Summary of scientific achievements

1 Name: Jerzy Legut

2 Scientific degrees:

- Ph. D. in Mathematics, Wroclaw University of Science and Technology 1984 (June), Thesis topic: *Games of Fair Divisions.* Supervisor: prof. dr hab. Rastislav Telgarsky
- M. Sc. in Mathematics, speciality: applied mathematics, Wroclaw University of Science and Technology 1981 (July), Thesis topic: *Games of Fair Divisions and Lyapunov Theorem.* Supervisor: prof. dr hab. Rastislav Telgarsky

3 Information on previous employment in scientific institutions:

- 2016-: Wroclaw University of Science and Technology, Faculty of Pure and Applied Mathematics, Lecturer
- 1984-1994 The Technical University of Wroclaw, Institute of Mathematics, Lecturer

4 General description of my scientific achievements

My scientific achievements consist of four parts:

- game theoretical approach to problems of fair division
- applications of fair division results to mathematical economics
- investigating properties of the range of nonatomic vectors measures and their applications to the theory of fair division
- methods of optimal partitioning of a measurable space and their applications to fair division theory and decision theory

The main results of the two first part of my scientific activity were achieved in years 1984 - 1994. Most of these results were published in papers being in the base of JCR (Journal Citation Reports) journals. Some of them were presented on two international game theory conferences in USA (1988, 1991). I was also invited by various universities (in USA, Israel, Holland) to give lectures on my results obtained that time. I established cooperation with mathematicians from Holland as a result of which two joint articles were written and published.

After twenty years break in my scientific activity I returned to work on the fair division problems. I have concentrated on the two last parts of my subject interests listed above. My postdoctoral dissertation deals with the achievements in these topics. My results were published in 7 articles, which three of them are in the base of JCR. The estimated [1] total scoring of all papers written by me and with co-authors according to the scoring

[1] Since the scoring of journals published before 2010 are not available I took into account in my calculations the earliest available data.

system of the Ministry of Science and Higher Education is about 375. In this number the total scoring of papers constituting the scientific postdoctoral achievement is equal 125. According to the Web of Science the total number of citations for my articles is 32 (not including the self citations).

I continue my the scientific research on the fair division problems together with other authors. The latest results of the research were presented in unpublished yet articles ([35, 36, 37]).

5 The indication of the scientific achievements

(a) The title of the scientific achievements

Methods of optimal partitioning of a measurable space

(b) The list of articles constituting the scientific achievements

H1. Legut J. and Wilczyński M.: *How to obtain a range of a nonatomic vector measure in* $\mathbb{R}^2$, J. Math. Anal. Appl. 394, 102-111 (2012)

H2. Dall'Aglio M., Legut J., Wilczyński M.: *On Finding Optimal Partitions of a Measurable Space*, Mathematica Applicanda, vol. 43(2), 193-206 (2015)

H3. Legut J.: *Optimal Fair Division for Measures with Piecewise Linear density Functions,* International Game Theory Review, vol. 19, No. 2, 175009, (2017)

H4. Legut J.: *Connecting two points in the range of a vector measure*, Colloquium Mathematicum, vol. 153, No. 2, 163-167 (2018)

H5. Legut J.: *How to obtain an equitable optimal fair division* Ann. Oper. Res. published on line, https://doi.org/10.1007/s10479-018-3053-2 , (2018)

H6. Legut J.: *On a method of obtaining an approximate solution of an exact fair division problem,* Mathematica Applicanda, vol. 46 (2), 245-256 (2018)

H7. Jóźwiak I. and Legut J.: *Minimax decision rules for identifying an unknown distribution of a random variable,* Proceedings of 39th International Conference on Information Systems Architecture and Technology, ISAT 308-317 (2018)

(c) A discussion of the mentioned articles and obtained results with presentation of possible applications

Contents

5.1 Summary of the main results contained in the listed articles

The most important scientific achievements presented in the articles deal with methods of obtaining optimal partition of a measurable space $\{\mathcal{X}, \mathcal{B}\}$ according to given nonatomic measures $\{\mu_i\}_{i=1}^n$. Different kinds of optimality notions are considered. Below I present summaries of the main results of each paper.

Paper [H1] deals with a constructive method of effecting a range of a two-dimensional nonatomic vector measure. The authors showed how to obtain a function which describes the boundary of the convex and compact range of a vector measure defined by given density functions. According to my knowledge it is the first result concerning construction of a range of vector measure for arbitrary density functions. This construction can be applied in theory of fair division for obtaining different kinds of divisions. There are some examples presented in this paper illustrating the construction and its applications.

Paper [H2] presents an algorithm for finding "almost" optimal partitions of the unit interval $[0,1)$ according to given nonatomic measures $\{\mu_i\}_{i=1}^n$. This algorithm is based on approximation of measurable density functions by simple functions. The optimal partition for measures defined by simple functions is obtained by linear programming method. The authors used this method to give lower and upper bounds for the optimal value of optimal partitioning. An example for three players illustrating this method is also presented. Discussion of the number of cuts needed for finding the optimal partitions is given.

Paper [H3] presents a nonlinear programming method for finding an optimal fair division of the unit interval $[0,1)$ among n players which preferences are described by nonatomic probability measures $\{\mu_i\}_{i=1}^n$ defined by piecewise linear (PWL) density functions. Presented algorithm can be applied for obtaining "almost" optimal fair divisions for measures with arbitrary density functions approximable by PWL functions. The number of cuts of $[0,1)$ needed for obtaining such divisions is given.

Paper [H4] considers some properties of a range of a nonatomic vector measure $\{\mu_i\}_{i=1}^n$ defined on measurable subsets $\mathcal{B}$ of the unit interval $[0,1]$. Denote by $\mathcal{U}(k)$ a collection of all sets being a union of at most k pairwise disjoint subintervals of $[0,1]$. The author showed that if $A \in \mathcal{U}(k)$ then the line segment connecting the origin point $(0,...,0) \in \mathbb{R}^n$ and the point $\mu(A)$ is contained in $\mu(\mathcal{U}(n+k-1))$. Moreover, it is proved that if $B, C \in \mathcal{U}(k)$ then the line segment connecting the points $\mu(B)$ and $\mu(C)$ is a subset of $\mu(\mathcal{U}(2n+4k-3))$. This result is used for showing yet another proof of the famous Lyapunov convexity theorem. The paper deals also with a discussion of the two-dimensional case for specific vector measures and gives some interesting conclusions.

Paper [H5] presents a nonlinear programming method for finding an equitable optimal fair division of the unit interval $[0,1)$ among n players. Players preferences are described by nonatomic probability measures $\{\mu_i\}_{i=1}^n$ with density functions having piecewise strict monotone likelihood ratio (SMLR) property. For example, this property is satisfied by polynomial functions of positive degree. The presented algorithm can be used to obtain also an equitable ε-optimal fair division in case of measures with arbitrary differentiable density functions. An example of an equitable optimal fair division for three players is

given.
Paper [H6] presents an algorithm of obtaining an approximate solution of an exact division problem of partitioning of the unit interval $[0,1]$. By an exact division is meant a partition $P=\{A_i\}_{i=1}^n$ such that $\mu_i(A_j)=1/n$ for all $i,j=1,...,n$ and $\cup_i A_i=[0,1]$. This partition is optimal in the sense of fair division theory. It is proportional, envy-free and equitable at the same time. An iterative procedure based on a theorem due to Alon [1] is presented. An example of implementing this algorithm is given.
Paper [H7] was published in proceedings of a conference indexed by Web of Science. It deals with applications of optimal partitioning of a measurable space to decision theory. The authors considered two-dimensional case of identifying an unknown distribution of a random variable and they also presented an example of finding a minimax decision rule for some distribution defined on the unit square.

5.2 Introduction

5.2.1 Definition of α-optimal partitions

First we present definitions, theorems and motivation for the development of the fair division theory.

Let $\{\mu_i\}_{i=1}^n$, $(n>1)$, denote nonatomic probability measures defined on a measurable space $\{\mathcal{X},\mathcal{B}\}$. By a partition $P=\{A_i\}_{i=1}^n$ of this space we mean a collection of $\mathcal{B}$-measurable pairwise disjoint subsets $A_1,\ldots,A_n$ of $\mathcal{X}$ whose union is equal to $\mathcal{X}$. Let $\mathscr{P}$ stand for the set of all measurable partitions $P=\{A_i\}_{i=1}^n$ of $\mathcal{X}$. Denote by

$$S_n=\{s=(s_1,\ldots,s_n)\in\mathbb{R}^n, s_i>0,\, i\in I,\, \sum_{i=1}^n s_i=1\},$$

the $(n-1)$-dimensional open simplex and let $\overline{S}_n$ be the closure of S_n in $\mathbb{R}^n$.
Let $\alpha=(\alpha_1,\ldots,\alpha_n)\in S_n$ and $I:=\{1,...,n\}$.

Definition 5.1. A partition $P^*=\{A_i^*\}_{i=1}^n\in\mathscr{P}$ is said to be α-optimal if

$$v^\alpha(\vec{\mu}):=\min_{i\in I}\left[\frac{\mu_i(A_i^*)}{\alpha_i}\right]=\sup_{P\in\mathscr{P}}\min_{i\in I}\left[\frac{\mu_i(A_i)}{\alpha_i}\right], \tag{5.1}$$

where the number $v^\alpha(\vec{\mu})$ denotes the best achievable value for the vector measure $\vec{\mu}=(\mu_1,\ldots,\mu_n)$ of partitioning of $\mathcal{X}$ proportional to coordinates of the vector $\alpha=(\alpha_1,\ldots,\alpha_n)\in S_n$.

The number $v^\alpha(\vec{\mu})$ (or v^α in short) we call the *α-optimal value* for the problem of α-optimal partitioning of a measurable space.

Definition 5.2. A partition $P=\{A_i\}_{i=1}^n\in\mathscr{P}$ is called an *equitable optimal* (or *optimal* in short) if it is *α-optimal* for $\alpha=(1/n,1/n,\ldots,1/n)\in S_n$.

The number $v := v^{\alpha}(\vec{\mu})/n$ is said to be *optimal value* for equitable optimal partitioning, i.e. for α-optimal partitioning of a measurable space, where $\alpha = (1/n, 1/n, \ldots, 1/n) \in S_n$. The existence of α-optimal partitions follows from the theorem of Dvoretzky et al. [18]:

Theorem 5.3. *If $\{\mu_i\}_{i=1}^n$ are nonatomic finite measures defined on the measurable space $\{\mathcal{X}, \mathcal{B}\}$ then the range $\vec{\mu}\,(\mathscr{P})$ of the mapping $\vec{\mu}\colon \mathscr{P} \to \mathbb{R}^n$ defined by*

$$\vec{\mu}\,(P) = (\mu_1(A_1), \ldots, \mu_n(A_n))\,, P = \{A_i\}_{i=1}^n \in \mathscr{P},$$

is convex and compact in $\mathbb{R}^n$.

5.2.2 A general form of α-optimal partitions

A general form of the α-optimal partition could be helpful in some cases for finding constructive methods of optimal partitioning of a measurable space. We can assume that all nonatomic measures $\{\mu_i\}_{i=1}^n$ are absolutely continuous with respect to the same measure ν (e.g. $\nu = \sum_{i=1}^n \mu_i$). Denote by $f_i = d\mu_i/d\nu$ the Radon-Nikodym derivatives, i.e.

$$\mu_i(A) = \int_A f_i\, d\nu, \text{ for } A \in \mathcal{B} \text{ and } i \in I.$$

For $\alpha = (\alpha_1, \ldots, \alpha_n) \in S_n$, $p = (p_1, ..., p_n) \in \overline{S}_n$ and $i \in I$, define the following measurable sets:

$$B_i(p) = \bigcap_{j=1, j\neq i}^{n} \left\{x \in \mathcal{X} : p_i\alpha_i^{-1} f_i(x) > p_j\alpha_j^{-1} f_j(x)\right\},$$

$$C_i(p) = \bigcap_{j=1}^{n} \left\{x \in \mathcal{X} : p_i\alpha_i^{-1} f_i(x) \geq p_j\alpha_j^{-1} f_j(x)\right\}$$

Legut and Wilczyński [33] using a minmax theorem of Sion (cf. [2]) proved the following:

Theorem 5.4. *For any $\alpha \in S_n$ there exists a point $p^* \in \overline{S}_n$ and a corresponding α-optimal partition $P^* = \{A_i^*\}_{i=1}^n$ satisfying*

(i) $B_i(p^*) \subset A_i^* \subset C_i(p^*)$,

(ii) $v^{\alpha}(\vec{\mu}) = \dfrac{\mu_1(A_1^*)}{\alpha_1} = \dfrac{\mu_2(A_2^*)}{\alpha_2} = \ldots = \dfrac{\mu_n(A_n^*)}{\alpha_n}$

Moreover, any partition $P^* = \{A_i^*\}_{i=1}^n$ which satisfies (i) and (ii) is α-optimal.

