2})^n (\beta * T_\rho f)^{*n} * \zeta * \beta(x)}{\sum_{n=0}^{\infty} (\frac{2\lambda r}{\sigma^2})^n (\beta * T_\rho f)^{*n} * \zeta * \beta(a)}. \tag{45}$$

3. *For $\sigma = 0$, $d > 0$ and $-pd \le x \le a$,*

$$h^d(x) = \frac{\sum_{n=0}^{\infty} (\frac{2\lambda r}{p})^n (T_\rho f)^{*n} * \varphi(x)}{\sum_{n=0}^{\infty} (\frac{2\lambda r}{p})^n (T_\rho f)^{*n} * \varphi(a)}. \tag{46}$$

4. *For $\sigma, d > 0$ and $x \le a$,*

$$h^d(x) = \frac{\sum_{n=0}^{\infty} (\frac{2\lambda r}{\sigma^2})^n (\beta * T_\rho f)^{*n} * \varphi_1(x)}{\sum_{n=0}^{\infty} (\frac{2\lambda r}{\sigma^2})^n (\beta * T_\rho f)^{*n} * \varphi_1(a)}, \tag{47}$$

where

$$\beta(x) := e^{-(\rho + \frac{2p}{\sigma^2})x}, \quad \zeta(x) := e^{\rho x}, \quad \varphi(x) := \zeta(x) - \frac{\lambda r}{p} \zeta * w_d(x),$$

ρ is the unique non-negative root of the Lundberg fundamental equation:

$$\frac{\sigma^2}{2}s^2 + ps - (\lambda + q) + \lambda r \int_0^\infty e^{-sx} f(x)dx = 0$$

and

$$\varphi_1(x) := \left(\rho + \frac{2p}{\sigma^2}\right) \zeta * \beta(x) + \beta(x) - \frac{2\lambda r}{\sigma^2} \zeta * \beta * w_d(x).$$

$$w_d(x) := \int_0^\infty \int_0^d e^{-qt} \sum_{k=0}^\infty r^k v_y(k,t) f(y+x) dt dy.$$

For $k \in \mathbb{N}$, $y > 0$, $t \geq 0$

$$v_y(k,t) = \frac{d}{dt}\mathbb{P}(N_{\tau_y^+} = k, \tau_y^+ \leq t | X_0 = 0).$$

The second method is based on the fluctuation theory for spectrally negative Lévy processes and an analogue of the exponential change of measure (2). Define

$$\left.\frac{d\mathbb{P}^{\alpha,r}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = r^{N_t} \exp\left(\alpha X_t - \psi_r(\alpha)t\right) = \exp\left(Z_t^\alpha - \psi_r(\alpha)t\right), \tag{48}$$

where $Z_t^\alpha := \alpha X_t + \log(r)N_t = \alpha x + \alpha pt - \sum_{i=1}^{N_t}(\alpha C_i - \log(r)) + \alpha\sigma B_t$. Then the following theorem was proved in [P6].

Theorem 17 (Theorem 3.9 in [P6])**.** *For any $\sigma \geq 0$ and $0 \leq x \leq a$, the function $h^d(x)$ can be expressed as follows:*

$$h^d(x) = \mathbb{E}_x\left[r^{N_{\tau_a^+}} e^{-q\tau_a^+}, \tau_a^+ < \tau^d\right] = \frac{\int_0^\infty W_r^{(q)}(x+z) z \mathbb{P}(X_d \in \mathrm{d}z)}{\int_0^\infty W_r^{(q)}(a+z) z \mathbb{P}(X_d \in \mathrm{d}z)}$$

and

$$h(x) = h^0(x) = \mathbb{E}_x\left[r^{N_{\tau_a^+}} e^{-q\tau_a^+}, \tau_a^+ < \tau^0\right] = \frac{W_r^{(q)}(x)}{W_r^{(q)}(a)}.$$

For $W_r^{(q)} : [0,\infty) \to [0,\infty)$ it holds that

$$\int_0^\infty e^{-\theta y} W_r^{(q)}(y) dy = \frac{1}{\psi_r(\theta) - q}, \qquad \theta > \Phi_r(q),$$

where $\psi_r(\theta) := \frac{1}{t}\log\mathbb{E}\left[e^{\theta X_t + \log(r)N_t}\right] = \frac{1}{t}\log\mathbb{E}\left[r^{N_t}e^{\theta X_t}\right]$ and $\Phi_r(q)$ is such that $\psi_r(\Phi_r(q)) = q$.

