Summary of professional achievements

1 Name and surname

Ilona Iglewska-Nowak

2 Scientific degrees

2002	MSc in mathematics, Institute of Mathematics, University of Potsdam (Germany), Master thesis *Wavelet correlation dimension and its applications*, supervisor: Prof. Dr. Matthias Holschneider
2007	Doctor of natural sciences, Institute of Mathematics, University of Potsdam (Germany), Doctoral thesis *Poisson wavelet frames on the sphere*, supervisor: Prof. Dr. Matthias Holschneider
2008	recognition of the degree as Doctor of mathematics, Faculty of Mathematics and Information Technology, Adam Mickiewicz University in Poznań

3 Information on previous employment in scientific institutions

Feb. 2002 – July 2007	Scientific Assistant in the Department of Applied Mathematics, Institute of Mathematics, University of Potsdam (Germany)
Oct. 2007 – Sep. 2008	Assistant Professor in the Department of Quantitative Methods, West Pomeranian Business School in Szczecin
Oct. 2008 – present	Assistant Professor in School of Mathematics, West Pomeranian University of Technology, Szczecin (until Dec. 2008 Technical University of Szczecin)

4 The indication of the scientific achievement

(a) The title of the scientific achievement

The theory of continuous wavelet transforms on n-dimensional spheres and discrete wavelet frames for $\mathcal{L}^2(\mathcal{S}^n)$

(b) The list of papers constituting the scientific achievement

[H1] I. Iglewska-Nowak, *Continuous wavelet transforms on n-dimensional spheres*, Appl. Comput. Harmon. Anal. 39 (2015), no. 2, 248–276.

[H2] I. Iglewska-Nowak, *Poisson wavelets on n-dimensional spheres*, J. Fourier Anal. Appl. 21 (2015), no. 1, 206–227.

[H3] I. Iglewska-Nowak, *Semi-continuous and discrete wavelet frames on n-dimensional spheres*, Appl. Comput. Harmon. Anal. 40 (2016), no. 3, 529–552.

[H4] I. Iglewska-Nowak, *Directional wavelets on n-dimensional spheres*, Appl. Comput. Harmon. Anal. 44 (2018), no. 2, 201–229.

[H5] I. Iglewska-Nowak, *Frames of directional wavelets on n-dimensional spheres*, Appl. Comput. Harmon. Anal. 43 (2017), no. 1, 148–161.

(c) A discussion of the above-mentioned papers and the obtained results, together with a discussion of their possible use

4.1 A historical overview

Investigation of data on higher dimensional spheres has become more and more important in the last decades. Statistical problems, computer vision, medical imaging, quantum chemistry, crystallography are some of the application areas.
The most difficult problem by defining of continuous spherical wavelets is the lack of a natural dilation operator. Currently, two essentially different definitions of a dilation are used in order to introduce a spherical wavelet transform.

4.1.1 Wavelets based on group-theoretical approach

The probably most popular one is that based on group-theoretical approach, introduced by Antoine and Vandergheynst in [4] (compare also [5] for the two-dimensional case). Dilation is performed in the tangent space to the sphere, onto which the wavelets are mapped via the stereographic projection from the south pole. The construction is quite technical, but the wavelets have many nice properties, most of them investigated in the two-dimensional case, e.g., existence of fast algorithms based on FFT and of a directional wavelet transform [2, 49, 64, 50], discrete wavelet frames [10, 1, 51]. (However, the idea can be hardly generalized to n-dimensions since the discretization is performed on an equiangular – with respect to the spherical variables – grid which causes a concentration of sampling points around the poles). A slightly different approach is presented in [63], where the same wavelets are reintroduced in a more straightforward way. It is also shown in that paper that the inverse stereographic projection of Euclidean wavelets leads to spherical ones.

4.1.2 Wavelets indexed by a scale parameter

A number of definitions is based on another concept, where a wavelet is a family of functions, indexed by a scale parameter $\rho \in \mathbb{R}_+$. The dilation is performed by a choice of this parameter. The most important definitions are desribed below.

The oldest one, based on the theory of singular integrals and approximate identities [6], was developed by Freeden, Windheuser *et al.* for a two-dimensional sphere in the 1990s [25, 28, 27, 26, 24]. In the beginning of 21^{st} century it was generalized by Bernstein *et al.* to three-dimensional [7, 8] and n-dimensional cases [20, 19]. An example are diffusive wavelets from Ebert's dissertation [19]. Approximate identities yield zonal wavelets (the most important examples are Gauss-Weierstrass wavelet and Abel-Poisson wavelet), but starting from 2009 also the nonzonal case was considered [20, 9].

Another example is the definition utilized by Holschneider and his coworkers [39, 40, 15] – the wavelet transform and the wavelet synthesis are given by the same formulae as in the works of Freeden *et al.* and Bernstein *at al.*, dilation relies on the parameter choice in a wavelet family. The main difference lies in the background: whereas in the previous approach one derives wavelets from approximate identities and shows that the wavelet transform and the inverse transform converge for any $\mathcal{L}^2$-signal, here a family of functions is called admissible if it proves that the wavelet analysis and synthesis converge. An example of a wavelet family satisfying this definition is Poisson wavelet family introduced in [40], compare also [41], and to my knowledge it is the only one which was implemented and applied [40, 38].

In 2006, one more class of wavelets indexed by a scale parameter, was presented. Needlets, introduced by Narcowich *et al.* in [53] (compare also [52]), have excellent point-wise localization and approximation properties and they yield a tight frame on the sphere. An important difference to the above described constructions is that needlets have a compact spectrum.

The construction proposed by Geller and Mayeli in [32, 33] somehow resembles needlets (differences are discussed in [32, Sec. 1.1]), the wavelets are kernels of the operator $f(t\Delta^*)$, where $0 \neq f \in \mathcal{S}(\mathbb{R}_+)$, $f(0) \neq 0$, and Δ^* denotes Laplace-Beltrami operator on a manifold. In the case of the sphere this leads to zonal wavelets of the form

$$K_\rho(\hat{e}, y) = \frac{1}{\Sigma_n} \sum_{l=0}^{\infty} f\left(\rho^2 l(l+2\lambda)\right) \frac{\lambda + l}{\lambda} C_l^\lambda(y) \tag{1}$$

(compare Section 4.3 for notation). As a particular example the authors investigate the wavelet given by $f(s) = se^{-s}$. It is worth noting that this wavelet is exactly the linear Gauss-Weierstrass wavelet (indexed by ρ^2) from the Freeden-Windheuser theory. A wavelet transform is called linear if the wavelet itself is not needed for reconstruction, otherwise, one has to do with the bilinear wavelet theory (this is the case in [32], compare [32, Proposition 5.4]). Thus, one cannot identify these constructions. In [48] the name *Mexican needlets* is introduced for kernels of the form (1) with $f(s) = s^r e^{-s}$, $r \in \mathbb{N}$. Bilinear Gauss-Weierstrass could be treated as a Mexican needlet of order $\frac{1}{2}$ if the theory was extended to functions with rational exponents. Due to excellent localization properties, kernels (1) yield a nearly tight frame [33], which is not a tight one because of uncompactness of their spectra. Statistical properties and applications of Mexican needlets are discussed in [46, 48], and their usefulness for characterization of the Besov spaces in [31, 35].
A generalization of this idea are *needlet-type spin wavelets* for investigation of sections of line bundles instead of scalar-valued functions [30]. Also in this case, nearly tight frames exist [34], their statistical properties are investigated e.g. in [29].

4.2 A summary of the results

In the papers [H1–H5] I developed the theory of wavelets derived from approximate identities. The work is based on the definition from [20], the most general one available so far, i.e., concerning n-dimensional spheres and wavelets that are not necessarily rotation-invariant (zonal). In [H1] I proved some useful properties of the wavelets and the wavelet transform, namely the Euclidean limit property (its importance is emphasized especially in the papers by Holschneider and his co-workers) and the isometry. Inspired by the works of Freeden *et al.* I developed a theory of nonzonal *linear* wavelets over the n-dimensional sphere. The theory presented in [H1] generalizes several known approaches. It is discussed in [H1, Section 5] that Holschneider's wavelets, Mexican needltes, and Ebert's diffusive wavelets are all special cases of wavelets derived from approximate identities. The only theory that is essentially different is the one of Antoine and Vandergheynst.
An interesting special case are Poisson wavelets. They are defined as certain derivatives of Poisson kernel for the sphere [15]. Although quite useful in applications [40, 38], Poisson wavelets seemed not to have a solid theoretical base. The definition of spherical wavelets introduced by Holschneider in [39] was quite broadly criticised for the *ad hoc* choice of the scale parameter, see e.g. [3, 5, 4, 14, 23, 49]. Its second weakness is the fact that wavelets are not defined intrinsically. In the case of zonal wavelets, it is explicitly stated that the wavelet reconstruction formula is valid *whenever the integral makes sense*, cf. [40, Sec. 2.2.2]. In [H2] I showed that the family of Poisson wavelets, generalized to functions over the n-dimensional sphere, satisfies stronger conditions of definitions from [20] or [H1], both in the bilinear and in the linear case. The proof of this stetement required a deep study of the properties of Poisson wavelets. The most important were localization results obtained by investigation how some irrational functions behave under derivation. I also found explicit formulae for

- the harmonic continuation of Poisson wavelets to functions over $\mathbb{R}^{n+1} \setminus \{r\hat{e}\}$, where $r\hat{e}$ denotes the source localization of the field given by the corresponding Poisson kernel, both as infinite series of rational functions of $e^{-\rho}$ and $\cos\vartheta$, and as a finite sum of rational functions of $e^{-\rho}$ and $\cos\chi$, where χ is an angle corresponding to the source localization of the field,

- Poisson wavelets as functions of the spherical variables,

- their Euclidean limit.

Additionally, I proved the polynomial decay of the Euclidean limits of Poisson wavelets in infinity.
For an efficient usage of a continuous wavelet transform, a discretization algorithm is needed. Frames have been constructed for two-dimensional spherical wavelets derived in [5], cf. [1, 10]. However, the phase-space discretization is performed on an equiangular grid, a solution that can hardly be applied in a higher dimension.
In the paper [H3] I showed that under some mild conditions n-dimensional spherical wavelets derived from approximate identities build semi-continuous frames. Moreover, for sufficiently dense grids Poisson wavelets on n-dimensional spheres constitute a discrete frame. It is a generalization of the results obtained in [45] for two-dimensional spherical wavelets.
The construction of semi-continuous frames is similar to that in [1, 10] for the two-dimensional sphere. As a next step, for each scale a discretization of the spherical parameter is performed such that the sampling points are quite uniformly distributed over the sphere. Finally, the sampling point positions are perturbed in such a way that the density of the resulting grid is controlled with respect to the scale and space parameter simultaneously. If density is big enough, the discrete set of wavelets is a frame for $\mathcal{L}^2(\mathcal{S}^n)$. The constraints on the wavelets are some estimations on their reproducing kernel and its gradient. Based on the precise estimations from [H2], I proved that they are satisfied by Poisson multipole wavelets.
In [H4] I introduced directional Poisson wavelets on n-dimensional spheres, being directional derivatives of Poisson kernel. Contrary to the wavelets investigated in the papers [H1–H3], directional wavelets are not rotation-invariant. Therefore, they are better suited for analyzing signals with directional details, such as a contour map of the Globe. The directional Poisson wavelets are a generalization of wavelets over $\mathcal{S}^2$ used by Hayn and Holschneider in [38] to the n-dimensional case, and certain linear combinations of them satisfy the definition given by Ebert *et al.* in [20] which can be regarded as a modified version of the definition from [H1]. To my best knowledge it was the first attempt to define a concrete wavelet family derived from approximate identities that is not rotation-invariant. It is an alternative approach to spherical curvelets and ridgelets presented in [58, 59], and its advantage is that no partitioning of the sphere is needed (which could be a problem anyway in more than two dimensions).