Legut and Wilczynski [33] proved that

$$v^{\alpha}(\vec{\mu}) = \max\left\{t \geq 1 : t(\alpha_1, \ldots, \alpha_n) \in \vec{\mu}\,(\mathscr{P})\right\}.$$

The above theorem gives a general form of α-optimal partition but unfortunately in general case of the densities f_i, $i \in I$, finding the numbers $p_1^*, ..., p_n^*$ is not easy.

5.2.3 Applications of α-optimal partitioning of a measureable space

Applications in the theory of fair division

The problem of α-optimal partitioning of a measurable space $(\mathcal{X}, \mathcal{B})$ can be viewed as a problem of fair division of an object $\mathcal{X}$ (e.g. a cake). Suppose a group $I = \{1, ..., n\}$ of numbered players are interested in fair division of a cake in such way, that each of them receives at least $1/n^{th}$ value of the cake according to his own estimation. Here each measure μ_i, $i \in I$, represents the individual evaluation of sets for the i-th player. In the literature of the fair division theory several notions of fair divisions $P = \{A_i\}_{i=1}^n \in \mathscr{P}$ are discussed.

Definition 5.5. A partition $P = \{A_i\}_{i=1}^n \in \mathscr{P}$ is called

1. a proportional division if $\mu_i(A_i) \geq 1/n$ for all $i \in I$,
2. an envy-free division if $\mu_i(A_i) \geq \mu_i(A_j)$ for all $i, j \in I$,
3. an exact division if $\mu_i(A_j) = 1/n$ for all $i, j \in I$,
4. an equitable division if $\mu_i(A_i) = \mu_j(A_j)$ for all $i, j \in I$.

The problem of fair division has been considered in many variants depending on the nature of goods to be divided and the fairness criteria. Different kinds of the players preferences and other criteria for evaluating the quality of the division has been analysed by various authors. The following three main directions are developed in the literature of the fair division theory:

- proving the existence of a partition of $\mathcal{X}$ satisfying given criteria (see e.g. Dubins and Spanier [16], Legut and Wilczyński [33, 34], Sagara [45], Weller [54]),
- providing procedures or algorithms for obtaining a fair divisions and applications of them to real-life situations (see e.g. Brams and Taylor [5, 6], Brams, Taylor and Zwicker [7, 8], Woodall [55]).
- giving the best possible estimation for the optimal value v and also α-optimal value $v^\alpha(\vec{\mu})$ and finding algorithms for obtaining α-optimal partitions (see Dall'Aglio and i Luca [15, 14], Dall'Aglio et al [H1], Hill et al [19], Legut [28], [H3], [H5])

Legut [29, 30] considered a problem of dividing a cake fairly among countably infinitely many players and proposed also a fair division model for continuum of players.
The results of the fair division theory can be applied in economics in the exchange and allocation of a heterogeneous commodities (cf. [31, 32, 40, 46]).
There are known many algorithms of obtaining proportional partitions (cf. [6]). A simple and well-known method for realizing the proportional division for two players is "for one to cut, the other to choose". Steinhaus in 1944 asked whether a fair procedure could be found for dividing a cake among n agents for $n > 2$. He found a solution for $n = 3$ and Banach and Knaster (cf. [26], [50], [51], [52]) showed that the solution for $n = 2$ can be extended to arbitrary n. Their result was modified by Dubins and Spanier [16]. In turn

Fink [20] gave an algorithm in which the number of players may be unknown. Brams and Taylor [5] found an interesting method of getting an envy free partition for which nobody would be better off with someone else's piece of cake.

The problem of the proportional fair division can be generalized if we assume that players do not have the same position in the game, but they have to divide the cake according to the individual shares $\alpha_1, \alpha_2, \ldots, \alpha_n$, where $\sum_{i\in I} \alpha_i = 1$. In this case a partition $P = \{A_i\}_{i=1}^n$ is $\alpha-$*fair*, if $\mu_i(A_i) \geq \alpha_i$ for all $i \in I$. Each player is interested in getting the biggest possible piece of cake according to his own evaluation. It means that we need to find the α-optimal value $v^\alpha(\vec{\mu})$ defined by (5.1) and effective methods of finding α-optimal partitions. The first estimation of the optimal value v was given by Elton et al. [19] and then Legut [28] generalized their result and gave better estimation for the α-optimal value $v^\alpha(\vec{\mu})$. An interesting algorithm for finding the bounds for the α-optimal value was found by Dall'Aglio and Luca [15]. In the literature of the fair division field there are known only few results concerning effective methods of finding optimal partitions. For example Dall'Aglio and Luca [14] found an algorithm for computing approximately optimal partition by construction of some maximin allocation in games of fair division. Most of the known methods of obtaining the optimal partitions were found by the author of this dissertation.

Applications for identifying an unknown distribution of a random variable

The problem of optimal partitioning of a measurable space $\{\mathcal{X}, \mathcal{B}\}$ is also considered in the classification problem (cf. [21, 24],[H7]). Suppose we are given a continuous random variable X having one of the known distribution described by density functions $f_i : [0,1] \to \mathbb{R}_+$ $i \in I$. We don't know which is the true distribution of X. We consider a classification problem (cf. [21]) in which after one observation of $X(\omega)$ (realisation of the random variable X) we are to decide which is the true distribution of X.

Definition 5.6. A partition $P = \{A_i\}_{i=1}^n \in \mathscr{P}$ is called a *decision rule* if in case of $X(\omega) \in A_i$, we guess that X has density function f_i, $i \in I$.

Our objective is to minimize the largest probability of misclassification

$$\max_{i\in I} \mathbb{P}(X \notin A_i |\, \mathrm{dist} X = f_i),$$

over all measurable partitions $P = \{A_i\}_{i=1}^n \in \mathscr{P}$. Denote by

$$R = \inf\left\{\max_{i\in I} \mathbb{P}(X \notin A_i |\, \mathrm{dist} X = f_i) : \{A_i\}_{i=1}^n \in \mathscr{P}\right\},$$

a minimal possible risk of misclassification. We obtain (cf. [24],[H7])

$$R = \inf\left\{\max_{i\in I}(1 - \mu_i(A_i)) : \{A_i\}_{i=1}^n \in \mathscr{P}\right\} = 1 - \sup\left\{\min_{i\in I} \mu_i(A_i) : \{A_i\}_{i=1}^n \in \mathscr{P}\right\}.$$

Definition 5.7. A partition $P^* = \{A_i^*\}_{i=1}^n \in \mathscr{P}$ is said to be a minimax decision rule if

$$R = 1 - \min_{i\in I} \mu_i(A_i^*).$$

It is easy to see that the minimax decision rule $P^* = \{A_i^*\}_{i=1}^n \in \mathscr{P}$ defined above is at the same time the equitable optimal partition in the sense of Definition 5.2.

5.3 Methods of optimal partitioning of the unit interval among two players

In this section we present a method of finding the α-optimal value $v^\alpha(\vec{\mu})$ and α-optimal partitions for two-dimensional case using some properties of the range of a vector measure $\vec{\mu} = (\mu_1, \mu_2)$. For $P = \{A_i\}_{i=1}^2 \in \mathscr{P}$ (cf. Theorem 5.3) we denote $\vec{\mu}(P) = (\mu_1(A_1), \mu_2(A_2))$ and for $A \in \mathcal{B}$ we denote $\vec{\mu}(A) = (\mu_1(A), \mu_2(A))$.
We need a method of obtaining the range $\vec{\mu}(\mathscr{P})$ with properties described in Theorem 5.3. It follows from the famous Lyapunov [39] convexity theorem that for two nonatomic measures $\{\mu_i\}_{i=1}^2$ the range

$$\vec{\mu}(\mathcal{B}) = \{(\mu_1(A), \mu_2(A)) \in [0,1]^2 : \ A \in \mathcal{B}\}$$

of a vector measure $\vec{\mu} = (\mu_1, \mu_2)$ is compact and convex. The set $\vec{\mu}(\mathscr{P})$ is a symmetric transformation of the set $\vec{\mu}(\mathcal{B})$ with respect to the line $x = \frac{1}{2}$, i.e.

$$\vec{\mu}(\mathscr{P}) = \left\{(x, y) \in [0,1]^2 : (1-x, y) \in \ \vec{\mu}(\mathcal{B})\right\}. \tag{5.2}$$

Now we show how to find a nondecreasing function $G : [0,1] \to [0,1]$ describing the boundary of the set $\vec{\mu}(\mathscr{P})$ as follows

$$\vec{\mu}(\mathscr{P}) = \left\{(x, y) \in [0,1]^2 : 1 - G(x) \leq y \leq G(1-x)\right\}. \tag{5.3}$$

It follows from the compactness of the range $\vec{\mu}(\mathcal{B})$ that for any $t \in [0,1]$ there exists a set $D(t) \in \mathcal{B}$ such that

$$\mu_2(D(t)) = \max\{\mu_2(A) : \mu_1(A) = t, \ A \in \mathcal{B}\}.$$

We denote by X_1 and X_2 random variables, defined on some probability space $(\Omega, \mathscr{F}, \mathbb{P})$, which are distributed according to f_1 and f_2, respectively. Without loss of generality we consider open interval $(0,1)$ instead of $[0,1]$. The corresponding distribution functions we denote by F_1 and F_2. Let $\mathbb{I}_A$ denote the indicator function of a set $A \in \mathcal{B}$. Legut and Wilczyński [H1] used the Neyman-Pearson lemma (cf. [38]) to find the function G describing the set $\vec{\mu}(\mathscr{P})$ and presented these methods in the following:

Proposition 5.8. *Assume that* $\{x : f_2(x) > 0\} \subset \{x : f_1(x) > 0\} = (0,1)$ *and denote* $r(x) = (f_2(x)/f_1(x))\,\mathbb{I}_{\{f_1(x)>0\}}$, $x \in (0,1)$. *Then the following statements hold:*

1. *If the ratio* $r(x)$ *is decreasing in* x *on* $(0,1)$, *then* $D(x) = \left(0, F_1^{-1}(x)\right)$ *and hence* $G(x) = F_2(F_1^{-1}(x))$.
2. *If the ratio* $r(x)$ *is increasing in* x *on* $(0,1)$, *then* $D(x) = \left(F_1^{-1}(1-x), 1\right)$ *and hence* $G(x) = 1 - F_2(F_1^{-1}(1-x))$.

3. *If the ratio $r(x)$ is symmetric about $x_0 = 1/2$ and decreasing in x on $(0, 1/2)$, then* $D(x) = \left(0, F_1^{-1}(x/2)\right) \cup \left(F_1^{-1}(1 - x/2), 1\right)$ *and hence* $G(x) = F_2\left(F_1^{-1}(x/2)\right) + 1 - F_2\left(F_1^{-1}(1 - x/2)\right)$.
4. *If the ratio $r(x)$ is symmetric about $x_0 = 1/2$ and increasing in x on $(0, 1/2)$, then* $D(x) = \left(F_1^{-1}\left(\frac{1-x}{2}\right), F_1^{-1}\left(\frac{1+x}{2}\right)\right)$ *and hence* $G(x) = F_2\left(F_1^{-1}(\frac{1+x}{2})\right) - F_2\left(F_1^{-1}(\frac{1-x}{2})\right)$.

Legut and Wilczyński [H1] found a method of obtaining the function G for more general densities f_1, f_2. We use the symbol $\mathscr{R}(f_1, f_2)$ instead of $\vec{\mu}(\mathcal{B})$:

$$\mathscr{R}(f_1, f_2) = \left\{ \left(\int_A f_1\, dt, \int_A f_2\, dt \right) \;:\; A \in \mathcal{B} \right\}.$$

Legut and Wilczyński [H1] showed that for any Lebesgue densities f_1, f_2 on $\{(0,1), \mathcal{B}\}$ there exist Lebesgue densities f_1^*, f_2^* on $\{(0,1), \mathcal{B}\}$ such that:

1. f_1^* is the density of the uniform distribution on $(0, 1)$,
2. f_2^* is nonincreasing in x on $(0, 1)$,
3. $\mathscr{R}(f_1, f_2) = \mathscr{R}(f_1^*, f_2^*)$.