Finally, like in [P1], we proved the optimality of the barrier strategy $\pi^{a^*,d}$ with the value function additionally discounted by the number of jumps of the compound Poisson process.

5.2 Parisian ruin (Papers [P2]-[P5])

This is a series of papers considering Parisian ruin in various models. In [P2] we obtained fluctuation identities for spectrally negative Lévy process considered until Parisian ruin with a lower ultimate bankrupt barrier $-a < 0$. For example, such as exit identities from a given interval or a half-line. The obtained results were expressed by scale functions of the process X and their Laplace transforms.

In [P3] the following discrete model was considered:

$$R_n = u + n - S_n, \tag{49}$$

where $u > 0$ and

$$S_n = \sum_{i=1}^{n} C_i, \qquad n = 1, 2, 3, \ldots$$

We assume that random variables C_i $(i = 1, 2, \ldots)$ are i.i.d. and we also assume that the process drift equals to 1. We denote $\mathbb{P}(C_1 = k) = p_k$ for $k = 0, 1, 2, \ldots$, and we assume that $\mu = \mathbb{E}(C_1) < 1$, hence $R_n \to +\infty$ a.s. Define the Parisian ruin moment

$$\tau^d = \inf\{n \in \mathbb{N} : n - \sup\{s < n : R_s > 0\} > d, R_n \leq 0\},$$

for a fixed time period $d \in \{1, 2, \ldots\}$. The main results of this paper are expressions for the Parisian ruin probabilities $\mathbb{P}_u(\tau^d < t)$ and $\mathbb{P}_u(\tau^d < \infty)$. In [P3] these are the following theorems.

Theorem 18 (Theorem 1 in [P3])**.** *For $u \geq 1$, the recursive representation of the Parisian non-ruin probability until finite-time t is as follows.*
For $t \leq d + 1$ we have $\mathbb{P}_u(\tau^d \geq t) = 1$.
For $t \geq d + 2$:

$$\mathbb{P}_u(\tau^d \geq t) = \mathbb{P}_u(\tau^0 \geq t - d) + \sum_{s=1}^{t-d-1} \sum_{\omega=1}^{d} \sum_{z=0}^{\omega-1} \mathbb{P}_u(\tau^0 = s, -R_{\tau^0} = z)\mathbb{P}(\tau_{z+1} = \omega)\mathbb{P}_1(\tau^d \geq t - \omega - s),$$

where

$$\mathbb{P}(\tau_x = \omega) = \frac{x}{\omega}\mathbb{P}(R_\omega = x) = \frac{x}{\omega} p^{*\omega}_{\omega - x}$$

and $\{p^{*t}_n, n \in \mathbb{N}\}$ denotes the t-th convolution of the law of C_1.

The probabilities $\mathbb{P}_u(\tau^0 \geq t)$, $\mathbb{P}_u(\tau^0 = s, -R_{\tau^0} = z)$ are given in the following lemmas.

Lemma 5 (Lemma 1 in [P3])**.** *We have $\mathbb{P}_u(\tau^0 = 1) = \mathbb{P}(Y_1 \geq u + 1)$ and for $t \geq 1$:*

$$\mathbb{P}_u(\tau^0 \geq t + 1) = \sum_{j=0}^{u+t-1} p^{*t}_j - \sum_{j=u+1}^{u+t-1} p^{*(j-u)}_j \left(\sum_{k=j}^{u+t-1} \frac{t + u - k}{t + u - j} p^{*(t+u-j)}_{k-j} \right),$$