In the paper, I investigated some properties of directional Poisson wavelets, among others, I derived recursive formulae for their Fourier coefficients and explicit representations as functions of spherical variables (for some wavelets), as well as an explicit formula for their Euclidean limits.
The goal of my research was to construct fully discrete wavelet frames built of nonzonal wavelets. The price to be paid was that the wavelet definition from [H1] is weakened. In the paper [H5] I presented a continuous wavelet transform over the n-dimensional sphere which is invertible only by the frame methods. Based on that definition, a wide class of both zonal and nonzonal wavelets is constructed, which contains all the so far studied wavelet families (derived from approximate identities). Further, it is shown that fully discrete frames exist. The frame constructions on n-dimensional spheres known to me are all based on zonal wavelets [33] [H3], therefore, this study seems to be the first approach to discretize a directional wavelet transform.

4.3 Preliminaries

4.3.1 Functions on the sphere

$\mathcal{S}^n$ denotes the n-dimensional unit sphere in the $n+1$-dimensional Euclidean space $\mathbb{R}^{n+1}$ with the rotation-invariant measure $d\sigma$ normalized such that

$$\Sigma_n = \int_{\mathcal{S}^n} d\sigma = \frac{2\pi^{(n+1)/2}}{\Gamma\big((n+1)/2\big)}.$$

The surface element $d\sigma$ is explicitly given by

$$d\sigma = \sin^{n-1}\vartheta_1\, \sin^{n-2}\vartheta_2 \dots \sin\vartheta_{n-1} d\vartheta_1\, d\vartheta_2 \dots d\vartheta_{n-1} d\varphi,$$

where $(\vartheta_1, \vartheta_2, \dots, \vartheta_{n-1}, \varphi) \in [0,\pi]^{n-1} \times [0, 2\pi)$ are the spherical coordinates satisfying

$$\begin{aligned}
x_1 &= \cos\vartheta_1,\\
x_2 &= \sin\vartheta_1 \cos\vartheta_2,\\
x_3 &= \sin\vartheta_1 \sin\vartheta_2 \cos\vartheta_3,\\
&\dots\\
x_{n-1} &= \sin\vartheta_1 \sin\vartheta_2 \dots \sin\vartheta_{n-2} \cos\vartheta_{n-1},\\
x_n &= \sin\vartheta_1 \sin\vartheta_2 \dots \sin\vartheta_{n-2} \sin\vartheta_{n-1} \cos\varphi,\\
x_{n+1} &= \sin\vartheta_1 \sin\vartheta_2 \dots \sin\vartheta_{n-2} \sin\vartheta_{n-1} \sin\varphi.
\end{aligned}$$

$\langle x, y\rangle$ or $x \cdot y$ stands for the scalar product of vectors with origin in O and endpoint on the sphere. As long as it does not lead to misunderstandings, these vectors are identified with points on the sphere.
The $\mathcal{L}^2(\mathcal{S}^n)$-norm is given by

$$\|f\|_2 = \left[\frac{1}{\Sigma_n} \int_{\mathcal{S}^n} |f(x)|^2\, d\sigma(x)\right]^{1/2},$$

and the scalar product of $f, g \in \mathcal{L}^2(\mathcal{S}^n)$ is defined by

$$\langle f, g\rangle_{\mathcal{L}^2(\mathcal{S}^n)} = \frac{1}{\Sigma_n} \int_{\mathcal{S}^n} \overline{f(x)}\, g(x)\, d\sigma(x),$$

such that $\|f\|_2^2 = \langle f, f\rangle$. A function is called zonal if its value depends only on $\vartheta = \vartheta_1 = \langle \hat{e}, x\rangle$, where $\hat{e}$ is the north pole of the sphere

$$\hat{e} = (1, 0, 0, \dots, 0).$$

It is invariant with respect to the rotation about the axis through O and $\hat{e}$. Whenever it does not lead to mistakes, I shall write

$$f(x) = f(\cos\vartheta_1).$$

The Gegenbauer polynomials C_l^λ of order $\lambda \in \mathbb{R}$ and degree $l \in \mathbb{N}_0$ are defined in terms of their generating function

$$\sum_{l=0}^{\infty} C_l^\lambda(t)\, r^l = \frac{1}{(1 - 2tr + r^2)^\lambda}, \qquad t \in [-1, 1].$$

A set of the Gegenbauer polynomials $\left\{C_l^\lambda\right\}_{l\in\mathbb{N}_0}$ builds a complete orthogonal system on $[-1, 1]$ with weight $(1-t^2)^{\lambda-1/2}$. Consequently, it is an orthogonal basis for zonal functions on the $2\lambda+1$-dimensional sphere. The numbers n and λ, related by

$$n = 2\lambda + 1,$$

will be used interchangeably.
Let Q_l denote a polynomial on $\mathbb{R}^{n+1}$ homogeneous of degree l, i.e., such that $Q_l(az) = a^l Q_l(z)$ for all $a \in \mathbb{R}$ and $z \in \mathbb{R}^{n+1}$, and harmonic in $\mathbb{R}^{n+1}$, i.e., satisfying $\nabla^2 Q_l(z) = 0$. Then, $Y_l(x) = Q_l(x)$, $x \in \mathcal{S}^n$, is called a hyperspherical harmonic of degree l. The set of the hyperspherical harmonics of degree l is denoted by $\mathcal{H}_l(\mathcal{S}^n)$. The number of linearly independent hyperspherical harmonics of degree l is denoted by $N = N(n, l)$. Hyperspherical harmonics of distinct degrees are orthogonal to each other. The addition theorem states that

$$C_l^\lambda(x \cdot y) = \frac{\lambda}{\lambda + l} \sum_{\kappa=1}^{N} \overline{Y_l^\kappa(x)}\, Y_l^\kappa(y) \tag{2}$$

for any orthonormal set $\{Y_l^\kappa\}_{\kappa=1,2,\dots,N(n,l)}$ of the hyperspherical harmonics of degree l on $\mathcal{S}^n$. The orthonormal basis for $\mathcal{L}^2(\mathcal{S}^n) = \overline{\bigoplus_{l=0}^\infty \mathcal{H}_l}$ I was working with consists of hyperspherical harmonics given by

$$Y_l^k(x) = A_l^k \prod_{\tau=1}^{n-1} C_{k_{\tau-1}-k_\tau}^{\frac{n-\tau}{2}+k_\tau}(\cos\vartheta_\tau) \sin^{k_\tau}\vartheta_\tau \cdot e^{\pm i k_{n-1}\varphi} \tag{3}$$

with $l = k_0 \geq k_1 \geq \cdots \geq k_{n-1} \geq 0$, k being a sequence $(k_1, \ldots, \pm k_{n-1})$ of integer numbers, and normalization constants A_l^k, compare [62, Sec. IX.3.6, formulae (4) and (5)]. The set of nonincreasing sequences k in $\mathbb{N}_0^{n-1} \times \mathbb{Z}$ with elements bounded by l is denoted by $\mathcal{M}_{n-1}(l)$. Every $\mathcal{L}^1(\mathcal{S}^n)$-function f can be expanded into a Laplace series of the hyperspherical harmonics by

$$S(f;x) \sim \sum_{l=0}^{\infty} Y_l(f;x).$$

For zonal functions one has the representation

$$Y_l(f;t) = \widehat{f}(l)\, C_l^{\lambda}(t), \qquad t = \cos\vartheta,$$

with the Gegenbauer coefficients

$$\widehat{f}(l) = c(l,\lambda) \int_{-1}^{1} f(t)\, C_l^{\lambda}(t) \left(1 - t^2\right)^{\lambda - 1/2} dt,$$

where $c(l, \lambda)$ is a constant depending only on l and λ, compare [6, p. 207]. The series

$$\sum_{l=0}^{\infty} \widehat{f}(l)\, C_l^{\lambda}(t) \tag{4}$$

is called the Gegenbauer expansion of f. For $f, h \in \mathcal{L}^1(\mathcal{S}^n)$, h zonal, their convolution $f * h$ is defined by

$$(f * g)(x) = \frac{1}{\Sigma_n} \int_{\mathcal{S}^n} f(y)\, g(x \cdot y)\, d\sigma(y).$$

Further, any function $f \in \mathcal{L}^2(\mathcal{S}^n)$ has a unique representation as a mean-convergent series

$$f(x) = \sum_{l=0}^{\infty} \sum_{k \in \mathcal{M}_{n-1}(l)} a_l^k\, Y_l^k(x), \qquad x \in \mathcal{S}^n, \tag{5}$$

where

$$a_l^k = a_l^k(f) = \frac{1}{\Sigma_n} \int_{\mathcal{S}^n} \overline{Y_l^k(x)}\, f(x)\, d\sigma(x) = \left\langle Y_l^k, f \right\rangle,$$

for proof cf. [62]. a_l^k are called Fourier coefficients of the function f. Convolution with a zonal function can be then written as

$$f * g = \sum_{l=0}^{\infty} \sum_{k \in \mathcal{M}(_{n-1}l)} \frac{\lambda}{\lambda + l}\, a_l^k(f)\, \widehat{g}(l)\, Y_l^k$$

and for zonal functions the following relation

$$\widehat{f}(l) = A_l^0 \cdot a_l^0(f) \tag{6}$$

between Fourier and the Gegenbauer coefficients holds.
The set of rotations of $\mathbb{R}^{n+1}$ is denoted by $SO(n+1)$. It is isomorphic to the set of square matrices of degree $n+1$ with determinant 1. The n-dimensional sphere can be identified with the class of left cosets of $SO(n+1)$ mod $SO(n)$,

$$\mathcal{S}^n = SO(n+1)/SO(n),$$

cf. [62, Sec. I.2].
The zonal product of arbitrary $\mathcal{L}^2(\mathcal{S}^n)$-functions f and h is defined by

$$(f \hat{*} g)(x \cdot y) = \int_{SO(n+1)} f(\Upsilon^{-1}x)\, h(\Upsilon^{-1}y)\, d\nu(\Upsilon), \qquad x, y \in \mathcal{S}^n,$$

and it has the representation

$$(f \hat{*} g)(x \cdot y) = \sum_{l=0}^{\infty} \sum_{k \in \mathcal{M}_{n-1}(l)} \frac{a_l^k(f)\, a_l^k(g)}{N(n,l)} \frac{\lambda + l}{\lambda} C_l^{\lambda}(x \cdot y). \tag{7}$$

4.3.2 Singular integrals and approximate identities

Singular integrals over n-dimensional spheres were introduced in [6] by Berens *et al.*, inspired by some previous papers concerning the theory of singular integrals on the real line [11], unit circle [12, 61], in k-dimensional Euclidean space [13] or on the k-dimensional torus [55].

Definition 1 *Denote by $\mathcal{L}^1_\lambda([-1,1])$ the class of fuctions over $[-1,1]$ integrable with respect to the weight function $t \mapsto (1-t^2)^{\lambda-1/2}$. Let $\{\mathcal{K}_\rho\}_{\rho\in\mathbb{R}_+} \subseteq \mathcal{L}^1_\lambda\big([-1,1]\big)$ be a family of kernels such that*

$$\widehat{\mathcal{K}_\rho}(0) = c(0,\lambda) \int_{-1}^{1} \mathcal{K}_\rho(t)\big(1-t^2\big)^{\lambda-1/2} dt = 1. \tag{8}$$

Then

$$I_\rho(f) = f * \mathcal{K}_\rho \tag{9}$$

is called a spherical singular integral. The family $\{\mathcal{K}_\rho\}$ is called the kernel of a singular integral.

Remark. $\mathcal{L}^1_\lambda$ can be identified with the class of integrable zonal functions over the $2\lambda+1$-dimensional sphere.
Approximate identities (without using this notion) were studied in [6], originally understood as singular integrals having an additional property that

$$\lim_{\rho\to 0^+} \|I_\rho f - f\|_{\mathcal{X}} = 0, \tag{10}$$

where $\mathcal{X}(\mathcal{S}^n)$ denotes the space $\mathcal{L}^p(\mathcal{S}^n)$, $p \in [1,\infty)$, or $\mathcal{C}(\mathcal{S}^n)$. This definition is used e.g. in [7, 8, 9, 28, 27, 24, 3]. However, condition (8) is necessary neither for the approximation property (10) nor for the definition of spherical wavelets. Moreover, it is not satisfied by many wavelet families. Therefore, similarly as the authors of [20], I use the following definition.