Define the following function $\overline{H} : \mathbb{R} \to [0, 1]$ by

$$\overline{H}(y) = \mathbb{P}\left(f_2(X_1) \;>\; y f_1(X_1)\right) = \mu_1(\{x \;:\; f_2(x) > y f_1(x)\}) = \int\limits_{\{x\,:\, f_2(x) > y f_1(x)\}} f_1(x)\, dx. \tag{5.4}$$

Throughout the rest of this section we denote by f_1^* the uniform density on $(0, 1)$, i.e. we set $f_1^*(x) = \mathbb{I}_{(0,1)}(x)$, $x \in \mathbb{R}$. We use the symbol f_2^* for any Lebesgue probability density function on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$, which satisfies

$$f_2^*(x) = \overline{H}^{\,-1}(x) \quad \text{for all} \quad x \in (0, 1),$$

where function $\overline{H}^{\,-1}$ is defined by

$$\overline{H}^{\,-1}(x) = \inf\{y \geq 0 \;:\; \overline{H}(y) \leq x\} \quad \text{for all} \quad 0 < x < 1. \tag{5.5}$$

Legut and Wilczyński [H1] proved the following:

Theorem 5.9. *Let f_1, f_2 be fixed probability Lebesgue densities on $\{(0,1), \mathcal{B}\}$ and let f_1^* and f_2^* be the corresponding probability Lebesgue densities defined above. Then $\mathscr{R}(f_1, f_2) = \mathscr{R}(f_1^*, f_2^*)$. Moreover,*

$$\mathscr{R}(f_1^*, f_2^*) = \left\{(x, y) \in \mathbb{R}^2 \;:\; 0 \leq x \leq 1,\; 1 - G(1 - x) \leq y \leq G(x)\right\},$$

where the function $G : [0, 1] \to [0, 1]$ has the form

$$G(x) = \int\limits_{\{t : f_1(t) = 0\}} f_2(t)\, dt + \int_0^x f_2^*(t)\, dt \quad \text{for all} \quad x \in [0, 1]. \tag{5.6}$$

The above theorem can be used to obtain the range $\vec{\mu}(\mathscr{P})$ by determining its boundaries (5.3), where the function G is given by (5.6). Legut and Wilczyński [H1] used Theorem 5.9 to obtain α-optimal value and α-optimal partitions in two-dimensional case. They proved the following:

Theorem 5.10. *Let μ_1, μ_2 be probability measures on $\{(0,1), \mathcal{B}\}$ with Lebesgue densities f_1, f_2 and let $\alpha = (\alpha_1, \alpha_2) \in S_2$. Then $v^\alpha = \frac{x_\alpha}{\alpha_1}$, where x_α is the root of the equation $\frac{\alpha_2}{\alpha_1} x = G(1-x)$. Moreover, the α-optimal partition has the form $\{\mathcal{X} \setminus A_2^\alpha, A_2^\alpha\}$, where A_2^α is any set of μ_1 measure $1 - x_\alpha$, which satisfies*

$$\{x : f_2(x) > y_\alpha f_1(x)\} \subset A_2^\alpha \subset \{x : f_2(x) \geq y_\alpha f_1(x)\}, \tag{5.7}$$

with $y_\alpha = \overline{H}^{-1}(1 - x_\alpha)$.

The following example illustrates an application of the above theorem.

Example 5.11. Consider two densities f_1, f_2, where f_1 is the uniform density on $(0,1)$ and the density f_2 is given by

$$f_2(x) = \mathbb{I}_{(0,\frac{1}{2})}(x)(-8x(x-1)) + \mathbb{I}_{[\frac{1}{2},1)}(x)8(x-1)^2.$$

Using Theorem 5.9 we obtain the function G (see (5.6)):

$$G(x) = x + \frac{1}{6}(1 - 4(x-1)x)^{3/2} - \frac{1}{6}, \tag{5.8}$$

describing the boundary of the set $\vec{\mu}(\mathscr{P})$ by (5.3). We find the α-optimal partition for $\alpha = (\frac{1}{3}, \frac{2}{3})$. First from the equation:

$$2x = G(1-x) = 1 - x + \frac{1}{6}(1 + 4x(1-x))^{\frac{3}{2}} - \frac{1}{6}$$

we obtain $x_\alpha \approx 0.433$ where G is given by (5.8). Hence,

$$v^\alpha(\vec{\mu}) = \frac{x_\alpha}{\alpha_1} \approx \frac{0,433}{1/3} = 1,299.$$

Finally we obtain the explicit form of the approximate α-optimal partition $P^\alpha = \{A_1, A_2\}$, where

$$A_1 \approx [0, 0.114) \cup (0.681, 1], \quad A_2 \approx [0.114, 0.681].$$

□

From now on to the end of this section we will considered partitioning of the closed unit interval $[0,1]$. One of the most interesting problem in fair division theory is to find a minimal number of cuts needed to obtain a partition, which is optimal in some sense (see e.g. [3, 5, 44]). In the case of α-optimal partition for two players this quantity can be easily evaluated, because the form of the set A_2^α (cf. (5.7)) depends on the number of sign changes of the function $f_2(x) - y_\alpha f_1(x)$, $x \in [0,1]$ (cf. Theorem 5.10). Legut and Wilczyński [H1] presented the following:

Proposition 5.12. *Let $\alpha = (\alpha_1, \alpha_2) \in S_2$ and $k \in \mathbb{N}$ be fixed. Then there exist measures μ_1, μ_2^k on $\{[0,1], \mathcal{B}\}$ for which the minimal number of cuts needed to obtain the α-optimal partition of $[0,1]$ equals $2k$.*

Determining the minimal number of cuts of $[0,1]$ for obtaining partitions $P = \{A_1, A_2\}$ satisfying optimality conditions may be expressed by determining the minimal number of subintervals which union form A_1 or A_2. Legut [H4] considered some properties of the range of nonatomic vector measure $\vec{\mu} = (\mu_1, ..., \mu_n)$ defined on measurable subsets $\mathcal{B}$ of the unit interval $[0,1]$. Denote by $\mathcal{U}(k)$ a collection of all sets being a union of at most k pairwise disjoint subintervals of $[0,1]$. Legut [H4] used a result of Stromquist and Woodall [53] to prove the following

Theorem 5.13. *If $A, B \in \mathcal{U}(k)$, $k \in \mathbb{N}$, then*

$$\langle \vec{\mu}(A), \vec{\mu}(B) \rangle \subset \vec{\mu}\left(\mathcal{U}(2n + 4k - 3)\right).$$

By $\langle \vec{\mu}(A), \vec{\mu}(B) \rangle$ we denoted the closed line segment connecting points $\vec{\mu}(A)$ and $\vec{\mu}(B)$. Legut [H4] used the above theorem for obtaining yet another proof of the Lyapunov convexity theorem. In special cases the following proposition can be used in two-dimensional case for estimating the number of cuts needed to obtain the α-optimal partitions:

Proposition 5.14. *Assume that for some $k \in \mathbb{N}$ all extreme points of the range $\vec{\mu}(\mathcal{B})$ are contained in $\vec{\mu}(\mathcal{U}(k))$. Then $\vec{\mu}(\mathcal{B}) = \vec{\mu}(\mathcal{U}(2n + 4k - 3))$.*

As it was mentioned earlier (5.2) the set $\vec{\mu}(\mathscr{P})$ is the symmetric transformation of the set $\vec{\mu}(\mathcal{B})$. It follows from Proposition 5.14 that if extreme points of $\vec{\mu}(\mathscr{P})$ can be constructed by partitions $P = \{A_1, A_2\}$, where one of the sets A_1, A_2 is a union of at most k pairwise disjoint subintervals of $[0,1]$ then it is possible to find α-optimal partition $P^\alpha = \{A_1^\alpha, A_2^\alpha\}$, where one of the sets A_1^α, A_2^α is a union of at most $2n + 4k - 3$ pairwise disjoint subintervals of $[0,1]$.

5.4 Methods of optimal partitioning of a measurable space for measures defined by different kinds of density functions

In this section we present methods of obtaining optimal partitions of the measurable space $\{[0,1), \mathcal{B}\}$ for arbitrary finite number of measures $\{\mu_i\}_{i=1}^n$ defined by different kinds of densities f_i, $i \in I$:

- simple functions
- piecewise linear functions
- functions with piecewise strictly monotone likelihood ratio (SMLR) property

5.4.1 Simple density functions

Assume that the measures $\{\mu_i\}_{i=1}^n$ defined on $\{[0,1),\mathcal{B}\}$ are given by simple functions, i.e.

$$f_i(x)=\sum_{j=1}^m h_{ij}\mathbb{I}_{[a_j,a_{j+1})}(x),$$

were $\{[a_j,a_{j+1})\}_{j=1}^m$ is a partition of the interval $[0,1)$ such that

$$[0,1)=\bigcup_{j=1}^m [a_j,a_{j+1}),\ \ a_1=0,\ a_{m+1}=1,\ \ a_{j+1}>a_j\ j=1,2,...,m, \tag{5.9}$$

and h_{ij}, $i\in I$, $j=1,...,m$, are nonnegative real numbers satisfying $\int_0^1 f_i dx=1$ for $i\in I$. For any natural number $k\geq n-1$ denote by $\mathscr{P}(k)$ the collection of all partitions of the unit interval $[0,1)$ which are obtained by using at most k cuts.
Dall'Aglio et al [H2] proved the following

Theorem 5.15. *Let $\alpha=(\alpha_1,\ldots,\alpha_n)\in S_n$. Then for the measures $\{\mu_i\}_{i=1}^n$ there exists an α-optimal partition $P^*=\{A_i^*\}_{i=1}^n\in\mathscr{P}(mn-1)$.*

The proof of the above theorem is constructive and is based on linear programming. Let z^* and $[x_{ij}^*]_{n\times m}$ be a solution of the following linear programming (LP) problem:

$$\max z \tag{5.10}$$

subject to constraints

$$z=\frac{1}{\alpha_i}\sum_{j=1}^m x_{ij}h_{ij}(a_{j+1}-a_j),\ \ i=1,2,...,n,$$

with respect to variables z, $[x_{ij}]_{n\times m}$ satisfying the following conditions:

$$\sum_{i=1}^n x_{ij}=1,\ \ j=1,2,...,m;$$

$$x_{ij}\geq 0,\qquad \text{for all}\quad i=1,2,...,n,\ \ j=1,2,...,m.$$

We construct a partition $P=\{A_i^*\}_{i=1}^n$ of the interval $[0,1)$ such that

$$\frac{\mu_i(A_i^*)}{\alpha_i}=z^*\quad \text{for}\quad i=1,2,...,n.$$

For $j=1,2,...,m$ we can find subpartitions $\{[b_i^j,b_{i+1}^{(j)})\}_{i=1}^n$ of intervals $[a_j,a_{j+1})$ with

$$[a_j,a_{j+1})=\bigcup_{i=1}^n [b_i^{(j)},b_{i+1}^{(j)}),$$

where $b_i^{(j)} \in [a_j, a_{j+1})$ are numbers satisfying the following conditions

$$\frac{b_{i+1}^{(j)} - b_i^{(j)}}{a_{j+1} - a_j} = x_{ij}^*,\ i = 1, 2, ..., n,$$

and $b_1^{(j)} = a_j, b_{n+1}^{(j)} = a_{j+1}$. If $x_{ij}^* = 0$ for some $i = 1, 2, ..., n$ then $b_{i+1}^{(j)} = b_i^{(j)}$ and we set $[b_i^{(j)}, b_{i+1}^{(j)}) = \emptyset$ in this case. Define a partition $P = \{A_i^*\}_{i=1}^n$ by

$$A_i^* = \bigcup_{j=1}^m [b_i^{(j)}, b_{i+1}^{(j)}),\ i = 1, 2, ..., n.$$

Dall'Aglio et al [H2] proved that the above partition is the α-optimal and can be obtained by using at most $nm - 1$ cuts of the unit interval $[0, 1)$. Presented method can be used for getting almost α-optimal partition for arbitrary density functions, which can be approximated by simple functions. An example of this method was presented by Dall'Aglio et al [H2].