Lemma 6 (Lemma 2 in [P3])**.** *For $s \geq 1$ we have:*

$$\begin{aligned}\mathbb{P}_u(\tau^0 = s, -R_{\tau^0} = z) &= \sum_{k=0}^{u+s-2} \mathbb{P}_u(\tau^0 > s - 1, S_{s-1} = k) p_{u+s-k+z} \\ &= \sum_{k=0}^{u+s-2} p^{*(s-1)}_k p_{u+s-k+z} - \sum_{k=u+1}^{u+s-2} \sum_{j=u+1}^{k} \frac{s - 1 + u - k}{s - 1 + u - j} p^{*(s-1+u-j)}_{k-j} p^{*(j-u)}_j p_{u+s-k+z}.\end{aligned}$$

The representation of ultimate Parisian ruin probability is given by:

Theorem 19 (Theorem 2 in [P3]). *For $u \geq 1$ we have*

$$\mathbb{P}_u(\tau^d < \infty) = (1-\mu) \sum_{j=u+1}^{\infty} p_j^{*(j-u)} - \big(1 - \mathbb{P}_1(\tau^d < \infty)\big) \sum_{z=0}^{d-1} \mathbb{P}_u(\tau^0 < \infty, -R_{\tau^0} = z)\mathbb{P}(\tau_{z+1} \leq d),$$

where

$$\mathbb{P}_u(\tau^0 < \infty) = (1-\mu) \sum_{j=u+1}^{\infty} p_j^{*(j-u)}$$

and

$$\mathbb{P}_1(\tau^d < \infty) = \frac{\mathbb{P}_1(\tau^0 < \infty) - \sum_{z=0}^{d-1} \mathbb{P}_1(\tau^0 < \infty, -R_{\tau^0} = z)\mathbb{P}(\tau_{z+1} \leq d)}{1 - \sum_{z=0}^{d-1} \mathbb{P}_1(\tau^0 < \infty, -R_{\tau^0} = z)\mathbb{P}(\tau_{z+1} \leq d)}.$$

The above formulas are recursive so by choosing specific process parameters one can make a precise numerical analysis as in Chapter 6 of [P3], where heavy-tailed jumps for the process R were also considered.

Additionally, in [P3] both asymptotics heavy- and light-tailed were estimated for the Parisian ruin probability in the case of $u \to \infty$.

The paper [P4] concerns the so-called limiting quasi-stationary distribution conditioned with respect to the Parisian stopping time. This means that the main goal of [P4] was to determine the following limit (also known in literature as a Yaglom's limit):

$$\lim_{t\uparrow\infty} \mathbb{P}_x(X_t \in B | \tau^\theta > t) = \mu_x^\theta(B), \qquad B \in \mathcal{B}([0,\infty)), \tag{50}$$

where τ^θ is a Parisian stopping time, defined by

$$\tau^\theta = \inf\{t > 0 : t - \sup\{s < t : X_s \geq 0\} \geq e_\theta, X_t < 0\},$$

where e_θ is an exponential random variable independent of X with intensity $\theta > 0$.

The ruin time τ^θ happens when process X stays negative longer than e_θ, which we will refer as implementation clock. We want to emphasize that in the definition of τ^θ there is not a single underlying random variable but a whole sequence of independent copies of a generic exponential random variable e_θ each one of them attached to a separate excursion below zero. The model with exponentially distributed delay has been also studied by [6, 38]. In [P4] we focus on asymmetric Lévy processes, which are either spectrally negative (i.e. the Lévy measure is supported on $(-\infty, 0)$) or spectrally positive (i.e. the Lévy measure is supported on $(0, \infty)$). For both asymmetric Lévy processes the limit (50) was obtained. The idea of the proof of the main results is based on finding the double Laplace transform of $\mathbb{P}_x(X_t \in dy, \tau^\theta > t)$ with respect to space and time. Then for some specific form of the Lévy measure using tauberian theorems and 'Heavyside' operation we identified the asymptotics of this probability as $t \to \infty$ (Theorems 7 and 11 in [P4]).