Definition 2 *Let a family $\{\mathcal{K}_\rho\}_{\rho\in\mathbb{R}_+}$ of integrable zonal functions satisfying* (10) *be given. Then, the family $\{\mathcal{K}_\rho *\}_{\rho\in\mathbb{R}_+}$ forms an approximate identity with kernel $\{\mathcal{K}_\rho\}_{\rho\in\mathbb{R}_+}$.*

Another useful characterization of approximate identities is given in the next theorem, cf. [20, Theorem 3.8].

Theorem 3 *Assume that the kernel $\{\mathcal{K}_\rho\}_{\rho\in\mathbb{R}_+} \subseteq \mathcal{L}^1_\lambda\big([-1,1]\big)$ is uniformly bounded in $\mathcal{L}^1_\lambda$-norm, i.e.,*

$$\int_{-1}^{1} |\mathcal{K}_\rho(t)| \big(1-t^2\big)^{\lambda-1/2} dt \leq \mathfrak{c} \tag{11}$$

uniformly in $\rho \in \mathbb{R}_+$ for a positive constant $\mathfrak{c}$. Then, the corresponding integral I_ρ, defined by (9) *is an approximate identity in $\mathcal{X}(\mathcal{S}^n)$ if and only if*

$$\lim_{\rho\to 0^+} \widehat{\mathcal{K}_\rho}(l) = \frac{\lambda+l}{\lambda} \tag{12}$$

for all $l \in \mathbb{N}_0$.

The properties of approximate identities are discussed in detail in [6].

4.3.3 Frames

Most statements in this section come from [45].

Definition 4 *A family of vectors $\{g_x,\, x\in X\} \subset \mathcal{H}$ in a Hilbert space $\mathcal{H}$ indexed by a measure space X with a positive measure μ is called a frame with weight μ if the mapping $x \mapsto g_x$ is weakly measurable, i.e., $x \mapsto \langle g_x, u\rangle$ is measurable, and if for some $\epsilon \in [0,1)$ we have*

$$(1-\epsilon)\, \|u\|^2 \leq \int_X |\, \langle g_x, u\rangle\, |^2 d\,\mu(x) \leq (1+\epsilon)\, \|u\|^2. \tag{13}$$

for all $u \in \mathcal{H}$. The numbers $1-\epsilon$ and $1+\epsilon$ are called the frame bounds. A frame is called tight if $\epsilon = 0$.

Let $\mathcal{H} = \mathcal{L}^2(X, d\mu)$ be a Hilbert space of functions over X with the reproducing kernel Π

$$u(x) = \int_X \Pi(x,y)\, u(y)\, d\mu(y).$$

The family of functions $\{g_x = \Pi(x,\cdot)\}$ with $x \in X$ is a tight frame with weight μ. Conversely, a tight frame $\{g_x,\, x \in X\}$ and a measure μ in a Hilbert space $\mathcal{H}$ are naturally associated with a reproducing kernel Hilbert space of functions in $\mathcal{L}^2(X, d\mu)$.

Theorem 5 *The mapping*

$$S : \mathcal{H} \to \mathcal{L}^2(X, d\mu), \qquad Su(x) = \langle g_x, u \rangle \tag{14}$$

is a partial isometry and the image $\mathcal{U}$ of this mapping is characterized by the reproducing kernel

$$\Pi(x, y) = \langle g_x, g_y \rangle \, .$$

That means, $u \in \mathcal{L}^2(X, d\mu)$ is in the range of S if and only if

$$\int_X \Pi(x, y)\, u(y)\, d\mu(y) = u(x).$$

The last integral is absolutely convergent since $\Pi(x, \cdot)$ is in $\mathcal{L}^2(X, d\mu)$. In particular:

Proposition 6 *Let $\{g_x,\, x \in X\}$ be a tight frame with weight μ on $\mathcal{H}$. A family $\{g_y,\, y \in \Lambda \subset X\}$ with measure m on Λ yields a frame for $\mathcal{H}$ if and only if $\{\Pi(y, \cdot),\, y \in \Lambda\}$, $\Pi(\xi, \eta) = \langle g_\xi, g_\eta \rangle$, is a frame for $S(\mathcal{H})$, with S given by* (14)*.*

Frames of the form $\{\Pi(y, \cdot)\}$ can be characterized as follows:

Theorem 7 *Let $\Lambda \subset X$ and let m be a measure on Λ, and μ be a measure on X. The family of functions $\{g_y = \Pi(y, \cdot),\, y \in \Lambda\} \subset \mathcal{L}^2(X, d\mu)$ is a frame with weight m for $\mathcal{U} = S(\mathcal{H})$ if and only if*

$$F(x, z) = \int_\Lambda \Pi(x, y)\, \Pi(y, z)\, dm(y) - \Pi(x, z) \tag{15}$$

is the kernel of a bounded operator $\mathbb{F}$ on $\mathcal{U}$ with $\|\mathbb{F}\| < 1$.

Since $\Pi(x, z) = \int_X \Pi(x, y)\, \Pi(y, z)\, d\mu(y)$, the theorem shows that the existence of frames is intimately linked to the existence of good quadrature rules for functions in $\mathcal{U}$. This general principle is used together with the following perturbation result.

Corollary 8 *Suppose, for a set Λ the family $\{g_y = \Pi(y, \cdot),\, y \in \Lambda\}$ is a weighted frame for $\mathcal{U}$ with weight m. If now for another set Υ one has for $\{g_y = \Pi(y, \cdot),\, y \in \Upsilon\} \subset \mathcal{U}$ and a weight υ that*

$$G(x, z) = \int_\Lambda \Pi(x, y)\, \Pi(y, z)\, dm(y) - \int_\Upsilon \Pi(x, y)\, \Pi(y, z)\, d\upsilon(y)$$

is the kernel of an operator $\mathbb{G}$ with operator norm $\|\mathbb{G}\| \leq 1 - \|\mathbb{F}\|$, where the kernel of $\mathbb{F}$ is given by (15)*, then $\{g_y,\, y \in \Upsilon\}$ is a frame with weight υ.*

More details on this topic can be found in [45], [17], and [16].

4.4 The spherical wavelet transform and its properties [H1]

The following two definitions originate from [20].

Definition 9 *Let $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ be a weight function. A family $\{\Psi_\rho\}_{\rho\in\mathbb{R}_+} \subseteq \mathcal{L}^2(\mathbb{S}^n)$ is called a bilinear spherical wavelet if it satisfies the following admissibility conditions:*

1. *for $l \in \mathbb{N}_0$*
$$\sum_{\kappa=1}^{N(n,l)} \int_0^\infty \left| a_l^\kappa(\Psi_\rho)\right|^2 \alpha(\rho)\, d\rho = N(n,l), \tag{16}$$

2. *for $R \in \mathbb{R}_+$ and $x \in \mathbb{S}^n$*
$$\int_{\mathbb{S}^n} \left| \int_R^\infty (\overline{\Psi_\rho} \hat{*} \Psi_\rho)(x\cdot y)\, \alpha(\rho)\, d\rho \right| d\sigma(y) \le \mathfrak{c} \tag{17}$$
with $\mathfrak{c}$ independent of R.

The factor $\Sigma_n = \int \mathcal{S}^n$ must be corrected (with respect to the formulae from [20]). The reason is that the addition theorem concerning the hyperspherical harmonics has been cited with a false constant in [6], a paper several further articles are based on. The proof of the addition theorem can be found in [22], where an old normalization convention for the hyperspherical harmonics had been used, and the theorem had been transferred without changing the adequate constant.

Definition 10 *Let $\{\Psi_\rho\}_{\rho\in\mathbb{R}_+}$ be a spherical wavelet. Then, the spherical wavelet transform*
$$\mathcal{W}_\Psi \colon \mathcal{L}^2(\mathcal{S}^n) \to \mathcal{L}^2(\mathbb{R}_+ \times SO(n+1))$$
is defined by
$$\mathcal{W}_\Psi f(\rho, \Upsilon) = \frac{1}{\Sigma_n} \int_{\mathcal{S}^n} \overline{\Psi_\rho(\Upsilon^{-1}x)}\, f(x)\, d\sigma(x).$$

Reduced to the two-dimensional zonal case, it yields the well-known results from the 1990s [24, 27, 28]. The novelty of the approach presented in [20] is the generalization to an arbitrary dimension, as well as consideration of wavelets that are not rotation-invariant. The wavelet transform defined in that way is invertible [20, Theorem 5.3]:
$$f(x) = \int_0^\infty \int_{SO(3)} (\mathcal{W}_\Psi f)(\rho, \Upsilon)\, \Psi_\rho(\Upsilon^{-1}x)\, d\nu(\Upsilon)\, \alpha(\rho)\, d\rho \qquad \text{a.e.}$$

The proof of this statement is based on Theorem 3. In a similar way I showed that the wavelet transform is an isometry [H1, Theorem 3.3]:
$$\langle \mathcal{W}_\Psi f, \mathcal{W}_\Psi g\rangle = \langle f, g\rangle\,,$$
where the scalar product in the wavelet phase space is given by
$$\langle F, G\rangle_{\mathcal{L}^2(\mathbb{R}_+\times SO(n+1))} = \int_0^\infty \int_{SO(n+1)} \overline{F(\rho,\Upsilon)}\, G(\rho,\Upsilon)\, d\nu(\Upsilon)\, \alpha(\rho)\, d\rho.$$

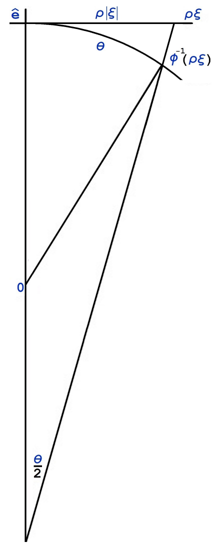

Figure 1: Stereographic projection

4.4.1 Euclidean limit

Euclidean limit property is the notion introduced in the context of spherical functions by Holschneider in [39]. By this expression it is meant that for small scales, i.e., in the case when the wavelet is concentrated in a small region of $\mathcal{S}^n$ that can be considered to be approximately flat, the wavelet behaves like one over the Euclidean space,

$$\Psi_\rho\left(\Phi^{-1}(\xi)\right) \approx \frac{1}{\rho^n} F\left(\frac{\xi}{\rho}\right) \qquad \text{for some } F \in \mathcal{L}^2(\mathbb{R}^n),$$

where Φ^{-1} denotes the inverse stereographic projection. More precisely ([H1, Theorem 3.4]):

Theorem 11 *Let a spherical wavelet $\{\Psi_\rho\} \subseteq \mathcal{L}^2(\mathcal{S}^n)$ with*

$$a_l^k(\Psi_\rho) = \frac{1}{l^{k_1-1} A_l^k}\, \mathcal{O}\left(\psi_k(l\rho)\right), \qquad \rho \to 0,$$

$l \in \mathbb{N}_0$, $k = (k_1, k_2, \ldots, k_{n-1}) \in \mathcal{M}_{n-1}(l)$, $k_1 \leq K$, be given, with $\psi_k \in \mathcal{L}^2(\mathbb{R}_+, t^{n-1}\, dt)$ – a piecewise smooth function satisfying

$$\rho^n \sum_{l=0}^{[c/\rho]} l^{n-1}\, \psi_k(l\rho) < \epsilon,$$

for some $c > 0$, $\epsilon \ll 1$, and $\rho < \rho_0$. Further, let

$$\lim_{\rho \to 0} a_l^k(\Psi_\rho) = 0$$

for $k_1 \geq K$. Then there exists a square integrable function $F : \mathbb{R}^n \to \mathbb{C}$ such that

$$\lim_{\rho \to 0} \rho^n \, \Psi_\rho \left(\Phi^{-1}(\rho\xi) \right) = F(\xi)$$

holds point-wise for every $\xi \in \mathbb{R}^n$.

The proof is technical and it involves the theory of special functions and the theory of Fourier transforms over $\mathbb{R}^n$.

4.4.2 Zonal wavelets

In my paper [H1] a separate subsection is devoted to rotation-invariant wavelets. In this case, $SO(3)$-rotations reduce to $\mathcal{S}^n$-translations (rotations), admissibility conditions are simplified and the inverse transform requires an integral over the sphere instead of an integral over the whole rotation group.