5.4.2 Piecewise linear density functions

In this section we show how to obtain an equitable optimal partition for measures with piecewise linear density functions. Let density functions $f_i : [0, 1) \to \mathbb{R}_+$, $i \in I$, be defined as follows:

$$f_i(x) = \sum_{j=1}^m (c_{ij}x + d_{ij})\mathbb{I}_{[a_j, a_{j+1})}(x), \qquad \int_0^1 f_i(x)\, dx = 1, \quad i \in I, \tag{5.11}$$

where $\{[a_j, a_{j+1})\}_{j=1}^m$ is a partition of the interval $[0, 1)$ such that

$$[0, 1) = \bigcup_{j=1}^m [a_j, a_{j+1}),\ \ a_1 = 0,\ a_{m+1} = 1,\ \ a_{j+1} > a_j\ j = 1, ..., m. \tag{5.12}$$

We assume that

$$c_{ij}x + d_{ij} \geq 0 \quad \text{for all} \quad x \in [a_j, a_{j+1}),\ i \in I,\ j = 1, ..., m.$$

We consider nonatomic probability measures $\{\mu_i\}_{i=1}^n$ given by

$$\mu_i(A) = \int_A f_i dx, \text{ for } A \in \mathcal{B},\ i \in I. \tag{5.13}$$

Without loss of generality we consider only left side closed and right side open intervals unless they are otherwise defined. Consider partitions of each interval $[a_j, a_{j+1})$, $j = 1, ..., m$ into n subintervals by cuts in points $x_k^{(j)}, k = 1, ..., n-1, j = 1, ..., m$ such that

$$[a_j, a_{j+1}) = \bigcup_{k=1}^n [x_{k-1}^{(j)}, x_k^{(j)}),$$

where $x_0^{(j)} = a_j$, $x_n^{(j)} = a_{j+1}$, $x_{k+1}^{(j)} \geq x_k^{(j)}$, $k = 1, ..., n-1$, $j = 1, ..., m$.
If $x_{k-1}^{(j)} = x_k^{(j)}$ for some $k = 1, ..., n$ we set $[x_{k-1}^{(j)}, x_k^{(j)}) = \emptyset$.
For simplicity we will also denote $B_{kj} := [x_{k-1}^{(j)}, x_k^{(j)})$, $k = 1, ..., n$, $j = 1, ..., m$.
Now we construct an assignment of each subintervals B_{kj} to each player $i \in I$.
Let p_j, q_j, $j = 1, ..., m$ be integers satisfying $0 \leq p_j \leq q_j \leq n$ and

$$\#\{i : i \in I, c_{ij} < 0\} = p_j,$$
$$\#\{i : i \in I, c_{ij} = 0\} = q_j - p_j,$$
$$\#\{i : i \in I, c_{ij} > 0\} = n - q_j,$$

where by $\#A$ we denote the number of elements of a finite set A.
For each interval $[a_j, a_{j+1})$, $j = 1, ..., m$, we define permutations $\sigma_j : I \to I$, $j = 1, ..., m$ satisfying the following conditions:

1. If $p_j > 0$ we define $\sigma_j(k) \in \{i : i \in I, c_{ij} < 0\}$ for $k = 1, ..., p_j$ such that
$$\frac{d_{\sigma_j(k)j}}{c_{\sigma_j(k)j}} \geq \frac{d_{\sigma_j(k+1)j}}{c_{\sigma_j(k+1)j}}, \quad k = 1, ..., p_j - 1. \tag{5.14}$$
2. If $q_j - p_j > 0$ we define $\sigma_j(k) \in \{i : i \in I, c_{ij} = 0\}$ for $k = p_j + 1, ..., q_j$ such that
$$\sigma_j(k) \leq \sigma_j(k+1), \quad k = p_j + 1, ..., q_j - 1. \tag{5.15}$$
3. If $n - q_j > 0$ we define $\sigma_j(k) \in \{i : i \in I, c_{ij} > 0\}$ for $k = q_j + 1, ..., n$ such that
$$\frac{d_{\sigma_j(k)j}}{c_{\sigma_j(k)j}} \geq \frac{d_{\sigma_j(k+1)j}}{c_{\sigma_j(k+1)j}}, \quad k = q_j + 1, ..., n - 1. \tag{5.16}$$

Permutations σ_j, $j = 1, ..., m$, define one-to-one assignment of the subintervals $B_{ij} \subset [a_j, a_{j+1})$, $i \in I, j = 1, ..., m$ such that player $i \in I$ receives subinterval $B_{\sigma_j^{-1}(i)j}$.
Finally we obtain a partition $\{B_i\}_{i=1}^n$of the unit interval defined by

$$B_i = \bigcup_{j=1}^{m} B_{\sigma_j^{-1}(i)j}, \; i \in I.$$

The following theorem proved by Legut [H3] presents an algorithm for obtaining an equitable optimal fair division.

Theorem 5.16. *Let a collection of numbers z^*, $\{x_k^{*(j)}\}, k = 1, ..., n-1$, $j \in J$, be a solution of the following nonlinear programming (NLP) problem*

$$\max\, z \tag{5.17}$$

subject to quadratic constraints

$$z = \sum_{j=1}^{m} \mu_i(B_{\sigma_j^{-1}(i)j}) = \sum_{j=1}^{m} \int_{B_{\sigma_j^{-1}(i)j}} f_i dx, \quad i = 1, ..., n,$$

with respect to variables z, $\{x_k^{(j)}\}$, $k = 1, ..., n-1, j \in J$, *satisfying the following inequalities*

$$0 = a_1 \leq x_1^{(1)} \leq ... \leq x_{n-1}^{(1)} \leq a_2, \tag{5.18}$$
$$a_2 \leq x_1^{(2)} \leq ... \leq x_{n-1}^{(2)} \leq a_3,$$
$$...$$
$$a_m \leq x_1^{(m)} \leq ... \leq x_{n-1}^{(m)} \leq a_{m+1} = 1.$$

Then the partition $\{A_i^*\}_{i=1}^n \in \mathscr{P}$ *of the unit interval* $[0,1)$ *defined by*

$$A_i^* = \bigcup_{j=1}^{m} A_{\sigma_j^{-1}(i)j}, \; i \in I, \tag{5.19}$$

where

$$A_{\sigma_j^{-1}(i)j} = \left[x^{*(j)}_{\sigma_j^{-1}(i)-1}, x^{*(j)}_{\sigma_j^{-1}(i)}\right), \; i \in I, \tag{5.20}$$

and $x_0^{*(j)} = a_j$, $x_n^{*(j)} = a_{j+1}$, $j = 1, ..., m$, *is an equitable optimal fair division and* $v = z^*$ *is the optimal value.*

The result of Theorem 5.15 is a special case of the above theorem for density functions (5.11) with $c_{ij} = 0$ for all $i, j \in I$. As in case of simple density functions the method presented in Theorem 5.16 can be applied for obtaining almost equitable optimal divisions for arbitrary densities approximated by functions with piecewise linear property. Theorem 5.16 can be easily generalized for obtaining α-optimal partitions for any $\alpha \in S_n$.
Legut [H3] presented an example illustrating the method given in Theorem 5.16:

Example 5.17. Consider problem of fair division for three players. Assume that each player $i = 1, 2, 3$ estimates the measurable subsets of the unit interval $[0,1)$ using measures μ_i defined respectively by density functions

$$f_i(x) = \sum_{j=1}^{3} (c_{ij}x + d_{ij}) I_{[a_j, a_{j+1})}(x), \; i = 1, 2, 3,$$

where the numbers c_{ij} and d_{ij} are elements of the matrices

$$[c_{ij}] = \begin{bmatrix} -2 & 1 & -1 \\ -1 & 1 & 1 \\ -\frac{1}{2} & 0 & -2 \end{bmatrix} \quad [d_{ij}] = \begin{bmatrix} 2 & 0 & \frac{5}{4} \\ \frac{5}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{7}{4} & \frac{13}{4} \end{bmatrix},$$

and $a_1 = 0$, $a_2 = \frac{1}{2}$, $a_3 = \frac{3}{4}$, $a_4 = 1$. Now we construct an optimal equitable division. For

$c_{ij} \neq 0$ define new matrix $[e_{ij}]$ with elements $e_{ij} = \dfrac{d_{ij}}{c_{ij}}$ (the element e_{32} is not defined)

$$[e_{ij}] = \begin{bmatrix} -1 & 0 & -\frac{5}{4} \\ -\frac{5}{4} & \frac{1}{4} & \frac{1}{4} \\ -1 & & -\frac{13}{8} \end{bmatrix}$$

Divide the players into three groups depending on the sign of c_{ij} separately on each interval:
$[0, \frac{1}{2}) : \{1, 2, 3\}, \{\emptyset\}, \{\emptyset\}$ for $c_{ij} < 0$,
$[\frac{1}{2}, \frac{3}{4}) : \{\emptyset\}, \{3\}, \{1, 2\}$ for $c_{ij} = 0$,
$[\frac{3}{4}, 1) : \{1, 3\}, \{\emptyset\}, \{2\}$ for $c_{ij} > 0$.
Analysing the columns of the matrix $[e_{ij}]$ we define permutations $\sigma_j : \{1, 2, 3\} \to \{1, 2, 3\}$, $j = 1, 2, 3$, satisfying conditions (5.14), (5.15) and (5.16). Ranking players according to the ratio e_{ij} in each of the three groups we obtain
$\left[0, \dfrac{1}{2}\right) : \dfrac{d_{11}}{c_{11}} = \dfrac{d_{31}}{c_{31}} > \dfrac{d_{21}}{c_{21}}$ and $\sigma_1(1) = 1$, $\sigma_1(2) = 3$, $\sigma_1(3) = 2$,
$\left[\dfrac{1}{2}, \dfrac{3}{4}\right) : \dfrac{d_{22}}{c_{22}} > \dfrac{d_{12}}{c_{12}}$ and $\sigma_2(1) = 3$, $\sigma_2(2) = 2$, $\sigma_2(3) = 1$,
$\left[\dfrac{3}{4}, 1\right) : \dfrac{d_{13}}{c_{13}} > \dfrac{d_{33}}{c_{33}}$ and $\sigma_3(1) = 1$, $\sigma_3(2) = 3$, $\sigma_3(3) = 2$.
Because of the equality $\dfrac{d_{11}}{c_{11}} = \dfrac{d_{31}}{c_{31}}$ we can also alternatively define $\sigma_1(1) = 2$, $\sigma_1(2) = 1$, $\sigma_1(3) = 2$. Permutations σ_j, $j = 1, 2, 3$, establish the assignment of subintervals $\{[x_{k-1}^{(j)}, x_k^{(j)})\}_{k=1}^{3}$, ($x_0^{(j)} = a_j$, $x_n^{(j)} = a_{j+1}$) to each player $i = 1, 2, 3$ such that
player $i = 1$ receives set $B_1 = [0, x_1^{(1)}) \cup [x_2^{(2)}, \frac{3}{4}) \cup [\frac{3}{4}, x_1^{(3)})$,
player $i = 2$ receives set $B_2 = [x_2^{(1)}, \frac{1}{2}) \cup [x_1^{(2)}, x_2^{(2)}) \cup [x_2^{(3)}, 1)$ and
player $i = 3$ receives set $B_3 = [x_1^{(1)}, x_2^{(1)}) \cup [\frac{1}{2}, x_1^{(2)}) \cup [x_1^{(3)}, x_2^{(3)})$.
Formulate the NLP problem (5.26) as follows

$$\max z$$

subject to constraints

$$\begin{aligned} z &= \int_0^{x_1^{(1)}} f_1(x)dx + \int_{x_2^{(2)}}^{3/4} f_1(x)dx + \int_{3/4}^{x_1^{(3)}} f_1(x)dx \\ &= -\frac{3}{8} + 2x_1^{(1)} - [x_1^{(1)}]^2 - \frac{[x_2^{(2)}]^2}{2} + \frac{5x_1^{(3)}}{4} - \frac{[x_1^{(3)}]^2}{2}, \\ z &= \int_{x_2^{(1)}}^{1/2} f_2(x)dx + \int_{x_1^{(2)}}^{x_2^{(2)}} f_2(x)dx + \int_{x_2^{(3)}}^{1} f_2(x)dx \end{aligned}$$

$$= \frac{5}{4} - \frac{5x_2^{(1)}}{4} + \frac{[x_2^{(1)}]^2}{2} - \frac{x_1^{(2)}}{4} - \frac{[x_1^{(2)}]^2}{2} + \frac{x_2^{(2)}}{4} + \frac{[x_2^{(2)}]^2}{2} - \frac{x_2^{(3)}}{4} - \frac{[x_2^{(3)}]^2}{2},$$

$$z = \int_{x_1^{(1)}}^{x_2^{(1)}} f_3(x)dx + \int_{1/2}^{x_1^{(2)}} f_3(x)dx + \int_{x_1^{(3)}}^{x_2^{(3)}} f_3(x)dx$$

$$= -\frac{7}{8} - \frac{x_1^{(1)}}{2} + \frac{[x_1^{(1)}]^2}{4} + \frac{x_2^{(1)}}{2} - \frac{[x_2^{(1)}]^2}{4} + \frac{7x_1^{(2)}}{4} - \frac{13x_1^{(3)}}{4} + [x_1^{(3)}]^2 + \frac{13x_2^{(3)}}{4} - [x_2^{(3)}]^2,$$

with respect to the variables z, $\{x_k^{(j)}\}$, $k = 1, 2, j = 1, 2, 3$, and satisfying the inequalities

$$0 \le x_1^{(1)} \le x_2^{(1)} \le \frac{1}{2} \le x_1^{(2)} \le x_2^{(2)} \le \frac{3}{4} \le x_1^{(3)} \le x_2^{(3)} \le 1.$$

Using some software we get the following approximate solution

$$z \approx 0.465276,\ x_1^{(1)} = x_2^{(1)} \approx 0.26852,\ x_1^{(2)} = x_2^{(2)} = x_1^{(3)} = 0.75 \text{ and } x_2^{(3)} \approx 0.766019.$$

Hence the optimal value $v \approx 0.465276$ and the optimal partition $\{B_1, B_2, B_3\}$ is given by $B_1 = [0, x_1^{(1)})$, $B_2 = [x_2^{(1)}, \frac{1}{2}) \cup [x_2^{(3)}, 1)$, $B_3 = [\frac{1}{2}, x_2^{(3)})$.