In [P5] the Cramér-Lundberg process (1) is considered. For this model the joint distribution of the Parisian ruin time and the number of claims until Parisian ruin is obtained. We consider both finite and infinite time horizon. This approach with the classical ruin moment has already been considered in [20, 22, 37, 52]. Formally, for the Parisian ruin moment τ^d define

$$w_x^d(k,t) := \frac{d}{dt}\psi_x^d(k,t), \quad k \in \mathbb{N}, t \geq 0,$$

where

$$\psi_x^d(k,t) = \mathbb{P}(N_{\tau^d} = k, \tau^d \leq t | X(0) = x), \quad k \in \mathbb{N}, t \geq d.$$

Further, let $p_x^d(k)$ denotes the probability that there have been exactly k claims up to Parisian ruin, so that

$$p_x^d(k) \;=\; \mathbb{P}(N_{\tau^d} = k, \tau^d \leq \infty | X(0) = x) = \int_d^\infty w_x^d(k,t)dt.$$

The main result of [P5] are recursive expressions for the density $w_x^d(k,t)$ and the probability $p_x^d(k)$ (Theorems 2.3 and 2.7 in [P5]). To obtain the formulas we used the strong Markov property and also the form of the infinitesimal generator for the Cramér-Lundberg process. Next we got the integro-differential equation and then by solving it we got the expressions for $w_x^d(k,t)$ and $p_x^d(k)$. In the last part of paper, it was assumed that the jumps of the considered process were Erlang distributed. Then using a new iterative algorithm proposed in [P5], the above density and probability were found numerically for the selected process parameters.

5.3 Optimality of multi-refraction dividend strategies (Paper [P7])
Paper [P7] concerns the dividend payments in multi-refracted model and uses the fluctuation formulas obtained in [H4]. The Lévy process $X = \{X_t; t \geq 0\}$ models the surplus of a company in the absence of control. We assume that it is *spectrally positive* or equivalently it has no negative jumps and is not a subordinator. An admissible strategy $\pi := (L^\pi(t), R^\pi(t); t \geq 0)$ is a set of non-decreasing, right-continuous, and adapted processes such that $L^\pi(0-) = R^\pi(0-) = 0$. In addition, we consider $\delta_i > 0$ for $i = 1, 2$, and we require that L^π and R^π are absolutely continuous with respect the Lebesgue measure and are of the form $L^\pi(t) = \int_0^t l^\pi(s)ds$ $R^\pi(t) = \int_0^t r^\pi(s)ds$; $t \geq 0$, with l^π, r^π restricted to take values in $[0, \delta_1]$ and $[0, \delta_2]$ uniformly in time. We denote by $V^\pi = X_t - L^\pi(t) + R^\pi(t)$ the controlled process associated to the strategy π. L^π is the cumulative amount of dividends and R^π is that of injected capital, which are made to reduce the risk of ruin. Assuming that $\beta > 1$ is the cost per unit injected capital, $\rho \in \mathbb{R}$ the terminal payoff (if $\rho \geq 0$)/penalty (if $\rho \leq 0$) at ruin, and $q > 0$ the discount factor. Define the ruin moment $\tau_0^\pi := \inf\{t > 0 : V^\pi(t) < 0\}$ and the following integral functional

$$v_\pi(x) := \mathbb{E}_x\left[\int_0^{\tau_0^\pi} e^{-qt} l^\pi(t)dt - \beta\int_0^{\tau_0^\pi} e^{-qt} r^\pi(t)dt + \rho e^{-q\tau_0^\pi}\right], \quad x \geq 0. \tag{51}$$

Hence our aim is to compute $v(x) := \sup_{\pi\in\mathcal{A}} v_\pi(x)$, where $\mathcal{A}$ is the set of all admissible strategies that satisfy the constraints described above. It was proved in [P7] that a multi-refracted process (for $k = 2$), which was previously defined and characterized in [H4], is a solution of the above problem, i.e. under a multi-refraction strategy, the resulting process becomes precisely a multi-refracted process.

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