Definition 12 *A subfamily $\{\Psi_\rho\}_{\rho \in \mathbb{R}_+}$ of the space $\mathcal{L}^1_\lambda\big([-1,1]\big)$ is called zonal spherical wavelet if it satisfies the following admissibility conditions:*

1. *for $l \in \mathbb{N}_0$*
$$\int_0^\infty \left| \widehat{\Psi_\rho}(l) \right|^2 \alpha(\rho) \, d\rho = \left(\frac{\lambda + l}{\lambda} \right)^2, \tag{18}$$

2. *for $R \in \mathbb{R}_+$*
$$\int_{-1}^1 \left| \int_R^\infty \big(\overline{\Psi_\rho} * \Psi_\rho \big)(t) \, \alpha(\rho) \, d\rho \right| \left(1 - t^2 \right)^{\lambda - 1/2} dt \leq \mathfrak{c} \tag{19}$$
with $\mathfrak{c}$ independent of R.

Definition 13 *Let $\{\Psi_\rho\}_{\rho \in \mathbb{R}_+}$ be a spherical wavelet. Then, the spherical wavelet transform*

$$\mathcal{W}_\Psi : \mathcal{L}^2 \left(\mathcal{S}^n \right) \to \mathcal{L}^2 \left(\mathbb{R}_+ \times \mathcal{S}^n \right)$$

is defined by

$$\mathcal{W}_\Psi f(\rho, x) = \left(f * \overline{\Psi_\rho} \right)(x). \tag{20}$$

The wavelet transform is an isometry and it is invertible in $\mathcal{L}^2$-sense by

$$f(x) = \frac{1}{\Sigma_n} \int_0^\infty \int_{\mathcal{S}^n} \mathcal{W}_\Psi f(\rho, y) \, \Psi_{\rho, y}(x) \, d\sigma(y) \, \alpha(\rho) \, d\rho. \tag{21}$$

In this way, the theory of zonal wavelets studied by many authors [25, 28, 27, 24, 8, 9, 19] is generalized to n-dimensional spheres.

4.4.3 Wavelets corresponding to an approximate identity

So far, the theory of approximate identities was utilized for the proof of invertibility of the spherical wavelet transform. The superposition of a wavelet transform and wavelet synthesis is shown to be an approximate identity. Conversely, to a suitable kernel of an approximate identity, a spherical wavelet can be associated, cf. [20, Theorem 6.1].

Theorem 14 *Let a kernel $\{\Phi_R\}_{R\in\mathbb{R}_+}$ of a uniformly bounded approximate identity be given with the Gegenbauer coefficients which are differentiable with respect to R and monotonically decreasing in R. Moreover, assume that*

$$\lim_{R\to\infty} \widehat{\Phi_R}(l) = 0$$

for $l \in \mathbb{N}$. Then, the associated spherical wavelet $\{\Psi_\rho\}_{\rho\in\mathbb{R}_+}$ is given by

$$\widehat{\Psi_\rho}(l) = \left(-\frac{1}{\alpha(\rho)}\,\frac{d}{d\rho}\left|\widehat{\Phi_\rho}(l)\right|^2\right)^{1/2}$$

for $l \in \mathbb{N}_0$, $\rho \in \mathbb{R}_+$.

There exists a one-to-one correspondence between approximate identities having properties listed in Theorem 14 and spherical wavelets. This is the reason for calling wavelets satisfying the conditions of Definition 9 or Definition 13 *wavelets derived from approximate identities*. This should help distinguish them from *wavelets derived group-theoretically* by Antoine and Vandergheynst [5, 4].

4.4.4 Linear wavelets

Additionally to the above discussed research concerning bilinear wavelets, I introduced nonzonal linear wavelets over n-dimensional spheres similarly to [24] and in analogy to the ideas from [20] (compare [H1, Section 4]):

Definition 15 *Let $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ be a weight function. A family $\{\Psi_\rho^L\}_{\rho\in\mathbb{R}_+} \subseteq \mathcal{L}^2(\mathcal{S}^n)$ is called linear spherical wavelet if it satisfies the following admissibility conditions:*

1. *for $l \in \mathbb{N}_0$*
$$A_l^0 \cdot \int_0^\infty a_l^0(\Psi_\rho^L)\,\alpha(\rho)\,d\rho = \frac{\lambda + l}{\lambda},$$
2. *for $R \in \mathbb{R}_+$ and $x \in \mathcal{S}^n$*
$$\int_{\mathcal{S}^n}\left|\int_R^\infty \Psi_\rho^L(x\cdot y)\,\alpha(\rho)\,d\rho\right| d\sigma(y) \le \mathfrak{c} \tag{22}$$
with $\mathfrak{c}$ independent of R.

Definition 16 *Let* $\{\Psi_\rho^L\}_{\rho\in\mathbb{R}_+}$ *be a linear spherical wavelet. Then, the linear spherical wavelet transform*

$$\mathcal{W}_\Psi^L\colon \mathcal{L}^2(\mathcal{S}^n) \to \mathcal{L}^2\left(\mathbb{R}_+ \times SO(n+1)\right)$$

is defined by

$$\mathcal{W}_\Psi^L f(\rho,\Upsilon) = \frac{1}{\Sigma_n}\int_{\mathcal{S}^n} \Psi_\rho^L(g^{-1}x)\, f(x)\, d\sigma(x).$$

Theorem 17 *(Reconstruction formula) Let* $\{\Psi_\rho^L\}_{\rho\in\mathbb{R}_+}$ *be a linear wavelet and* $f\in\mathcal{L}^2(\mathcal{S}^n)$*. Then*

$$f(x) = \int_0^\infty \int_{SO(n)} \mathcal{W}_\Psi^L f(\rho,\hat{x}\tilde{g})\, d\nu(\tilde{g})\, \alpha(\rho)\, d\rho.$$

in $\mathcal{L}^2$*-sense, where* $\hat{x}$ *denotes any fixed element of* $SO(n+1)$ *satisfying* $\hat{x}\hat{e} = x$*.*

The proof is technical and it involves properties of the regular representations of $SO(n) \subseteq SO(n+1)$ in $\mathcal{L}^2(\mathcal{S}^n)$.
In this case, no wavelet is needed for the reconstruction. This is the reason for calling this wavelet transform *linear*.

4.4.5 Relationship to other spherical wavelets

A big advantage of the approach presented in [H1] is its generality. In the last section of [H1] I showed that several other constructions of continuous spherical wavelets, namely Holschneider's wavelets, Mexican needlets, and Ebert's diffusive wavelets, satisfy the conditions of Definition 9. Thus, there exist two essentially different continuous wavelet transforms over the sphere: the one based on the theory of approximate identities [H1], and the one based on group-theoretical approach [5, 4].

4.5 Poisson wavelets [H2-H3]

An example of wavelet families are Poisson wavelets, introduced (for the two-dimensional case) by Holschneider *et al.* in [40] and in more detail investigated in [41, 42, 45]. Poisson wavelets have proved to be very useful in applications, cf.[40, 15]; further, in my doctoral thesis I showed that their discrete frames exist [42], compare also [45]; they also possess a directional (i.e., nonzonal) counterpart [38]. Their big advantage over other wavelet families is an explicit representation as a function of the spherical variables ϑ and φ. However, Poisson wavelets had been introduced as wavelet families satisfying spherical wavelet definition given in [39] that did not find much favor in mathematical community. These reasons motivated me to investigate Poisson wavelet families from a perspective of wavelets derived from an approximate identity.
In [H2] I generalized the definition of Poisson wavelets to n-dimensional spheres (Definition 3.1).

Definition 18 *Poisson wavelet of order m, $m \in \mathbb{N}$, at a scale ρ, $\rho \in \mathbb{R}_+$, is given recursively by*

$$
\begin{aligned}
\Psi_\rho^1 &= \rho r \partial_r p_{r\hat{e}}, \qquad r = e^{-\rho}, \\
\Psi_\rho^{m+1} &= \rho r \partial_r \Psi_\rho^m,
\end{aligned} \tag{23}
$$

where $p_{r\hat{e}}$ is Poisson kernel for the unit sphere,

$$
p_\zeta(y) = \frac{1}{\Sigma_n} \frac{1-|\zeta|^2}{|\zeta - y|^{n+1}} = \frac{1}{\Sigma_n} \frac{1-r^2}{(1-2r\cos\vartheta + r^2)^{(n+1)/2}} \tag{24}
$$

for ζ, $y \in \mathbb{R}^{n+1}$,

$$
r = |\zeta| < |y| = 1
$$

and ϑ – the angle between the vectors ζ and y, i.e.,

$$
r\cos\vartheta = \zeta \cdot y.
$$

In analogy to the results published in [41] I derived the Gegenbauer expansion of Poisson wavelets, i.e., the representation as a series

$$
f = \sum_{l=0}^{\infty} \widehat{f}(l)\, \mathcal{C}_l^\lambda,
$$

to be equal to

$$
\Psi_\rho^m(y) = \frac{1}{\Sigma_n} \sum_{l=0}^{\infty} \frac{\lambda + l}{\lambda} (\rho l)^m e^{-\rho l}\, \mathcal{C}_l^\lambda(\cos\vartheta),
$$

[H2, Lemma 3.2]. I also showed that the wavelets possess a representation as a linear combination of the hyperspherical harmonics centered in the point where the field source (multipole) is located. Moreover, I proved that a harmonic continuation exists to functions over the space with the source point excluded [H2, Proposition 4.1].

Proposition 19 *Poisson wavelets g_ρ^m, $m \in \mathbb{N}$, can be uniquely harmonically continued to functions over $\mathbb{R}^{n+1} \setminus \{r\hat{e}\}$. They are given by*

$$
g_\rho^m(x) = \frac{\rho^m}{\Sigma_n} \sum_{l=0}^{m+1} l! \left(\alpha_l^m + \frac{\alpha_l^{m+1}}{\lambda} \right) e^{-\rho l}\, \frac{C_l^\lambda(\cos\chi)}{|x - r\hat{e}|^{l+2\lambda}}, \tag{25}
$$

where $r = e^{-\rho}$,

$$
\cos\chi = \frac{x - r\hat{e}}{|x - r\hat{e}|} \cdot \hat{e}
$$

and the coefficients α_l^m are recursively given by

$$
\begin{aligned}
\alpha_0^0 &= 1, \\
\alpha_0^m &= 0 \qquad \text{for } m \geq 1, \\
\alpha_m^l &= 0 \qquad \text{for } l > m, \\
\alpha_l^{m+1} &= l\alpha_l^m + \alpha_{l-1}^m.
\end{aligned}
$$

I also derived explicit formulae for Poisson wavelets as irrational functions of $\cos\vartheta$. This is one of the features that makes them suited for applications. Note that Gauss-Weierstrass wavelets [24, Section 10], respectively Mexican needlets [48], as well as all the discrete wavelets investigated in [24, Section 11] are given only as Laplace series.