□

5.4.3 Density functions with piecewise strictly monotone likelihood ratio property

In this section we present an algorithm for obtaining an equitable optimal fair division for large class of density functions. Suppose we are given n nonatomic probability measures μ_i, $i \in I$, defined on the measurable space $\{[0, 1), \mathcal{B}\}$. We need the following

Assumption 5.18. The measures μ_i, $i \in I$, are absolutely continuous with respect to the Lebesgue measure λ defined on $\{[0, 1), \mathcal{B}\}$ and additionally

$$\text{supp}(\mu_i) = [0, 1), \quad i \in I.$$

Let f_i, $i \in I$, denote the Radon-Nikodym derivatives of the measures μ_i with respect to the Lebesgue measure λ. Define absolutely continuous and strictly increasing functions $F_i : [0, 1] \to [0, 1]$ by

$$F_i(t) = \int_{[0,t)} f_i \, d\lambda, \quad t \in [0, 1], \quad i \in I. \tag{5.21}$$

We need yet another crucial assumption.

Assumption 5.19. There exists a partition $\{[a_j, a_{j+1})\}_{j=1}^m$ of the interval $[0, 1)$, where $a_1 = 0$, $a_{m+1} = 1$, such that the densities f_i satisfy strictly monotone likelihood ratio (SMLR) property on each interval $[a_j, a_{j+1})$, $j \in J := \{1, ..., m\}$, i.e. for any $i, k \in I$, $i \ne k$, the ratios $\frac{f_i(x)}{f_k(x)}$ are strictly monotone on each interval $[a_j, a_{j+1})$.

The following proposition could be helpful for some density functions f_i, $i \in I$, to check whether the Assumption 5.19 is satisfied.

Proposition 5.20. *If the density functions f_i, $i \in I$, are differentiable and the set*

$$Q := \{x \in (0,1) : f_i'(x) f_k(x) = f_i(x) f_k'(x),\ i, k \in I,\ i \neq k\} \tag{5.22}$$

is finite then Assumption 5.19 is satisfied.

If the densities f_i, $i \in I$, are polynomial functions of positive degree, the assumptions of Proposition 5.20 and also Assumption 5.19 are obviously satisfied. Consider the problem of the equitable optimal fair division for two players with the following density functions $f_1(x) = x \sin \frac{1}{x} + c$, with the constant c satisfying $\int_0^1 f_1(x)dx = 1$, and $f_2(x) = \mathbb{I}_{[0,1)}(x)$ for $x \in [0,1)$. It is easy to verify, that in this case the set Q is infinite.
For construction of the optimal partition we need the following:

Proposition 5.21. *Suppose the densities f_i satisfy Assumption 5.19. Then for any numbers θ_1, θ_2 satisfying $a_j \leq \theta_1 < \theta_2 < a_{j+1}$, $j \in J$, and any $i, k \in I$, $i \neq k$ the one of the two following inequalities*

$$\frac{F_i(t) - F_i(\theta_1)}{F_i(\theta_2) - F_i(\theta_1)} < \frac{F_k(t) - F_k(\theta_1)}{F_k(\theta_2) - F_k(\theta_1)} \tag{5.23}$$

$$\frac{F_i(t) - F_i(\theta_1)}{F_i(\theta_2) - F_i(\theta_1)} > \frac{F_k(t) - F_k(\theta_1)}{F_k(\theta_2) - F_k(\theta_1)} \tag{5.24}$$

holds for each $t \in (\theta_1, \theta_2)$.

The inequalities (5.23) and (5.24) mean that there is a strict relative convexity relationship between the functions F_i and F_k, $i \neq k$, defined by (5.21). If the inequality (5.23) holds, then F_i is strictly convex with respect to F_k . This property is equivalent to the strict convexity of the composite function $F_i \circ F_k^{-1}$ on the interval $(F_k(a_j), F_k(a_{j+1}))$ (cf. [41]). It follows from a result of Shisha and Cargo [49] (Theorem 1) that $F_i \circ F_k^{-1}$ is strictly convex on $(F_k(a_j), F_k(a_{j+1}))$ if and only if the ratio $\dfrac{f_i(x)}{f_k(x)}$ is strictly increasing on (a_j, a_{j+1}). Hence the reverse implication in Proposition 5.21 is also true.

The relative convexity is one of many various generalizations of convexity started in 1931 by Jessen [23]. They were developed by Popoviciu [42] and Beckenbach [4] and continued later by Karlin [25] especially for applications in approximation theory.

The relation of strict relative convexity induces on each interval (a_j, a_{j+1}) a strict partial ordering of the functions F_i (cf. [41]). Let $F_i \prec_j F_k$ denote that F_i is strictly convex with respect to F_k on (a_j, a_{j+1}). For each $j \in J$ define permutation $\sigma_j : I \to I$, such that

$$F_{\sigma_j(k+1)} \prec_j F_{\sigma_j(k)},$$

for $k = 1, ..., n-1$. Hence for all $t \in (a_j, a_{j+1})$ we have

$$\frac{F_{\sigma_j(k+1)}(t) - F_{\sigma_j(k+1)}(a_j)}{F_{\sigma_j(k+1)}(a_{j+1}) - F_{\sigma_j(k+1)}(a_j)} < \frac{F_{\sigma_j(k)}(t) - F_{\sigma_j(k)}(a_j)}{F_{\sigma_j(k)}(a_{j+1}) - F_{\sigma_j(k)}(a_j)}. \tag{5.25}$$

The following theorem proved by Legut ([H5]) presents an algorithm for obtaining an equitable optimal fair division for density functions satisfying Assumption 5.18 and 5.19:

Theorem 5.22. *Let a collection of numbers z^*, $\{x_k^{*(j)}\}, k = 1, ..., n-1, j \in J$, be a solution of the following nonlinear programming (NLP) problem*

$$\max z \tag{5.26}$$

subject to constraints

$$z = \sum_{j=1}^{m} \left[F_i(x^{(j)}_{\sigma_j(i)}) - F_i(x^{(j)}_{\sigma_j(i)-1}) \right] \quad i = 1, ..., n, \tag{5.27}$$

with respect to variables z, $\{x_k^{(j)}\}$, $k = 1, ..., n-1, j \in J$, satisfying the following inequalities

$$\begin{gathered} 0 = a_1 \le x_1^{(1)} \le ... \le x_{n-1}^{(1)} \le a_2, \\ a_2 \le x_1^{(2)} \le ... \le x_{n-1}^{(2)} \le a_3, \\ ... \\ a_m \le x_1^{(m)} \le ... \le x_{n-1}^{(m)} \le a_{m+1} = 1. \end{gathered} \tag{5.28}$$

Then the partition $\{A_i^*\}_{i=1}^n \in \mathscr{P}$ of the unit interval $[0,1)$ defined by

$$A_i^* = \bigcup_{j=1}^{m} \left[x^{*(j)}_{\sigma_j(i)-1}, x^{*(j)}_{\sigma_j(i)} \right), \; i \in I, \tag{5.29}$$

where $x_0^{*(j)} = a_j$, $x_n^{*(j)} = a_{j+1}$, $j \in J$, is an equitable optimal fair division for the measures μ_i, $i \in I$ and $v = z^*$ is the optimal value.

If for some $i \in I$ and $j \in J$, the equality $x^{*(j)}_{\sigma_j(i)-1} = x^{*(j)}_{\sigma_j(i)}$ holds we set $\left[x^{*(j)}_{\sigma_j(i)-1}, x^{*(j)}_{\sigma_j(i)} \right) = \emptyset$ in the union of intervals (5.29).

The following example presents the method described in the above theorem.

Example 5.23. Consider a problem of fair division for three players $I = \{1,2,3\}$ estimating measurable subsets of the unit interval $[0,1)$ using measures μ_i, $i = 1,2,3$, defined respectively by the following density functions

$$f_1(x) = 12\left(x - \frac{1}{2}\right)^2, f_2(x) = 2x, \; f_3(x) = \mathbb{I}_{[0,1)}(x), \quad x \in [0,1).$$

We use the algorithm described in Theorem 5.22 to obtain an equitable optimal fair division. First we need to divide the interval $[0,1)$ into some subintervals on which the densities f_i, $i = 1,2,3$, separably satisfy SMLR property. For this reason we find the set Q defined by (5.22). It is easy to check that $Q = \{\frac{1}{2}\}$ and hence by Proposition 5.21 the densities f_i, $i = 1, 2, 3$, satisfy the SMLR property on intervals $[0, \frac{1}{2})$ and $[\frac{1}{2}, 1)$. Denote cumulative strictly increasing distribution functions by $F_i(t) = \int_0^t f_i(x)\,dx$, $i = 1, 2, 3$. Then we have

$$F_1(t) = 4t^3 - 6t^2 + 3t,\ F_2(t) = t^2,\ F_3(t) = t, \quad t \in [0, 1).$$

Based on the inequalities (5.25) we establish the proper order of assignments of subintervals of the intervals $[0, \frac{1}{2})$ and $[\frac{1}{2}, 1)$ to each player as follows: we take midpoints $\{\frac{1}{4}\}$ and $\{\frac{3}{4}\}$ of the two intervals and verify that

$$\frac{F_1(1/4) - F_1(0)}{F_1(\frac{1}{2}) - F_1(0)} > \frac{F_3(1/4) - F_3(0)}{F_3(\frac{1}{2}) - F_3(0)} > \frac{F_2(1/4) - F_2(0)}{F_2(\frac{1}{2}) - F_2(0)},$$

and

$$\frac{F_3(3/4) - F_3(0)}{F_3(1) - F_3(\frac{1}{2})} > \frac{F_2(3/4) - F_2(0)}{F_2(1) - F_2(\frac{1}{2})} > \frac{F_1(3/4) - F_1(0)}{F_1(1) - F_1(\frac{1}{2})}.$$

Hence, we obtain permutations

$$\sigma_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \quad \text{and} \quad \sigma_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}.$$

Now we are ready to formulate an NLP problem as in Theorem 5.22:

$$\max z$$

subject to constraints

$$z = F_1(x_1^{(1)}) - F_1(0) + F_1(1) - F_1(x_2^{(2)}),$$

$$z = F_2(\tfrac{1}{2}) - F_2(x_1^{(2)}) + F_2(x_2^{(2)}) - F_2(x_2^{(1)}),$$

$$z = F_3(x_1^{(2)}) - F_3(x_1^{(1)}) + F_3(x_2^{(1)}) - F_3(\tfrac{1}{2}),$$

with respect to the variables $z, \{x_k^{(j)}\}$ $k = 1, 2, j = 1, 2$, satisfying the following inequalities

$$0 \le x_1^{(1)} \le x_1^{(2)} \le \tfrac{1}{2} \le x_2^{(1)} \le x_2^{(2)} \le 1.$$

Solving the above NLP problem we obtain

$$z^* \approx 0.4843,\ x_1^{*(1)} \approx 0.1426,\ x_1^{*(2)} = a_2 = 0.5,\ x_2^{*(1)} \approx 0.6269, x_2^{*(2)} \approx 0.9367.$$

Hence, we get the equitable optimal fair division $\{A_i^*\}_{i=1}^3 \in \mathscr{P}$ of the unit interval $[0, 1)$, where

$$A_1^* = [0, x_1^{*(1)}) \cup [x_2^{*(2)}, 1), \quad A_2^* = [x_2^{*(2)}, x_2^{*(1)}) \quad \text{and} \quad A_3^* = [x_2^{*(1)}, x_1^{*(1)}).$$

The optimal value $v = z^* \approx 0.4843$.

□

Legut [H5] used the method presented in Theorem 5.22 for obtaining also equitable ε-optimal divisions in case where the set Q defined by (5.22) is countably infinite. The definition of an equitable ε-optimal division is given below.

Definition 5.24. A partition $P^\varepsilon = \{A_i^\varepsilon\}_{i=1}^n \in \mathscr{P}$ is said to be an equitable ε-optimal fair division if for all $i \in I$

$$\mu_i(A_i^\varepsilon) > v - \varepsilon,$$

where v is the optimal value.

5.5 A metod of obtaining an approximate solution of an exact fair division problem

In this section we consider a different kind of optimality. We mean the optimality in the sense of fair division theory. We present an algorithm of obtaining an approximate solution of an exact fair division which is at the same time proportional, envy-free and equitable (see Definition 5.5). The existence of exact fair divisions can be immediately derived from a theorem of Hobby and Rice [22] generalized by Alon [1]:

Theorem 5.25. *Let $\mu_1, ..., \mu_n$ be non-atomic measures defined on measurable subsets of the unit interval $[0, 1]$. Then it is possible to cut the interval in $(m-1)n$ places and partition the $(m-1)n+1$ resulting subintervals into m families $\mathcal{F}_1, ..., \mathcal{F}_m$ such that $\mu_i(\cup\mathcal{F}_j) = \frac{1}{m}$ for all $i = 1, ..., n$ and $j = 1, ..., m$. The number $(m-1)n$ is best possible.*

By $\cup\mathcal{F}_j$ we denote the union of all subintervals belonging to the family $\mathcal{F}_j$.
Unfortunately, it is not easy to use Theorem 5.25 directly for obtaining exact divisions. We need to find $n(n-1)$ unknown numbers solving equations systems for all possible configurations of the families $\mathcal{F}_1, ..., \mathcal{F}_n$ with constrains $\mu_i(\cup\mathcal{F}_j) = \frac{1}{n}$, $i, j = 1, ..., n$. Therefore, Legut [H6] proposed an iterative algorithm for obtaining approximate solution of this problem.
Let $\mathcal{F}_1, ..., \mathcal{F}_m$ be the families satisfying Theorem 5.25. Hence $\sum_{j\in I} |\mathcal{F}_j| \leq (m-1)n+1$, where $|\mathcal{F}_j|$ denote the number of subintervals belonging to the family $\mathcal{F}_j$, $j \in I$. Let $q_n(m) = \left\lfloor \frac{(m-1)n+1}{m} \right\rfloor$, $m = 2, ..., n$, where " $\lfloor x \rfloor$ " denotes the greatest integer less than or equal to x. It is easy to check, that

$$\min_j |\mathcal{F}_j| \leq q_n(m). \tag{5.30}$$

Let $F_i : [0, 1] \to [0, 1]$ be continuous and nondecreasing functions defined by $F_i(t) = \mu_i([0, t))$, $t \in [0, 1]$, $i \in I$. We construct an approximate exact fair division $P = \{A_i\}_{i=1}^n$ in $n-1$ steps starting from $m = n$ backwards down to $m = 2$.
Step 1. $(m = n)$
Since $q_n(n) = n - 1$ we consider two finite sequences of $2(n-1)$ numbers $\{x_k\}$ and $\{y_k\}$, $k = 1, ..., n-1$, satisfying

$$0 \leq x_k \leq y_k \leq 1, \quad \text{for all} \quad k = 1, ..., n-1, \tag{5.31}$$

$$y_k < x_{k+1} \quad \text{for all} \quad k = 1, ..., n-2 \quad \text{and} \tag{5.32}$$

$$\sum_{k=1}^{n-1} [F_i(y_k) - F_i(x_k)] = \sum_{k=1}^{n-1} \mu_i([x_k, y_k)) = \frac{1}{n}, \quad i = 1, ..., n. \tag{5.33}$$

It follows from Theorem 5.25 and (5.30) that the system (5.33) of n equations has at least one solution with respect to the variables x_k, y_k, $k = 1, ..., n-1$. Denote by $\{x_l^{(1)}\}, \{y_l^{(1)}\}$, $l = 1, ..., n-1$ a solution of the equation system (5.33) satisfying (5.31) and (5.32). Let $1 \leq r_1 \leq n-1$ denote the number of pairs $(x_l^{(1)}, y_l^{(1)})$ satisfying $x_l^{(1)} < y_l^{(1)}$. Denote

$$l_k = \min\{l : l > l_{k-1},\, x_l^{(1)} < y_l^{(1)}\}, \quad k = 1, ..., r_1,$$

where $l_0 = 0$. Define $a_k^{(1)} = x_{l_k}^{(1)}$, $b_k^{(1)} = y_{l_k}^{(1)}$ and

$$A_1 = \bigcup_{k=1}^{r_1} [a_k^{(1)}, b_k^{(1)}).$$

It follows from (5.33) that $\mu_i(A_1) = 1/n$, for all $i \in I$. We have found the first set belonging to the exact fair division $P = \{A_i\}_{i=1}^n$.

Step 2. $(m = n-1)$

Let $\{u_k^{(1)}\}, \{w_k^{(1)}\}$, $k = 1, ..., s_1$, be sequences of numbers satisfying

$$0 \leq u_k^{(1)} < w_k^{(1)} \leq 1,$$

$$w_k^{(1)} < u_{k+1}^{(1)} \quad \text{for all} \quad k = 1, ..., s_1 - 1,$$

such that

$$C_1 = [0, 1] \setminus A_1 = \bigcup_{k=1}^{s_1} [u_k^{(1)}, w_k^{(1)}).$$

It can be verified that $r_1 - 1 \leq s_1 \leq r_1 + 1$.

In case of $a_1^{(1)} > 0$ and $b_{r_1}^{(1)} < 1$ we have $s_1 = r_1 + 1$ and

$$u_1^{(1)} = 0, w_1^{(1)} = a_1^{(1)},\, u_{r_1}^{(1)} = b_{r_1}^{(1)}, w_{r_1}^{(1)} = 1,$$

$$u_k^{(1)} = b_k^{(1)},\, w_k^{(1)} = a_{k+1}^{(1)}, \quad \text{for} \quad k = 2, ..., r_1 - 1.$$

Define a finite sequence of numbers $0 = e_0^{(1)} < e_1^{(1)} < ... < e_{s_1-1}^{(1)} < e_{s_1}^{(1)} = 1$ satisfying

$$e_k^{(1)} = \frac{1}{1 - \lambda(A_1)} \sum_{j=0}^{k} (w_j^{(1)} - u_j^{(1)}), \quad k = 1, ..., s_1 - 1,$$

where λ denote the Lebesgue measure defined on $\{[0, 1], \mathcal{B}\}$.

Let $g_1 : C_1 \to [0, 1]$ be the one-to-one correspondence such that

$$\text{if} \quad x \in [u_k^{(1)}, w_k^{(1)}) \quad \text{then} \quad g_1(x) = e_{k-1}^{(1)} + \frac{x - u_k^{(1)}}{1 - \lambda(A_1)},\, k = 1, ..., s_1.$$

Define for $i \in I$ continuous and nondecreasing functions $F_i^{(1)} : [0,1] \to [0,1]$ by

$$F_i^{(1)}(t) = \frac{n}{n-1}\left(F_i(g_1^{-1}(t)) - \mu_i(A_1 \cap [0,t))\right). \tag{5.34}$$

Let $\mu_1^{(1)}, ..., \mu_n^{(1)}$ be non-atomic probability measures defined on measurable subsets of the unit interval $[0,1]$ generated by functions $F_i^{(1)}$ i.e. for any $0 \le x \le y \le 1$ we set $\mu_i^{(1)}([x,y)) = F_i^{(1)}(y) - F_i^{(1)}(x)$. Now we apply Theorem 5.25 for measures $\mu_1^{(1)}, ..., \mu_n^{(1)}$ and $m = n-1$. There exist families $\mathcal{F}_1^{(1)}, ..., \mathcal{F}_{n-1}^{(1)}$ such that $\mu_i(\cup \mathcal{F}_j^{(1)}) = \frac{1}{n-1}$ for all $i \in I$ and $j = 1, ..., n-1$. It follows from Theorem 5.25 and (5.30) that there exists a solution $\{x_l^{(2)}\}, \{y_l^{(2)}\}$, $l = 1, ..., q_n(n-1)$, of the equation system

$$\sum_{k=1}^{q_n(n-1)} [F_i^{(1)}(y_k) - F_i^{(1)}(x_k)] = \sum_{k=1}^{q_n(n-1)} \mu_i^{(1)}([x_k, y_k)) = \frac{1}{n-1}, \tag{5.35}$$

for $i \in I$ with respect to variables x_k, y_k, $k = 1, ..., q_n(n-1)$, satisfying

$$0 \le x_k \le y_k \le 1, \quad \text{for all} \quad k = 1, ..., q_n(n-1) \quad \text{and}$$

$$y_k < x_{k+1} \quad \text{for all} \quad k = 1, ..., q_n(n-1) - 1.$$

Let $1 \le h_1 \le q_n(n-1)$ denote the number of pairs $(x_l^{(2)}, y_l^{(2)})$ satisfying $x_l^{(2)} < y_l^{(2)}$. Denote

$$l_k = \min\{l : l > l_{k-1},\ x_l^{(2)} < y_l^{(2)}\}, \quad k = 1, ..., h_1,$$

where $l_0 = 0$. Define $c_k^{(1)} = x_{l_k}^{(2)}$, $d_k^{(1)} = y_{l_k}^{(2)}$ and

$$B_1 = \bigcup_{k=1}^{h_1} [c_k^{(1)}, d_k^{(1)}) \subset [0,1].$$

It follows from (5.35) that $\mu_i^{(1)}(B_1) = \frac{1}{n-1}$ for all $i \in I$. Denote $A_2 = g_1^{-1}(B_1)$. Since g_1 is the piecewise linear function then A_2 is a union of intervals and is given by

$$A_2 = \bigcup_{k=1}^{r_2} [a_k^{(2)}, b_k^{(2)}) \subset C_1,$$

for some numbers $0 \le a_k^{(2)} < b_k^{(2)} \le 1$, with $r_2 = h_1 + z_1$, where z_1 is the number of points $\{e_k^{(1)}\}$, $k = 1, ..., s_1 - 1$, belonging to $\bigcup_{k=1}^{h_1}(c_k^{(1)}, d_k^{(1)})$. It is easy to see that if for some k_0

$$\bigcup_{k=1}^{s_1-1} \{e_k^{(1)}\} \cap (c_{k_0}^{(1)}, d_{k_0}^{(1)}) = \emptyset,$$

then $g_1^{-1}((c_{k_0}^{(1)}, d_{k_0}^{(1)}))$ is a single interval. Suppose now that

$$\bigcup_{k=1}^{s_1-1}\{e_k^{(1)}\} \cap (c_{k_0}^{(1)}, d_{k_0}^{(1)}) = \{e_{k_1}^{(1)}, ..., e_{k_p}^{(1)}\}, \text{ where } 1 \leq p \leq s_1 - 1.$$

In this case $g_1^{-1}((c_{k_0}^{(1)}, d_{k_0}^{(1)}))$ consists of $p+1$ subintervals.
It can be calculated that

$$\mu_i(A_2) = \frac{1}{n} \quad \text{for all} \quad i \in I.$$

If $n > 3$ we proceed with the next steps in similar way.
Step 3. **($m = n - 2$)**
Let $\{u_k^{(2)}\}, \{w_k^{(2)}\}$, $k = 1, ..., s_2$, be sequences of numbers satisfying

$$0 \leq u_k^{(2)} < w_k^{(2)} \leq 1,$$

$$w_k^{(2)} < u_{k+1}^{(2)} \quad \text{for all} \quad k = 1, ..., s_2 - 1,$$

such that

$$C_2 = [0,1] \setminus (A_1 \cup A_2) = \bigcup_{k=1}^{s_2}[u_k^{(2)}, w_k^{(2)}).$$