Proposition 20 *Poisson wavelets of order $m \in \mathbb{N}$ are represented by*

$$g_\rho^m(y) = \frac{\rho^m}{\Sigma_n} D_{\lambda+m+1} \sum_{k=0}^{m} R_k^m(r) \cos^k \vartheta, \tag{26}$$

where

$$D_j = D_j(r,\vartheta) = \frac{r}{(1-2r\cos\vartheta + r^2)^j}$$

and R_k^m are polynomials of degree $2m-k+1$, explicitly given by

$$R_k^m(r) = \sum_{j=0}^{[(2m-k+1)/2]} a_j^{m,k} r^{2j+(k-1)_{mod\,2}},$$

where the coefficients $a_j^{m,k}$ satisfy the recursion

$$\begin{aligned}
a_j^{m+1,0} &= b_j^{m+1,0}, && j = 0,\dots,m+1,\\
a_j^{m+1,k} &= b_j^{m+1,k} + c_j^{m+1,k}, && k = 1,\dots,m,\quad j = 0,\dots,m+1-\left[\tfrac{k-1}{2}\right],\\
a_j^{m+1,m+1} &= c_j^{m+1,m+1}, && j = 0,\dots,\left[\frac{m+1}{2}\right],
\end{aligned}$$

with

$$\begin{aligned}
a_0^{1,0} &= -(n+3), & a_1^{1,0} &= n-1,\\
a_0^{1,1} &= n+1, & a_1^{1,1} &= -(n-3)
\end{aligned}$$

and

$$\begin{aligned}
b_0^{m+1,k} &= 2\,a_0^{m,k},\\
b_j^{m+1,k} &= 2\,(j+1)\,a_j^{m,k} + 2\,(j-\lambda-m-1)\,a_{j-1}^{m,k}, \qquad j = 1,\dots,m-k/2,\\
b_{m+1-k/2}^{m+1,k} &= -(2\lambda+k)\,a_{m-k/2}^{m,k},\\
c_0^{m+1,k} &= 2\,(\lambda+m)\,a_0^{m,k-1},\\
c_j^{m+1,k} &= 2\,(\lambda+m)\,a_j^{m,k-1} - 2\cdot 2j\,a_j^{m,k-1}\\
&= 2\,(\lambda+m-2j)\,a_j^{m,k-1} \qquad j = 1,\dots,m+1-k/2,
\end{aligned}$$

for an even k and

$$
\begin{aligned}
b_0^{m+1,k} &= a_0^{m,k}, \\
b_j^{m+1,k} &= (2j+1)\, a_j^{m,k} + (2\,(j-\lambda-m)-3)\; a_{j-1}^{m,k}, \qquad j=1,\ldots,m-[k/2], \\
b_{m+1-[k/2]}^{m+1,k} &= -(2\lambda+2m+1)\, a_{m-[k/2]}^{m,k} + (2m-k+1)\, a_{m-[k/2]}^{m,k} \\
&= -(2\lambda+k)\, a_{m-[k/2]}^{m,k}, \\
c_0^{m+1,k} &= 0, \\
c_j^{m+1,k} &= 2\,(\lambda+m-2j+1)\, a_{j-1}^{m,k-1}, \qquad j=1,\ldots,m+1-[k/2],
\end{aligned}
$$

for an odd k.

The explicit representation was used to show that the wavelets are polynomially localized as functions of the geodesic distance ϑ [H2, Theorem 6.5]:

Theorem 21 *Let Ψ_ρ^m be Poisson wavelet of order m. Then there exists a constant $\mathfrak{c}$ such that*

$$
|\rho^n \Psi_\rho^m(\cos(\rho\vartheta))| \leq \frac{\mathfrak{c}\cdot e^{-\rho}}{\vartheta^{m+n}}, \quad \vartheta \in \left(0, \frac{\pi}{\rho}\right],
$$

uniformly in ρ. $m+n$ is the largest possible exponent in this inequality.

Further, using representations as finite sums of the hyperspherical harmonics, I found explicit expressions for the Euclidean limits of Poisson wavelets [H2, Theorem 7.1]:

Theorem 22 *The Euclidean limits of Poisson wavelets are given by*

$$
G^m(|\xi|) = \frac{1}{\Sigma_n \lambda}\,(m+1)!\,\frac{C_{m+1}^{\lambda}(1/\sqrt{1+|\xi|^2})}{(1+|\xi|^2)^{(m+n)/2}}. \tag{27}
$$

I also proved that they are polynomially localized [H2, Proposition 7.2]:

Proposition 23 *The functions G^m decay at infinity polynomially with degree*

$$
m+n+(m+1)_{mod\ 2}.
$$

Apart from the above listed properties of Poisson wavelets, all of them being generalizations of the two-dimensional results from [41, 42, 45], the main contribution of [H2] is the proof that Poisson wavelets (when properly normalized) satisfy the commonly accepted spherical wavelet definition (with measure $\alpha(\rho)=\frac{1}{\rho}$) from [H1] (cf.also [24]), both as bilinear and as linear wavelets [H2, Sections 8 and 9]. The difficulty of the proof that a function family is a wavelet lies in the estimation of the triple integral on the left-hand-side of (17) or (22). In the case of Poisson wavelets, the previously proved estimations were used in order to show that the inequalities (17) and (22) hold.

4.5.1 Frames of Poisson wavelets

One more reason why Poisson wavelets are so useful in applications is the existence of their discrete frames. A proof of this fact for Poisson wavelets over the two-dimensional sphere was the subject of my doctoral thesis, and in the paper [H3] I generalized these results to the n-dimensional case. The frame property of Poisson wavelets is a big advantage for data storage: the knowledge of wavelet transform values over a countable set of arguments yields the whole information about the analyzed function.

It can be shown by a simple calculation that a wide class of zonal spherical wavelets (Poisson wavelets included) constitutes a semi-discrete frame, i.e., such that the scale variable is discretized, and the spherical variable remains unaltered, cf. [H3, Theorem 3.2].

Theorem 24 *Let $\{\Psi_\rho :\ \rho\in\mathbb{R}_+\}$ be a wavelet family with*

$$\widehat{\Psi}_\rho(l)=\frac{l+\lambda}{\lambda}\cdot\gamma\big(\rho\cdot\tau(l)\big),$$

where τ is an arbitrary function and γ is such that $\int_0^\infty \left||\gamma^2|'(t)\right|\,dt<\infty$. Then, for any $\epsilon>0$ and $\delta>0$ there exists a constant q such that for any sequence $\mathcal{B}=(b_j)_{j\in\mathbb{N}_0}$ with $b_0\geq -\log q$ and $1<b_j/b_{j+1}<1+q\cdot\delta$ the family $\{\Psi_{b_j,x},\ b_j\in\mathcal{B},\ x\in\mathcal{S}^n\}$ with measure $\{\nu_j = C\cdot\log\frac{b_j}{b_{j+1}}\}$ is a semi-continuous frame for $\mathcal{L}^2(\mathcal{S}^n)$, satisfying the frame condition (13) *with the prescribed ϵ.*

Full discretization of frames is essentially based on the estimation of the error made by discretizing convolution of two wavelet kernels. The reproducing kernel Π of the wavelet transform with respect to zonal wavelets, satisfying

$$\begin{aligned}\mathcal{W}_\Psi f(a,x)&=\Pi(a,x;b,y)*\mathcal{W}_\Psi f(b,y)\\ &=\int_0^\infty\int_{\mathcal{S}^n}\Pi(a,x;b,y)\mathcal{W}_\Psi f(b,y)\,d\sigma(y)\alpha(b),\end{aligned}$$

is given by

$$\Pi(a,x;b,y)=C\cdot\big\langle\Psi_{(x,a)},\Psi_{(y,b)}\big\rangle_{\mathcal{L}^2(\mathcal{S}^n)}$$

with a constant C, and in the case of Poisson wavelets it can be expressed in terms of the wavelets themselves,

$$\Pi^m(a,x;b,y)=C\cdot\frac{(ab)^m}{(a+b)^{2m}}\,\Psi^{2m}_{a+b}(x\cdot y).$$

Thus, it is well polynomially localized as a function of the angular variable $x\cdot y$. This property as well as the boundedness of its gradient is utilized by the estimation of

$$\left|\sum_{(b,y)\in\Lambda}\Pi(a,x;b,y)\,\Pi(b,y;c,z)\,\mu(b,y)\right.\\ \left.-\widetilde{C}\sum_{b\in\mathcal{B}}\int_{\mathcal{S}^n}\Pi(a,x;b,y)\,\Pi(b,y;c,z)\,d\sigma(y)\,\nu(b)\right|,\tag{28}$$

see Theorem 7. The grid $\Lambda \subseteq \mathcal{B} \times \mathcal{S}^n$ is supposed to be of type (δ, Ξ). That means that at each scale $b = b_j$, there is a measurable partition of $\mathcal{S}^n$ $\mathcal{P}_b = \{\mathcal{O}_k^{(b)},\, k = 1, 2, \dots, K_b\}$ into simply connected sets such that the diameter of each set (measured in geodesic distance) is not larger than $\Xi\, b$. Each of these sets contains exactly one point of the grid, and the measure is given by $\mu(b, y) = \sigma(\mathcal{O}_k^{(b)})$.

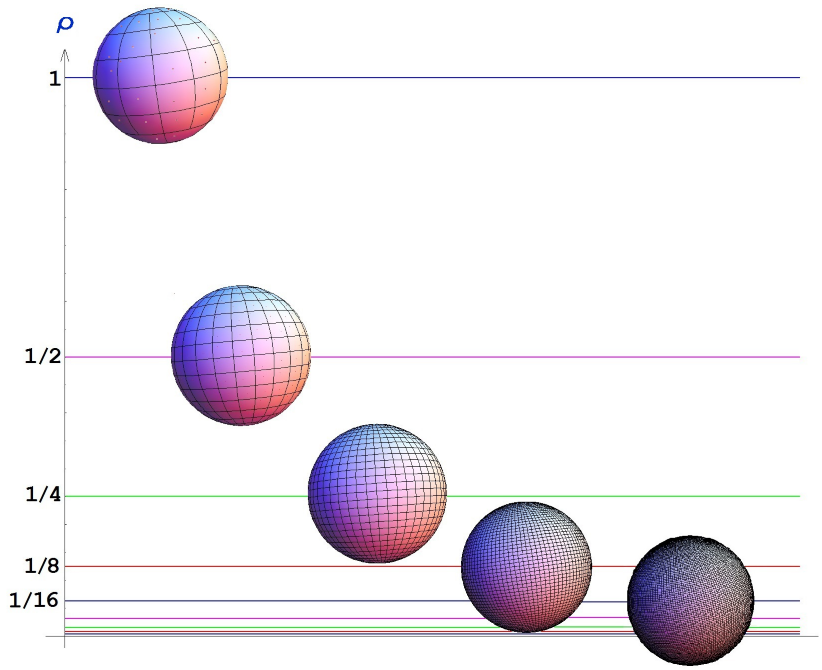

Figure 2: A example of a grid of type (δ, Ξ)

The error (28) is shown to be less than

$$\frac{1}{c^n}\, f\left(\frac{a}{c}, \frac{\angle(x, z)}{c}\right)$$

for some $f \in \mathcal{L}^1\left(\mathbb{R}_+ \times \mathbb{R}_+, (\frac{da}{a}, \vartheta^{n-1}\, d\vartheta)\right)$ with $\|f\|$ that is arbitrarily small if Ξ is small enough. The calculation requires a clever partition of the integration region as well as careful estimations of the kernels, compare the proof of [H3, Theorem 4.2]. According to Theorem 7, the fact that the integral of the error (28) exists and is less than a prescribed value (depending on some properties of the semi-discrete frame the construction is based on) implies the existence of a fully discrete frame

$$\{\Psi_{b,x},\, (b, x) \in \Lambda\}$$

with measure μ, compare [H3, Corollary 2.10].

The main thesis of [H3], the existence of fully discrete wavelet frames over n-dimensional spheres, is proven under some assumptions on the localization of the wavelet kernels. To my best knowledge, from the wavelet families constructed so far, only Poisson wavelets (of order $m \geq n+1$) satisfy these constraints, cf. [H3, Theorem 6.7]. However, [H3, Theorem 4.2] admits a wider class of wavelets. Further, this way of proving, i.e., error estimation of kernel convolution discretizing, although challenging, yields an additional information about the structure of the grid Λ, in particular proportionality of the set diameter in the partition $\mathcal{P}_b$ to the scale b.

4.6 Directional wavelets [H4]

As I mentioned before, one of the main achievements of Ebert *et al.* in [20] was the defining of nonzonal wavelets and nonzonal wavelet transform. However, no example was presented in the paper such that the applicability and usefulness of the definition were difficult to be estimated. In [H4] I was seeking to construct an example of a directional wavelet family. The idea my work was based on originates from [38], where the authors derive Poisson kernel when the source $r\hat{e}$ is moved orthogonally to the z-axis (complementary to the derivative along z-axis in (23)).

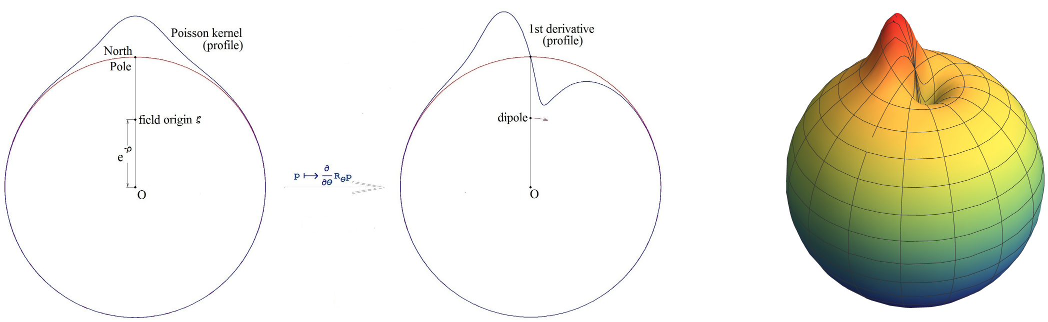

Figure 3: Directional derivative of Poisson kernel

Definition 25 [H4, Definition 3.1] *Directional Poisson multipole wavelet of order $d \in \mathbb{N}$ is defined as*

$$\Psi_\rho^{[d]}(x) = \rho^d \frac{\partial^d}{\partial \vartheta^d} \left(p_{\Upsilon_\vartheta^{-1}\zeta}(x) \right) \Big|_{\vartheta=0},$$

where Υ_ϑ denotes the rotation of $\mathbb{R}^{n+1}$ in the plane $(\widehat{e}, x_2)$ with rotation angle ϑ, and $\zeta = e^{-\rho}\,\hat{e}$.

In order to investigate wavelets arising in such a way I derived exact recursive formulae for directional derivatives of the hyperspherical harmonics [H4, Lemma 4.2].

Lemma 26 *Let $n \geq 3$ and $l \in \mathbb{N}$ be fixed. Then*

$$\frac{\partial}{\partial \vartheta} Y_l^{(k_1,0,\dots,0)}(\Upsilon_\vartheta x)\bigg|_{\vartheta=0} = \beta_{l,k_1-1}\, Y_l^{(k_1-1,0,\dots,0)}(x) - \beta_{l,k_1}\, Y_l^{(k_1+1,0,\dots,0)}(x) \tag{29}$$

for

$$\beta_{l,k_1} = \sqrt{\frac{(k_1+1)\,(2\lambda+k_1-1)\,(l-k_1)\,(2\lambda+l+k_1)}{(2\lambda+2k_1-1)\,(2\lambda+2k_1+1)}}, \tag{30}$$

$k_1 = 0, 1, \dots, l$, and

$$\beta_{l,-1} = 0.$$

Let $n = 2$ and $l \in \mathbb{N}$ be fixed. Define $\widetilde{Y}_l^k$ as

$$\widetilde{Y}_l^k = \begin{cases} Y_l^{-k} + Y_l^k & \text{for } k = 0, 1, \dots, l, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\frac{\partial}{\partial \vartheta} \widetilde{Y}_l^0(\Upsilon_\vartheta x)\bigg|_{\vartheta=0} = -\sqrt{l(l+1)}\, \widetilde{Y}_l^1(x) \tag{31}$$

and

$$\frac{\partial}{\partial \vartheta} \widetilde{Y}_l^k(\Upsilon_\vartheta x)\bigg|_{\vartheta=0} = \beta_{l,k-1}\, \widetilde{Y}_l^{k-1}(x) - \beta_{l,k}\, \widetilde{Y}_l^{k+1}(x) \tag{32}$$

for $k = 1, 2, \dots, l$.

The proof is technical and utilizes properties of the Gegenbauer polynoimials. Based on that lemma, I derived an algorithm for computing Fourier coefficients of a directional derivative of a zonal function [H4, Theorem 4.3].

Theorem 27 *Let a zonal function*

$$f = \sum_{l=0}^{\infty} a_l^0(f)\, Y_l^0$$

be given. Then

$$f^{(d)} := \frac{\partial^d}{\partial \vartheta^d}\, (f(\Upsilon_\vartheta x))\bigg|_{\vartheta=0} = \sum_{l=0}^{\infty} \sum_{j=0}^{\left[\frac{d}{2}\right]} a_l^{2j+d_{mod2}}(f^{(d)}) Y_l^{(2j+d_{mod2},0,\dots,0)}(x) \tag{33}$$

for $n \geq 3$ or

$$f^{(d)} = \sum_{l=0}^{\infty} \sum_{j=0}^{\left[\frac{d}{2}\right]} a_l^{2j+d_{mod2}}(f^{(d)}) \left(Y_l^{2j+d_{mod2}} + Y_l^{-(2j+d_{mod2})}\right) \tag{34}$$

for $n=2$ with coefficients $a_l^k(f^{(d)})$, obtained recursively via

$$\begin{aligned} a_l^{2j+1}(f^{(d)}) &= \beta_{l,2j+1} a_l^{2j+2}(f^{(d-1)}) - \beta_{l,2j} a_l^{2j}(f^{(d-1)}),\\ a_l^{2j}(f^{(d)}) &= 0 \end{aligned}$$

for an odd d and

$$\begin{aligned} a_l^0(f^{(d)}) &= -\beta_{l,0}\, a_l^1(f^{(d-1)}) \qquad (\textit{for } n \geq 3),\\ a_l^0(f^{(d)}) &= -2\beta_{l,0}\, a_l^1(f^{(d-1)}) \qquad (\textit{for } n = 2),\\ a_l^{2j}(f^{(d)}) &= \beta_{l,2j-1}\, a_l^{2j-1}(f^{(d-1)}) - \beta_{l,2j}\, a_l^{2j+1}(f^{(d-1)}),\\ a_l^{2j+1}(f^{(d)}) &= 0 \end{aligned}$$

for an even d, where $\beta_{l,k}$ are defined as in Lemma 26.

The structure of the coefficients β_{l,k_1} arising by the derivation causes that neither of the wavelet families known hitherto nor constructed in a similar manner, i.e., with Gabor coefficients being samples of a positive integer or half-integer power of a polynomial multiplied by an exponential function with a polynomial exponent, can satisfy the conditions of Definition 9. (The definition used in [38] is slightly different and admits a wider class of wavelets. Its main disadvantage is that the reconstruction is possible only up to a Fourier multiplier. Compare also the zonal version [39] and my discussion concerning Poisson wavelets [H2].) My idea to overcome this problem was to apply two distinct wavelet families for wavelet analysis and wavelet synthesis, a maneuver often used in wavelet analysis over the Euclidean space.

Definition 28 [H4, Definition 5.1] *Let $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ be a weight function. Families $\{\Psi_\rho\}_{\rho\in\mathbb{R}_+} \subseteq \mathcal{L}^2(\mathcal{S}^n)$ and $\{\Omega_\rho\}_{\rho\in\mathbb{R}_+} \subseteq \mathcal{L}^2(\mathcal{S}^n)$ are called an admissible wavelet pair if they satisfy the following conditions:*

1. *for $l \in \mathbb{N}_0$*

$$\sum_{\kappa=1}^{N(n,l)} \int_0^\infty \overline{a_l^\kappa(\Psi_\rho)}\, a_l^\kappa(\Omega_\rho)\, \alpha(\rho)\, d\rho = N(n,l), \tag{35}$$

2. *for $R \in \mathbb{R}_+$ and $x \in \mathcal{S}^n$*

$$\int_{\mathcal{S}^n} \left| \int_R^\infty (\overline{\Psi_\rho} \hat{*} \Omega_\rho)(x \cdot y)\, \alpha(\rho)\, d\rho \right| d\sigma(y) \leq \mathfrak{c} \tag{36}$$

   with $\mathfrak{c}$ independent of R.

The wavelet transform is given by the same formula as in [H1], cf. [H4, Definition 5.2], and it is invertible by

$$f(x) = \int_0^\infty \int_{SO(n+1)} \mathcal{W}_\Psi f(\rho, \Upsilon)\, \Omega_\rho(g^{-1}x)\, d\nu(\Upsilon)\, \alpha(\rho)\, d\rho.$$

Further, unless $\{\Psi_\rho\}$ and $\{\Omega_\rho\}$ are not equal to each other, the wavelet transform is not an isometry.
It is proved in [H4, Theorem 5.7] that certain linear combinations of directional derivatives of Poisson kernel and the kernel

$$h_\rho(x) = \sum_{l=0}^{\infty} e^{-\frac{\rho l^2}{2\lambda}} \mathcal{C}_l^\lambda \tag{37}$$

are an admissible wavelet pair with respect to $\alpha(\rho) = \frac{1}{\rho}$. Roughly speaking, directional wavelets derived from (37) are the reconstruction family for the directional Poisson wavelets. Additionally, in the paper [H4] explicit formulae for the Euclidean limit of the directional Poisson wavelets are derived [H4, Theorem 6.1].

Theorem 29 *The Euclidean limits of directional Poisson wavelets are given by*

$$G^{[d]}(\xi) = \frac{\partial^d}{\partial \xi_2^d} \frac{2}{\Sigma_n (1+|\xi|^2)^{\lambda+1}}, \tag{38}$$

$\xi = (\xi_2, \xi_3, \ldots, \xi_{n+1}) \in \mathbb{R}^n$.

The Appendix of [H4] contains an application of the recursive formula from Theorem 4.3, yielding an explicit formula for the second directional derivative of Poisson kernel,

$$\begin{aligned}\left(p_\zeta^\lambda\right)^{(2)}(x) = &- \frac{2(\lambda+1)\, e^{-\rho}\,(1-e^{-2\rho})}{\Sigma_n\,(1-2e^{-\rho}\cos\vartheta_1 + e^{-2\rho})^{\lambda+2}} \cos\vartheta_1 \\ &+ \frac{4(\lambda+1)(\lambda+2)\, e^{-2\rho}\,(1-e^{-2\rho})}{\Sigma_n\,(1-2e^{-\rho}\cos\vartheta_1 + e^{-2\rho})^{\lambda+3}} \sin^2\vartheta_1 \cos^2\vartheta_2.\end{aligned}$$

4.7 Discrete frames of nonzonal wavelets [H5]

The aim of my further research [H5] was a construction of discrete frames of directional wavelets. Due to the number of variables involved when nonzonal wavelets were considered, as well as the nonexistence of wavelet families of the common type being their own reconstruction families, an application of the method used in [H3] seemed impossible. The way to deal with this problem was to calculate directly, i.e., to estimate

$$\left| \int_X |\langle f_x, f\rangle|\, d\mu(x) - \|f\|^2 \right|$$

according to (13), when $\{f_x\}$ is properly chosen. This method requires a modification of the wavelet definition, cf. [H5, Definition 2.3].

Definition 30 *Let* $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ *be a weight function. The family* $\{\Psi_\rho\}_{\rho\in\mathbb{R}_+} \subseteq \mathcal{L}^2(\mathcal{S}^n)$ *is called a wavelet (family) of order* m *if it satisfies*

$$A \le \beta(l) \le B \tag{39}$$

for some positive constants A and B independent of $l \in \mathbb{N}_0$, $l > m$, and $\beta(l) = 0$ for $l \in \mathbb{N}_0$, $l \leq m$, where $\beta(l)$ is defined by

$$\beta(l) := \frac{\sum_{\kappa=1}^{N(n,l)} \int_0^\infty |a_l^\kappa(\Psi_\rho)|^2 \, \alpha(\rho) \, d\rho}{N(n,l)}.$$

The relaxation of the definition conditions by leaving the constraint (17) (respectively (36)) admits a big freedom in the wavelet construction. In [H5, Theorem 2.6] a wide class of functions with an infinite spectrum is proved to be wavelets.