It can be verified that $s_2 \leq 2h_1$. In case of

$$\bigcup_{k=1}^{s_1-1}\{e_k^{(1)}\} \cap \bigcup_{k=1}^{h_1}(c_k^{(1)}, d_k^{(1)}) = \emptyset,$$

we have $s_2 = 2h_1$.
Define a finite sequence of numbers $0 = e_0^{(2)} < e_1^{(2)} < ... < e_{s_2-1}^{(2)} < e_{s_2}^{(2)} = 1$ satisfying

$$e_k^{(2)} = \frac{1}{1 - \lambda(A_1 \cup A_2)} \sum_{j=0}^{k}(w_j^{(2)} - u_j^{(2)}), \quad k = 1, ..., s_2 - 1,$$

Let $g_2 : C_2 \to [0,1]$ be the one-to-one correspondence such that

$$\text{if} \quad x \in [u_k^{(2)}, w_k^{(2)}) \quad \text{then} \quad g_2(x) = e_{k-1}^{(2)} + \frac{x - u_k^{(2)}}{1 - \lambda(A_1 \cup A_2)}, \; k = 1, ..., s_2.$$

Define for $i \in I$ continuous and non-decreasing functions $F_i^{(2)} : [0,1] \to [0,1]$ by

$$F_i^{(2)}(t) = \frac{n}{n-2}\left(F_i(g_2^{-1}(t)) - \mu_i((A_1 \cup A_2) \cap [0,t))\right).$$

Let $\mu_1^{(2)}, ..., \mu_n^{(2)}$ be non-atomic probability measures defined on measurable subsets of the unit interval $[0,1]$ generated by functions $F_i^{(2)}$. It follows from Theorem 5.25 applied for measures $\mu_2^{(2)}, ..., \mu_n^{(2)}$ and $m = n-2$ that there exist families $\mathcal{F}_1^{(2)}, ..., \mathcal{F}_{n-2}^{(2)}$ such that $\mu_i(\cup \mathcal{F}_j^{(2)}) = \frac{1}{n-2}$ for all $i \in I$ and $j = 1, ..., n-2$. Then there exists a solution $\{x_l^{(3)}\}, \{y_l^{(3)}\}$, $l = 1, ..., q_n(n-2)$, of the equation system

$$\sum_{k=1}^{q_n(n-2)} [F_i^{(2)}(y_k) - F_i^{(2)}(x_k)] = \sum_{k=1}^{q_n(n-2)} \mu_i^{(2)}([x_k, y_k)) = \frac{1}{n-2}, \tag{5.36}$$

for $i \in I$ with respect to variables x_k, y_k, $k = 1, ..., q_n(n-2)$, satisfying

$$0 \le x_k \le y_k \le 1, \quad \text{for all} \quad k = 1, ..., q_n(n-2) \quad \text{and}$$

$$y_k < x_{k+1} \quad \text{for all} \quad k = 1, ..., q_n(n-2) - 1.$$

Let $1 \le h_2 \le q_n(n-2)$ denote the number of pairs $(x_l^{(3)}, y_l^{(3)})$ satisfying $x_l^{(3)} < y_l^{(3)}$. Denote

$$l_k = \min\{l : l > l_{k-1},\, x_l^{(2)} < y_l^{(2)}\}, \quad k = 1, ..., h_2,$$

where $l_0 = 0$. Define $c_k^{(2)} = x_{l_k}^{(3)}$, $d_k^{(2)} = y_{l_k}^{(3)}$ and

$$B_2 = \bigcup_{k=1}^{h_2} [c_k^{(2)}, d_k^{(2)}) \subset [0, 1].$$

Hence we have $\mu_i^{(2)}(B_2) = \frac{1}{n-2}$ for all $i \in I$. Denote

$$A_3 = g_2^{-1}(B_2) = \bigcup_{k=1}^{r_3} [a_k^{(3)}, b_k^{(3)}) \subset X_2,$$

for some numbers $0 \le a_k^{(3)} < b_k^{(3)} \le 1$, with $r_3 = h_2 + z_2$ where z_2 is the number of points $\{e_k^{(2)}\}$, $k = 1, ..., s_2 - 1$, belonging to $\bigcup_{k=1}^{h_2} (c_k^{(2)}, d_k^{(2)})$.
It can be verified that

$$\mu_i(A_3) = \frac{1}{n} \quad \text{for all} \quad i \in I.$$

Finally in step $n-1$ for $m = 2$ we obtain a set A_{n-1} with $\mu_i(A_{n-1}) = \frac{1}{n}$ for all $i \in I$. The last set A_n we get by $A_n = [0, 1] \setminus \bigcup_{j=1}^{n-1} A_j$.

If the non-atomic measures μ_i, $i \in I$, are defined by densities f_i being simple functions then the equation systems (5.33), (5.35) and (5.36) are linear and we can obtain accurate solutions. Unfortunately for arbitrary density functions using numerical methods we can obtain only approximate solutions of the problem of exact division of a cake. Moreover, it is not even possible to estimate the error of approximate computation of the equation systems (5.33), (5.35) and (5.36).
Legut [H6] presented the following example of obtaining approximate solution of exact fair division problem for three players.

Example 5.26. Assume that three players estimate measurable subsets of $[0,1]$ with measures μ_i, $i=1,2,3$, defined by

$$\mu_1([0,t)) = F_1(t) = t,$$
$$\mu_2([0,t)) = F_2(t) = t^2,$$
$$\mu_3([0,t)) = F_3(t) = \sqrt{t},$$

for $t \in [0,1]$. To find an exact fair division we need to proceed with two steps.

Step 1.

It follows from Theorem 5.25 that there exists three families $\mathcal{F}_j$, $j = 1,2,3$ of subintervals, such that $\mu_i(\cup\mathcal{F}_j) = \frac{1}{3}$ and $\sum_j |\mathcal{F}_j| \leq 7$. Then $\min_j |\mathcal{F}_j| \leq 2$. Consider the following equation system for variables

$0 \leq x_1 \leq y_1 < x_2 \leq y_2 \leq 1$

$$\begin{cases} F_1(y_1) - F_1(x_1) + F_1(y_2) - F_1(x_2) = y_1 - x_1 + y_2 - x_2 = \frac{1}{3}, \\ \\ F_2(y_1) - F_2(x_1) + F_2(y_2) - F_2(x_2) = y_1^2 - x_1^2 + y_2^2 - x_2^2 = \frac{1}{3}, \\ \\ F_3(y_1) - F_3(x_1) + F_3(y_2) - F_3(x_2) = \sqrt{y_1} - \sqrt{x_1} + \sqrt{y_2} - \sqrt{x_2} = \frac{1}{3}. \end{cases}$$

Solving the above non-linear equation system we obtain an approximate solution:

$$\begin{cases} x_1 = a_1 = 0, & y_1 = b_1 \approx 0.011394 \\ x_2 = a_2 \approx 0.356524, & y_2 = b_2 \approx 0.678463. \end{cases}$$

Let

$$A_1 = [0, b_1) \cup [a_2, b_2). \tag{5.37}$$

Hence we have $\mu_i(A_1) \approx \frac{1}{3}$ for all $i = 1,2,3$.

Step 2.

Now we construct the remaining sets A_2 and A_3. The first measure μ_1 is the Lebesgue measure, then $\lambda(A_1) = \mu_1(A_1) \approx \frac{1}{3}$. Denote

$$C_1 = [0,1] \setminus A_1 = [u_1, w_1) \cup [u_2, w_2] = [u_1, w_1) \cup [u_2, 1],$$

where $u_1 = b_1$, $w_1 = a_2$, $u_2 = b_2$, $w_2 = 1$. Define numbers $0 = e_0 < e_1 < e_2 = 1$ where

$$e_1 = \frac{3}{2}(w_1 - u_1) \approx 0.517695.$$

Define a function $g : C_1 \to [0,1]$ by

$$g(x) = \begin{cases} \frac{3}{2}(x - u_1) & \text{if} \quad x \in [u_1, w_1), \\ e_1 + \frac{3}{2}(x - u_2) & \text{if} \quad x \in [u_2, 1]. \end{cases}$$

Hence

$$g^{-1}(t) = \begin{cases} \frac{2}{3}t + u_1 & \text{if} \quad t \in [0, e_1), \\ \\ \frac{2}{3}(t - e_1) + u_2 & \text{if} \quad t \in [e_1, 1]. \end{cases}$$

Using (5.34) we construct continuous and nondecreasing functions $F_i^{(1)} : [0,1] \to [0,1]$, $i = 1,2,3$ by

$$F_1^{(1)}(t) = t, \quad \text{for} \quad t \in [0,1],$$

$$F_2^{(1)}(t) = \begin{cases} \frac{2}{3}t^2 + 2tu_1 & \text{if} \quad t \in [0, e_1), \\ \frac{2}{3}(t-e_1)^2 + 2(t-e_1)u_2 + \frac{3}{2}(w_1^2 - u_1^2) & \text{if} \quad t \in [e_1, 1]. \end{cases}$$

$$F_3^{(1)}(t) = \begin{cases} \frac{3}{2}\left(\sqrt{\frac{2}{3}t + u_1} - \sqrt{u_1}\right) & \text{if} \quad t \in [0, e_1), \\ \frac{3}{2}\left(\sqrt{\frac{2}{3}(t-e_1) + u_2} - \sqrt{u_1} - \sqrt{u_2} + \sqrt{w_1}\right) & \text{if} \quad t \in [e_1, 1]. \end{cases}$$

Denote by ν_i, $i = 1,2,3$, nonatomic probability measures defined on $[0,1]$ generated by functions $F_i^{(1)}$ respectively. Since $q_3(2) = 2$ we can cut the interval $[0,1]$ in 3 places to obtain two families $\mathcal{F}_j^{(1)}$, $j = 1,2$, such that $\nu_i(\cup \mathcal{F}_j^{(1)}) = \frac{1}{2}$ for all $i = 1,2,3$. Consider the following equation system for variables $0 \le y_1^{(1)} < x_2^{(1)} \le y_2^{(1)} \le 1$

$$F_i^{(1)}(y_1^{(1)}) + F_i^{(1)}(y_2^{(1)}) - F_i^{(1)}(x_2^{(1)}) = \frac{1}{2}, \quad i = 1,2,3. \tag{5.38}$$

Solving the equation system (5.38) we get an approximate solution

$$\begin{cases} d_1^{(1)} = y_1^{(1)} \approx 0.0617, \\ c_2^{(1)} = x_2^{(1)} \approx 0.333082, \\ d_2^{(1)} = y_2^{(1)} \approx 0.771382. \end{cases} \tag{5.39}$$

In this case we have $c_1^{(1)} = 0$. Denote $B_1 = [0, d_1^{(1)}) \cup [c_2^{(1)}, d_2^{(1)})$. It follows from (5.38) and (5.39) that

$$\nu_i(B_1) \approx \frac{1}{2} \quad \text{for all} \quad i = 1,2,3.$$

Let $A_2 = g^{-1}(B_1)$. It can be verified that

$$\mu_i(A_2) \approx \frac{1}{3}, \quad \text{for all} \quad i = 1,2,3.$$

Finally we obtain the approximate solution given by

$$\begin{cases} A_1 \approx [0, 0.011394) \cup [0.356524, 0.678463), \\ A_2 \approx [0.011394, 0.056060) \cup [0.233449, 0.356524) \cup [0.678463, 0.847588) \\ A_3 \approx [0.056060, 0.233449) \cup [0.847588, 1]. \end{cases}$$

□

6 Discussion of other scientific achievements

Besides this postdoctoral dissertation, my scientific achievements were presented in the following articles:

D1. Legut J. (1985): "Market Games with a Continuum of Indivisible Commodities", *International Journal of Game Theory,* 15, 1-7.

D2. Legut J. (1985): "The Problem of Fair Division for Countably Many Participants", *J. Math. Anal. Appl.*, 109, 83-89.

D3. Legut J. (1987): "A Game of Fair Division with a Continuum of Players". *Colloquium Mathematicum*, vol LIII, 323-331.

D4. Legut J. (1988): "A Game of Fair Division in Normal Form", *Colloquium Mathematicum*, vol LVI, 179-184.