Theorem 31 *Let $\{\Psi_\rho\} \subset \mathcal{L}^2(\mathcal{S}^n)$ be a zonal family of functions satisfying*

$$\widehat{\Psi}_\rho(l) = \left(\rho^a \, [q_\gamma(l)]^b\right)^c \, e^{-\rho^a \, [q_\gamma(l)]^b} \cdot \frac{l+\lambda}{\lambda} \qquad \text{for } l \in \mathbb{N}_0, \tag{40}$$

where q_γ is a polynomial of degree γ, strictly positive for a positive l, and a, b, c – some positive constants. Denote by $\Upsilon_{\widehat{\varsigma},\vartheta}$ the rotation of $\mathbb{R}^{n+1}$ in the plane $(\widehat{e}, \widehat{\varsigma})$, where $\widehat{\varsigma}$ is a rotation axis parallel to the tangent space at $\widehat{e}$, with rotation angle ϑ. Then for any $d \in \mathbb{N}$,

$$\left\{\Psi_\rho^{\widehat{\varsigma},d}(x)\right\} = \left\{\rho^{ad/(\nu b)} \, \frac{\partial^d}{\partial \vartheta^d} \, \Psi_\rho(\Upsilon_{\widehat{\varsigma},\vartheta} x)\bigg|_{\vartheta=0}\right\}$$

is a wavelet of order 0 according to Definition 30 with $\alpha(\rho) = \frac{1}{\rho}$.

What is lost, is the possibility to reconstruct the analyzed function by integration. However, wavelets satisfying Definition 30 constitute a continuous frame [H5, Theorem 2.5], thus, a reconstruction is possible by, e.g., an iterative algorithm. Further, for wavelets from Theorem 31 a scale discretization is possible [H5, Theorem 3.1].

Theorem 32 *Let $\{\Psi_\rho\} \subset \mathcal{L}^2(\mathcal{S}^n)$ be a wavelet family as in Theorem 31. Then, for any $\epsilon > 0$ there exist constants $\mathfrak{a}_0$ and X such that for any sequence $\mathcal{R} = (\rho_j)_{j \in \mathbb{N}_0}$ with $\rho_0 \geq \mathfrak{a}_0$ and $1 < \rho_j/\rho_{j+1} < X$ the family $\{\Psi_{\rho_j}(\Upsilon^{-1}\circ),\ \rho_j \in \mathcal{R},\ \Upsilon \in SO(n+1)\}$ is a semi-continuous frame for $\mathcal{L}^2(\mathcal{S}^n)$.*

And the main result, discretization of rotation parameter is proved for *each* wavelet family that possesses a semi-continuous frame. The rotation group $SO(n+1)$ is interpreted as the Cartesian product of $\mathcal{S}^J$, $J = 1, 2, \ldots, n$, and discretization is performed separately on each S^J [H5, Definition 4.1 and Theorem 4.2].

Definition 33 *Let n be given. We say Λ is a grid of type $(\delta_n, \delta_{n-1}, \ldots, \delta_1)$ if it is a discrete measurable set of rotations in $SO(n+1)$, constructed iteratively in the following way. There is a measurable partition $\mathcal{P}_n = \{\mathcal{O}^n_{\alpha_n} : \alpha_n = 1, \ldots, K_n\}$ of $\mathcal{S}^n$ into simply connected sets such that the diameter of each set (measured in geodesic distance) is not*

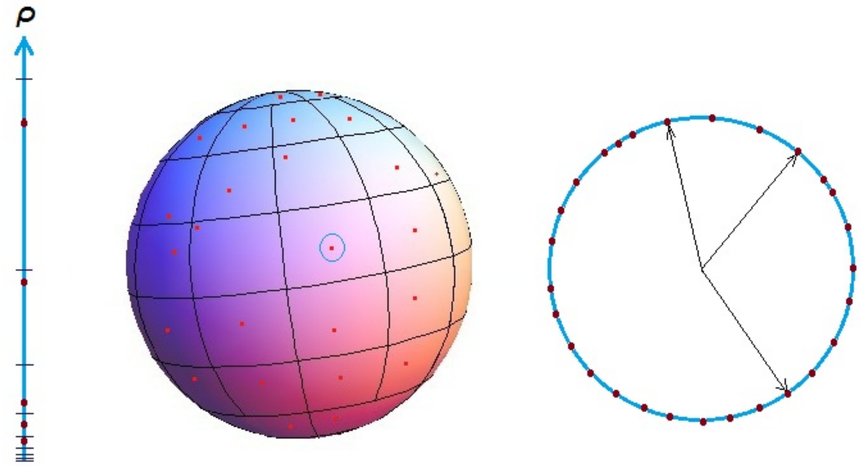

Figure 4: An example of a sequence $\mathcal{R}$ and a grid of type (δ_2, δ_1)

larger than δ_n. *Each of these sets contains exactly one point* $x^n_{\alpha_n}$, $\alpha_n = 1, \ldots, K_n$. *For* $J < n$ *let* $(\alpha_n, \alpha_{n-1}, \ldots, \alpha_{J+1})$ *be a fixed multi-index and*

$$\mathcal{P}_J = \mathcal{P}_j(\alpha_n, \alpha_{n-1}, \ldots, \alpha_{J+1}) = \{\mathcal{O}^J_{\alpha_J} : \alpha_J = 1, \ldots, K_J\}$$

a measurable partition of $\mathcal{S}^J$ *into* $K_J = K_J(\alpha_n, \alpha_{n-1}, \ldots, \alpha_{J+1})$ *simply connected sets of diameter not larger than* δ_J, *each of them containing exactly one point* $x^J_{\alpha_J} = x^J_{(\alpha_n, \ldots, \alpha_J)}$. *Then,* Λ *is the set of rotations given by*

$$\Upsilon_{(\alpha_n, \ldots, \alpha_1)} = \Upsilon^1(x^1_{\alpha_1})\, \Upsilon^2(x^2_{\alpha_2}) \ldots \Upsilon^n(x^n_{\alpha_n})$$

with the measure

$$\lambda(\Upsilon_{(\alpha_n, \ldots, \alpha_1)}) := \prod_{J=1}^{n} \lambda^J\left(x^J_{\alpha_J}\right), \qquad \lambda^J\left(x^J_{\alpha_J}\right) := \sigma_J\left(\mathcal{O}^J_{\alpha_J}\right).$$

For a single $\mathcal{S}^J$,

$$\Upsilon^J(x^J) = \Upsilon_1(\vartheta^J_1)\, \Upsilon_2(\vartheta^J_2) \ldots \Upsilon_{J-1}(\vartheta^J_{J-1})\, \Upsilon_J(\varphi^J), \qquad J = 1,\, 2, \ldots,\, n,$$

for

$$x^J = (\vartheta^J_1, \ldots, \vartheta^J_{J-1}, \varphi^J) \in \mathcal{S}^J,$$

where $\Upsilon_\iota(\vartheta)$, *is the rotation in the plane* $(x_\iota, x_{\iota+1})$ *with the rotation angle* ϑ, *and* ϑ^J_ι *and* φ^J, $\iota = 1, 2, \ldots, J-1$, *are the Euler angles of* Υ.

Theorem 34 *Let Ψ_ρ be a $\mathcal{C}^1$-wavelet family with the property that $\{\Psi_{\rho_j,x},\, j \in \mathbb{N}_0,\, x \in \mathcal{S}^n\}$ is a semi-continuous frame. Then, for each $j \in \mathbf{N}_0$ there exist sequences $(\delta_n^j, \delta_{n-1}^j, \ldots, \delta_1^j)$ such that*

$$\{\Psi_{\rho_j}(\Upsilon^{-1}_{(\alpha_n^j,\ldots,\alpha_1^j)} \circ),\, j \in \mathbf{N}_0,\, \Upsilon_{(\alpha_n^j,\ldots,\alpha_1^j)} \in \Lambda^j\}$$

is a frame for $\mathcal{L}^2(\mathcal{S}^n)$, provided that Λ^j is a grid of type $(\delta_n^j, \delta_{n-1}^j, \ldots, \delta_1^j)$.

In the proof, the boundedness of the $\mathcal{L}^2$-norms of the wavelet and its surface gradient on each scale ρ_j is used. It is shown that discretization can be performed fine enough such that the total error satisfies the required estimations.

5 Description of other scientific achievements

Besides the five papers, which constitute a mono-thematic series of publications, after obtaining the doctor degree, I published eight articles, and another four have been submitted to journals. The total number of my publications is 13, the number of citations, according to the Web of Science database ('Sum of the Times Cited' on 11/22/2018), is 62 (16 without self-citations), and the h-index (Hirsch index) is 5. The total *impact factor* of the journals for the five publications included in the *scientific achievement* is equal to 11.306, according to the Journal Citation Reports, and the total *impact factor* of the journals for all publications equals 21.529, see Table 1.

(a) The list of papers that have not been included into *the scientific achievement*

[P1] M. Holschneider and I. Iglewska-Nowak, *Poisson wavelets on the sphere*, J. Fourier Anal. Appl. 13 (2007), no. 4, 405–419.

[P2] I. Iglewska-Nowak and M. Holschneider, *Frames of Poisson wavelets on the sphere*, Appl. Comput. Harmon. Anal. 28 (2010), no. 2, 227–248.

[P3] I. Iglewska-Nowak and M. Holschneider, *Irregular Gabor frames*, Kyushu J. Math. 67 (2013), no. 1, 237–247.

[P4] I. Iglewska-Nowak, *Multiresolution on n-dimensional spheres*, Kyushu J. Math., 70 (2016), no. 2, 353–374.

[P5] I. Iglewska-Nowak, *On the uncertainty product of spherical wavelets,* Kyushu J. Math. 71 (2017), no. 2, 407–416.

[P6] I. Iglewska-Nowak, *Uncertainty of Poisson wavelets*, Kyushu J. Math. 71 (2017), no. 2, 349–362.

Table 1: Impact factor of the journals according to Journal Citation Report from the publication year (or 2017 for publications from 2018)

paper	journal	year of publication	impact factor
[H1]	Appl. Comput. Harmon. Anal.	2015	2.094
[H2]	J. Fourier Anal. Appl.	2015	0.912
[H3]	Appl. Comput. Harmon. Anal.	2016	2.634
[H4]	Appl. Comput. Harmon. Anal.	2018	2.833
[H5]	Appl. Comput. Harmon. Anal.	2017	2.833
[P1]	J. Fourier Anal. Appl.	2007	1.125
[P2]	Appl. Comput. Harmon. Anal.	2010	3.144
[P3]	Kyushu J. Math.	2013	0.25
[P4]	Kyushu J. Math.	2016	0.375
[P5]	Kyushu J. Math.	2017	0.478
[P6]	Kyushu J. Math.	2017	0.478
[P7]	Appl. Comput. Harmon. Anal.	2018	2.833
[P8]	Int. J. Wavelets Multiresolution Inf. Process.	2018	0.54

[P7] I. Iglewska-Nowak, *Angular multiselectivity with spherical wavelets*, Appl. Comput. Harmon. Anal. 45 (2018), no. 3, 729–741.

[P8] I. Iglewska-Nowak, *Uncertainty product of the spherical Gauss-Weierstrass wavelet*, Int. J. Wavelets Multiresolution Inf. Process., Int. J. Wavelets Multiresolut. Inf. Process. 16 (2018), no. 4, 1850030, 14 pp.

[Pre1] I. Iglewska-Nowak, *A continuous spherical wavelet transform for $\mathcal{C}(\mathcal{S}^n)$*, arXiv: https://arxiv.org/abs/1806.07881.

[Pre2] I. Iglewska-Nowak, *Spin weighted wavelets on the sphere*, arXiv: https://arxiv.org/abs/1804.04947.

[Pre3] I. Iglewska-Nowak, *Uncertainty product of the spherical Abel-Poisson wavelet*, arXiv: https://arxiv.org/abs/1806.07883.

[Pre4] I. Iglewska-Nowak, *On the uncertainty product of spherical functions*, arXiv: https://arxiv.org/abs/1806.07880.

(b) A discussion of the results included in the aforementioned papers

5.1 The content of the doctoral thesis [P1–P3]

Proof of the existence of discrete frames of Poisson wavelets over $\mathcal{S}^2$ was the subject of my doctoral thesis assigned to me by my supervisor prof. Matthias Holschneider. The papers [P1] and [P2] present the results being the substance of the thesis. The strategy is analogous to the one developed later by me for wavelets over n-dimensional spheres and is precisely described in the previous section of this report.
In the paper [P3] the method of error estimation by discretizing the kernel convolution is applied to the Gabor transform in order to show that discrete frames exist. In the proof, the fact is utilized that an integrable majorant of the kernel exists, and an example of such a majorant is given. Additionally to these results that had been presented in my doctoral thesis, it is shown in [P3] that the existence of such a majorant implies boundedness of the gradient of the product of two kernels, a property that is utilized on the proof of the main result. In the case of spherical wavelets, boundedness of the wavelet kernel gradient must be explicitly required; here, the gradient of a single kernel is oscillating, therefore for the proof one uses the estimation of the kernel product. Finally, it is proven that the constraints on the kernel of the Gabor transform imply that the window function is from the Schwartz class (the converse implication is obvious). A numerical example demonstrates that this method yields a density bound sufficient for the sampled set to be a frame, however, one that is far from being optimal. This shows that the presented method should be applied rather as an existence proof.