D5. Legut J. (1988): "Inequalities for α-optimal partitioning of a measurable space", *Proc. of the American Math. Soc.* vol. 104, No. 3, 1249-1251.

D6. Legut J. and Wilczyński M. (1988): "Optimal Partitioning of a Measurable Space", *Proc. of the American Math. Soc.* vol. 104, 262-264.

D7. Legut J. (1990): "On Totally Balanced Games Arising from Cooperation in Fair Division", *Games and Economic Behavior*, 2, 47-60.

D8. Legut J. and Wilczyński M. (1990): "Optimal partitioning of a Measurable Space into Countably Many Sets", *Probability Theory and Related Fields* 86, 551-558.

D9. Legut J., Potters J.A.M. and Tijs S.H. (1994): "Economies with Land - A Game Theoretical Approach", *Games and Economic Behavior* vol. 6, Issue 3, 416-430.

D10. Legut J., Potters J.A.M. and Tijs S.H. (1995): "A transfer Property of Equilibrium Payoffs in Economies with Land", *Games and Economic Behavior* vol. 10, Issue 2, 355-375.

D11. Jóźwiak I. and Legut J. (1991): "Decision Rule for an Exponential Reliability Function" *Microelectron. Reliab.* vol. 31. 71-73.

The first four papers [D1]-[D4] deal with game theoretical approaches to the fair division problem and were the base of my doctor thesis.
Most of my results presented in the above articles were mentioned in a popular book entitled "Fair division-from cake-cutting to dispute resolution" which was written by Brams and Taylor [6]. In paper [D5] for the first time in the theory of fair division I introduced a notion of α-optimal partitions which generalizes the notion of equitable optimal partitions. For these partitions I defined an α-optimal value for which I found better estimation than it was obtained by Elton et. al. [19]. Moreover, this estimation was achieved using simple geometrical method. This method were used later by other authors for improving my estimation (cf. [11], [13], [14], [15], [45],).

One of my most significant scientific result was achieved together with Wilczyński and was presented in paper [D6]. We used a minmax theorem of Sion (see [2]) to give the form of α-optimal partitions. This result was later very helpful for finding algorithms of optimal partitioning of measurable spaces (cf. [H2], [H3], [H5],[H7]) and was also discussed by other authors analysing the α-optimal partitions (cf. [11], [14], [15], [45]).
In paper [D7] I proposed a method of analysing a secondary division of an object $\mathcal{X}$ using the theory of cooperative games. In this method players form coalitions to improve the initial partition and then a cooperative game is defined. It turned out that these games are totally balanced and then have nonempty core. A method of obtaining an imputation from this core is found. A characterization and some properties of such class of games are presented. This results were considered in the literature of fair division and cooperative games theory (cf. [9], [10], [12], [13]).
In paper [D8] a notion of an optimal partition of a measurable space into countably many sets according to given nonatomic measures is defined. The existence of such partition is proved. Bounds for the optimal value are given and the set of optimal partitions is characterized. Finally, an example related to statistical decision theory is presented.
In paper [D9] a cooperative game v_E associated with an economy with land E (an economy of Debreu-type in which land is the unique commodity) is defined. The set of all TU-games of type v_E is investigated and the set of equilibrium payoffs (in the TU-sense) of the economy E is described as a subset of the core of v_E. The authors proved that equilibrium payoffs can be extended to population monotonic allocation schemes in the sense of Sprumont. The results of this article were mentioned in other papers (cf. [14], [15], [43], [46], [47])
Paper [D10] deals with an exchange economy of Debreu type with only one commodity (land). The authors investigate NTU-games connected to these kinds of economies. The main result of this paper is that equilibrium payoffs is the NTU-model are connected to equilibrium payoff in the TU-model considered in [D9] by b-transfer - a concept introduced by Shapley [48].
The main result of the paper [D6] was used to obtain a minimax decision rule for the exponential reliability function. This result was presented in article [D11]. The authors gave an example which was solved using computer algorithm.

References

[1] Alon, N. *Splitting Necklaces*, Advanced in Math. 63, 247-253 (1987)

[2] Aubin, J.P.: *Mathematical Methods of Game and Economic Theory,* North-Holland Publishing Company. (1980)

[3] Barbanel, J.: *On the geometry of cake division,* J. Math. Anal. Appl. 264, 639-656 (2001)

[4] Beckenbach, E. F., *Generalized convex functions,* Bull. Am. Math. Soc., 43, 363-371 (1937)

[5] Brams S. J. and Taylor A.D.: *An envy-free cake division protocol,* Am. Math. Mon. 102, 9-18 (1995)

[6] Brams S. J. and Taylor A.D.: Fair division-From cake-cutting to dispute resolution, Cambridge University Press, (1996)

[7] Brams S. J., Taylor A.D., Zwicker W. S.: *Old and new moving-knife schemes,* Math. Intellingencer 17, 30-35 (1995)

[8] Brams S. J., Taylor A.D., Zwicker W. S.: *A moving-knife solution to the four-person envy-free cake division problem,* Proc. Amer. Math. Soc. 125, 547-554 (1997)

[9] Csóka, P., Herings, P.J.J., Kóczy, L.: *Stable allocations of risk,* Games Econom. Behav. 67, no. 1, 266-276 (2009)

[10] Csóka, P., Herings, P.J.J., Kóczy, L.: *Balancedness conditions for exact games,* Math. Methods Oper. Res. 74, 41-52 (2011)

[11] Dall' Aglio M.: *The Dubins-Spanier optimization problem in fair division theory.* J. Comput. Appl. Math. 130, no. 1-2, 17-40 (2001)

[12] Dall'Aglio, M., Branzei, R, Tijs, S.: *Cooperation in dividing the cake,* TOP 17, no. 2, 417-432. (2009)

[13] Dall'Aglio M. and Hill T,: *Maximin share and minimax envy in fair-division problems.* J. Math. Anal. Appl. 281 , no. 1, 346-361 (2003)

[14] Dall'Aglio M. and Di Luca: *Finding maxmin allocations in cooperative and competitive fair division,* Ann. Oper. Res. 223, 121–136 (2014)

[15] Dall'Aglio M. and Di Luca: *Bounds for α-Opimal Partitioning of a Measurable Space Based on Several Efficient Partitions,* J. Math. Anal. Appl. 425, no. 2, 854–863 (2015)

[16] Dubins, L. and Spanier E.: *How to cut a cake fairy,* Am. Math. Mon. 68, 1-17 (1961)

[17] Demko, S. and Hill, T.: *Equitable Distribution of Indivisible Objects,* Mathematical Social Sciences 16, 145-158 (1988)

[18] Dvoretzky, A., Wald A. and Wolfowitz, J.: *Relations among certain ranges of vector measures,* Pacific J. Math. 1, 59-74 (1951)

[19] Elton, J. Hill, T. and Kertz, R.: (1986), *Optimal partitioning ineaqualities for non-atomic probability measures,* Trans. Amer. Math.Soc., 296, 703-725 (1986)

[20] Fink, A. M.: *A note on the fair division problem,* Math. Magazine, 37, 341-342 (1964)

[21] Hill, T. and Tong, Y.: *Optimal-partitioning ineaqualities in classification and multi hypotheses testing* Ann. Stat., 17, 1325-1334 (1989)

[22] Hobby C.R. and Rice J.R.*A moment problem in L_1 approximation* Proc. Amer. Math. Soc. 16, 665-670 (1965)

[23] Jessen, B. *Bemaerkiner om konvekse functioner og uligheder imellem midelvaerdier,* Matematisk tidsskfift. B., 17-28 (1931)

[24] Jóźwiak I. and Legut J.: *Decision Rule for an Exponential Reliability Function* , Microelectron. Reliab. vol. 31, 71-73 (1991)

[25] Karlin, S. and Novikoff, A., *Generalized convex inequalities,* Pacific J. Math., 13, 1251-1279 (1963)

[26] Knaster, B.: *Sur le probleme du partage pragmatique. de H. Steinhaus,* Ann. Soc. Polon. Math. 19, 228-230 (1946)

[27] Legut J.: *The Problem of Fair Division for Countably Many Participants* , J. Math. Anal. Appl., 109, 83-89 (1985)

[28] Legut, J.: *Inequalities for* α*- optimal partitioning of a measurable space.* Proc. Amer. Math. Soc., 104, 1249-1251 (1988)

[29] Legut J.: *The problem of fair division for countably many participants.* J. Math. Anal. Appl. 109, no. 1, 83–89 (1985)

[30] Legut J.: *A game of fair division with continuum of players.* Colloq. Math. 53 , no. 2, 323–331 (1987)

[31] Legut J.: *Market games with a continuum of indivisible commodities.* Internat. J. Game Theory 15 no. 1, 1–7 (1986)

[32] Legut, J.; Potters, J. A. M.; Tijs, S. H.: *Economies with land—a game theoretical approach.* Games Econom. Behav. 6, no. 3 416–430 (1994)

[33] Legut, J. and Wilczyński, M.: *Optimal partitioning of a measurable space.* Proc. Amer. Math. Soc. 104, 262-264 (1988)

[34] Legut, J., Wilczyński, M.: *Optimal partitioning of a measurable space into countably many sets,* Probab. Theory Related Fields 86, (1990), no. 4, 551-558 (1990)

[35] Legut, J. and Wilczyński, M.: *How to obtain maximal and minimal subranges of two-dimensional vector measures,* unpublished paper available on line at: http://prac.im.pwr.wroc.pl/~legut/hab/L35.pdf

[36] Legut, J. and Wilczyński, M.: *On dividing a land with a river fairly,* unpublished paper available on line at: http://prac.im.pwr.wroc.pl/~legut/hab/L36.pdf

[37] Legut, J.: *Simple fair division of a square,* unpublished paper available on line at: http://prac.im.pwr.wroc.pl/~legut/hab/L37.pdf

[38] Lehmann, E. L., Joseph P. Romano. *Testing Statistical Hypotheses*, third edition. Springer Science+Business Media, Inc., New York. (1986)

[39] Lyapunov, A.: *Sur les fonctions-vecteurs completement additives,* Bulletin of the Academy of Sciences of the USSR, 4, 465-478 (1940)

[40] Nicoló, A. Perea, A. and Roberti, P.: *Equal opportunity equivalence in land division.* J. Spanish Econom. Assoc. 3 (1-2) 133-142 (2012)

[41] Palmer, J.A., Relative Convexity, unpublished paper (2003)

[42] Popoviciu, T., *Notes sur les fonctions d'ordre superieur*, Mathematica 12, 81-92 (1936)

[43] Reijnierse, H., Borm, P., Quant, M., Meertens, M.: *Processing games with restricted capacities* European J. Oper. Res. 202, no. 3, 773-780 (2010)

[44] Robertson, J.M. and Webb, W.A.: *Minimal number of cuts for fair division,* Ars Combin. 31, 191-197 (1991)

[45] Sagara, N.: *A characterization of α-maximin solutions of fair division problems,* Mathematical Social Sciences, vol 55, 273-280 (2008)

[46] Segal-Halevi, E., Nitzan, S., Hassidim, A. and Aumann, Y.: *Fair and square: Cake-cutting in two dimensions.* J. Math. Ecnom. 70, 1-28 (2017)

[47] Segal-Halevi, E. Sziklai, B. R.: *Resource-monotonicity and population-monotonicity in connected cake-cutting,* Math. Social Sci. 95 19-30 (2018)

[48] Shapley, L. S.: *Utility comparison and the theory of games,* The Shapley value, 307-319, Cambridge Univ. Press, Cambridge, 307-319 (1988)

[49] Shisha, O. and Cargo, G. T., *On comparable means*, Pacific J. Math. (3) 14, 1053-1058 (1964)

[50] Steinahus, H.: *The problem of fair division,* Econometrica 16, 101-104 (1948)

[51] Steinahus, H.: *Sur la division pragmatique* Econometrica 17 (Suppl), 315-319 (1949)

[52] Steinahus, H.: *Mathematical Snapshots* Oxford University Press, Oxford, 65-72 (1960)

[53] W. Stromquist, W. and Woodall, D. R.: *Sets on which several measures agree*, J. Math. Anal. Appl. 108, 241-248 (1985)

[54] D. Weller, D.: *Fair division of a measurable space,* J. Math. Econom. 14, 5-17 (1985)

[55] Woodall, D.R. *Dividing a cake fairly,* J. Math. Anal. Appl. 78, 233-247 (1980)