5.2 Polynomial wavelets and multiresolution analysis [P4]

A wavelet construction over $\mathcal{S}^n$, essentially different from the one derived from approximate identities, is proposed in [P4]. The approach is a generalization of the results concerning wavelets over $\mathcal{S}^2$ presented in [18]. Wavelets in [P4] are linear combinations of hyperspherical harmonics, a feature that on the one hand results in an oscillatory behavior of the wavelets themselves, on the other hand, ensures orthogonality of the wavelets and scaling functions (in this case, a wavelet is a difference of scaling functions on different scales), their reproducing property, as well as a kind of localization property. The wavelets and the scaling functions are sampled over equiangular grids. They satisfy a simple two-scale relation, and they are frames for the spaces that they span. The reconstruction algorithm is quite simple but instabilities can occur by its application caused by the sampling point concentration around the poles. One of the most important results is the definition of a multiresolution analysis of sampling spaces. To my best knowledge, this has the first attempt to define an MRA over $\mathcal{S}^n$. Additionally, a formula for the computation of the uncertainty product of zonal $\mathcal{S}^n$-functions is derived, and the uncertainty products of the wavelets as well as scaling functions are computed.

5.3 Results based on the theory of wavelets derived from approximate identities [P5–P8,Pre1–Pre3]

5.3.1 The uncertainty product

The results presented in the papers [H1–H5] open a wide field of research. Thanks to their generality directional spherical wavelets can be constructed or appraised according to various criteria. This is the subject of my further research.
A quality criterion of an analyzing function is its uncertainty product [54, 57]. An uncertainty product (constant) of a function is a measure for the trade-off between the spatial and frequency localization. In the case of continuous functions over the sphere it is given by

$$U(F)=\sqrt{\frac{1-\|\xi_O(F)\|^2}{\|\xi_O(F)\|^2}}\cdot\frac{\|\nabla_{\mathcal{S}^n}F\|_2}{\|F\|_2},$$

where

$$\xi_O(F)=\frac{1}{\|F\|_2^2}\int_{\mathcal{S}^n}x\,|F(x)|^2\,d\sigma(x)$$

for $F\in\mathcal{C}(\mathcal{S}^n)$. It is bounded from below by $\frac{n}{2}$ [57, 36]. It means that a function cannot be simultaneously sharp in space and frequency. This statement corresponds to the Heisenberg uncertainty principle in physics.
It was shown in [47] that the uncertainty product of Gauss-Weierstrass kernel over $\mathcal{S}^2$ approaches the minimum value when $\rho\to 0$. An interesting question is to classify other function families from Theorem 31 according to this benchmark or to compute the uncertainty product of the wavelets with finite spectrum, similarly as it was done in [P4] (for a different class of spherical wavelets, one with an a priori discrete set of scales). The paper [P5] contains general results concerning the uncertainty product of functions given by (40) [P5, Theorem 3.4].

Theorem 35 *Let $\{\Psi_\rho\}$ be a zonal wavelet family with (40), where $a>0$, $c>0$, and $q_\nu(l)=a_\nu l^\nu+a_{\nu-1}l^{\nu-1}+\cdots+a_1l+a_0$ is a polynomial of degree ν, positive and monotonously increasing for $l\geq 1$. The uncertainty product of Ψ_ρ for $\rho\to 0$ behaves like*

$$U(\Psi_\rho)\leq\mathcal{O}\left(\rho^{\frac{-a}{2\nu}}\right).$$

That means that boundedness of the uncertainty constant of spherical wavelets constructed in that way is in general not given. Further investigations show that the uncertainty product of certain functions satisfying the conditions of Theorem 35 is bounded for $\rho\to 0$. These functions are Gauss-Weirstrass wavelet $\{\Psi_\rho^G\}$ (being a derivative of Gauss-Weierstrass kernel) satisfying [P8]

$$U(\Psi_\rho^G)\leq\sqrt{2\left(1+\frac{6}{e}+\frac{16}{e^2}\right)}+O(1),\rho\to 0,$$

and Abel-Poisson wavelet $\{\Psi_\rho^A\}$ (that can be interpreted as Poisson wavelet of order $\frac{1}{2}$) with [Pre4]

$$\lim_{\rho\to 0} U(\Psi_\rho^A) = \frac{1}{2}\sqrt{\frac{(n+1)(n+2)(n^2-3n+3)}{n(n-1)}}.$$

The most interesting result is the one concerning Poisson wavelets $\{g_\rho^m\}$ [P6]. Their uncertainty product is not only bounded in limit $\rho \to 0$, but it also approaches the optimal value for some limiting cases. More exactly, for a fixed n, Poisson wavelet of order $[(n-1)/2]$ has the smallest limit of the uncertainty constant for $\rho \to 0$ among all Poisson wavelets,

$$\min_{m\in\mathbb{N}} \lim_{\rho\to 0} U(g_\rho^m) = \frac{1}{2}\sqrt{\frac{n(n-1)(2n-1)}{2n-3}}.$$

For $n \to \infty$, the value of this expresssion behaves like $\frac{n}{2}$, i.e., it tends to the optimal value. This shows that among of spherical functions, it is not only Gauss kernel that has this property, contrary to the Euclidean case.

5.3.2 Angular multiselectivity [P7]

Another feature of the wavelets that can be quantified is their angular selectivity. It is an interesting task to construct directional wavelets being sharp in analyzing directional components of a signal, i.e., sharp with respect to the polar variable [P7].

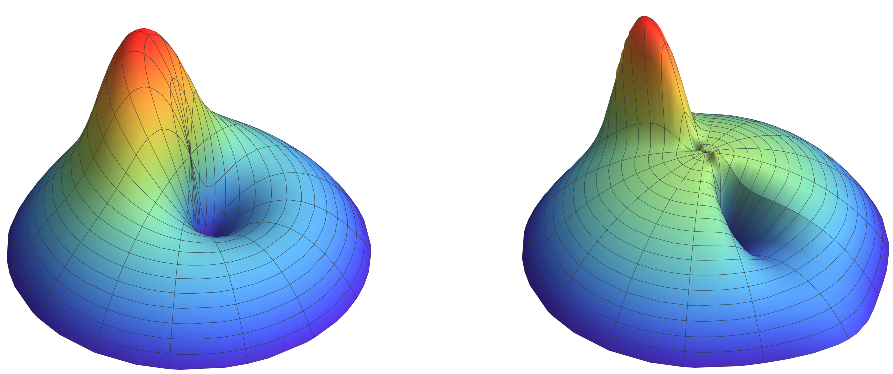

Figure 5: A wavelet with small and a wavelet with big angular selectivity

The first directional derivative of Poisson kernel is a function with separated variables,

$$g_\rho^1(\vartheta,\varphi) = \upsilon_\rho(\vartheta)\cdot\cos(\varphi).$$

In order to reach the above described goal I was trying to replace $\cos(\varphi)$ by

$$f_\tau(\varphi) = \sum_{j\in\mathbb{Z}} F_\tau(\varphi + 2j\pi), \quad \varphi \in [0, 2\pi),$$

where F_τ is given by

$$F_\tau(\phi) = e^{-\frac{\tau^2\phi^2}{2}} - e^{-\frac{\tau^2(\phi-\pi)^2}{2}}, \quad \phi \in \mathbb{R}.$$

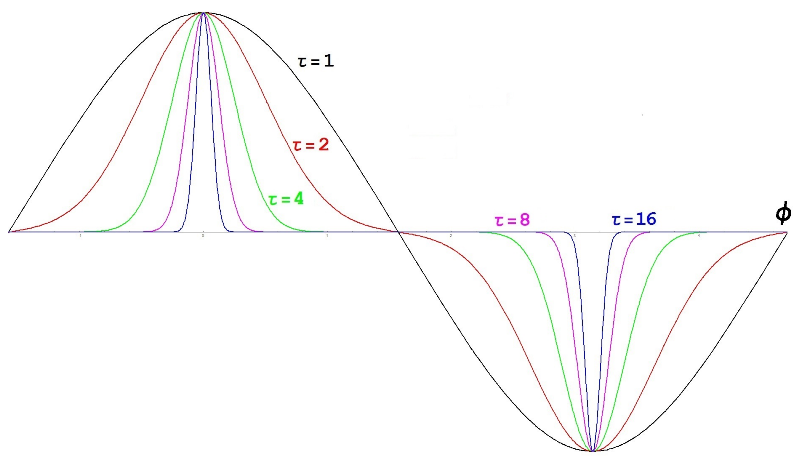

Figure 6: f_τ for distinct τ

However, the function $(\vartheta, \varphi) \mapsto \upsilon_\rho(\vartheta)\cdot\cos(\varphi)$ is not a wavelet. Therefore, it was necessary to find a function ω_ρ such that $(\vartheta, \varphi) \mapsto \omega_\rho(\vartheta) \cdot f_\tau(\varphi)$ satisfies the conditions of Definition 30. The way for doing it was a manipulation with the coefficients of various derivatives of Poisson kernel and verification whether both inequalities (16) and (17) are satisfied. I found two solutions:

$$\omega_\rho^{(1)}(\vartheta) = \rho \cdot \sin^5\vartheta \cdot r \cdot \frac{\partial}{\partial r}\left[r \cdot \frac{\partial}{\partial r} p_{r\hat{e}}(\cos\vartheta)\right],$$
$$\omega_\rho^{(2)}(\vartheta) = \rho \cdot \sin^5\vartheta \cdot r^2 \cdot \frac{\partial^2}{\partial r^2} p_{r\hat{e}}(\cos\vartheta)$$

with $r = e^{-\rho}$. In both cases, $\omega_\rho \cdot f_\tau$ is a wavelet for each value of $\tau \geq 1$. Thus, if the wavelet transform is discretized according to the scheme from [H5], the angular selectivity can be adapted to the features detected in each point of the analyzed signal by varying the parameter τ.

5.3.3 Spin weighted wavelets [Pre2]

Further, having cosmological applications in mind (in particular, the analysis of Cosmic Microwave Background polarization), I defined spin weighted spherical wavelets [Pre2], a generalization of needlet-type spin wavelets [30]. Polarization can be interpreted as a section of a line bundle on the sphere and needlet-type spin wavelets are a tool to analyze such sections of line bundles instead of ordinary functions. They are generalizations of the spherical needlets [32], being a special case of the spherical wavelets [H1]. Thus, the construction of spin-weighted spherical wavelets extends the class of functions that can be used in the analysis of CMB polarization.

5.3.4 Wavelet transform of continuous functions [Pre1]

Another interesting task is a multiresolution analysis based on the spherical wavelets (instead of the hyperspherical harmonics). Once defined, it could serve as a base to construct Schauder basis for *continuous* functions over $\mathcal{S}^n$ or at least $\mathcal{S}^2$, in analogy to the strategy presented in, e.g., [21]. A careful choice of functions spanning the single spaces is necessary such that the resulting Lebesgue constants are uniformly bounded, compare also [56]. In [Pre1] I constructed a wavelet in such a way that the inverse wavelet transform of a continuous function is convergent in the supremum norm.

5.4 The uncertainty product of spherical functions [Pre4]

My previous results concerning the uncertainty product of distinct classes of wavelets were restricted to rotation-invariant functions. One of the reasons was that the computation of the uncertainty product in the nonzonal case is quite sophisticated. In [Pre4] I derived a formula that expresses the uncertainty product of a continuous function in terms of its Fourier coefficients. Further, I applied it to the second directional derivative of Poisson wavelet g_ρ^1.

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Ilona Iglewska-Nowak