Summary of doctoral dissertation titled "Pricing time-capped American options". Author: Paweł Stępniak

This thesis considers the problem of pricing (both analytically and numerically) of American options with stochastic constraints imposed on the time to maturity. Fundamentally, the problem can be formulated as an optimal stopping problem of the following form:

$$\sup_{\tau \in \mathcal{T}, \tau \leq T} \mathbb{E}[e^{-r\tau \wedge \theta} G(S_{\tau \wedge \theta})],$$

where τ is the optimal stopping time, T is the positive (possibly infinite) time to maturity, S_t is a stochastic process describing the underlying asset price, G is the payout function and θ is the random time at which the contract is terminated. This time is referred to as a time-cap and may or may not be associated with the underlying asset performance.

The motivation for this thesis is to investigate a new and broad class of financial instruments that could potentially be introduced to the market. The development of capped options began in the early 1990s in Chicago, when the Chicago Board Options Exchange launched into the market capped European options on the S&P 100 and S&P 500. These contracts, which were immediately exercised when the underlying index exceeded a predetermined level (the cap), became a practical example of how payoff-limiting features could reduce the seller's risk while retaining investment appeal. Since then, many variations of capped options have been studied, but almost all of them apply the cap to the payoff function or the underlying asset price.

In this work, the approach is different: the cap is placed on the time to maturity. Each chosen time cap defines a new type of instrument, allowing for the study of a wide range of problems. Among the contracts considered are those where the time cap is given by the first exit time of the underlying asset from a specified interval, the last exit time from such an interval, and the first time when the drawdown — the ratio between the current price and the historical maximum — exceeds a certain threshold. The last exit time is not a stopping time, which would make this contract difficult to implement in practice; nevertheless, it is included here as an interesting analytical exercise that fits the overall scope of the thesis.

The price dynamics of the underlying asset is modeled using a spectrally negative geometric Lévy process, which allows for sudden downward jumps while retaining analytical tractability. The analytical part of the work employs the guess-and-verify method: an optimal stopping rule is proposed based on qualitative arguments, and then its validity is established using a verification theorem together with the Hamilton–Jacobi–Bellman system. In the numerical part, a modified version of the Least Squares Monte Carlo method, specifically adapted for the time-capped framework, is applied to illustrate and support the theoretical results. This thesis is organized as follows.

Chapter 1 provides a theoretical introduction to financial markets and options, as well as an overview of the probability space, Lévy processes, scale functions, and the key parameters used throughout the remainder of the thesis.

Chapter 2 focuses on the valuation of a perpetual American put option with a random maturity date given by the first exit time of the underlying asset price from a pre-specified set. While similar instruments have been studied in the past, the introduction of downward jumps in our setting allows for a Poisson-type drop below the lower boundary. In such a case, the contract is early exercised by the cap event, rather than terminated, which means the holder may still obtain a non-zero payoff. In the proposed solution, we conjecture that the optimal stopping time is the first moment when the asset price falls below a certain level, and then we determine the option price under this assumption. This conjecture is subsequently verified later in the chapter. Finally, in the last part of this chapter, we present a numerical analysis.

Chapter 3 examines options stopped by a drawdown event, defined as the first time when the ratio between the underlying asset price and its historical maximum exceeds a predetermined threshold. The central statement of this chapter is the proof that the optimal stopping time is the first moment when the asset price falls below the value of a certain function of its past historical maximum. We start from the analysis of the special case of the Black-Scholes market when there are no jumps in the asset price. Then, later, we analyze the case with additional exponential downward jumps in the asset price. In both scenarios, we show that the optimal strategy is either to wait until the price falls below a fixed barrier or until the option is stopped by the drawdown event. As in the previous case, the latter does not terminate the option but forces its early exercise, potentially resulting in a positive payoff for the holder.

Chapter 4 considers an option stopped at the moment when the underlying asset price crosses above a given threshold for the last time. Since this moment is not a stopping time, the contract would be difficult to implement in practice. However, it is very interesting from the mathematical point of view and produces intriguing analysis. As before, we use the guess-and-verify method proposing the optimal strategy and then verify it and derive the corresponding option price. This chapter is based on [1].

Chapter 5 presents the description and proof of convergence of a modified Least Squares Monte Carlo method, designed to price American options whose maturity is constrained by an arbitrary stopping time. The chapter concludes with numerical results for several selected instruments of this type. This chapter is based on [2].

References

[1] Palmowski, Z. and Stępniak, P. (2023) *Last-Passage American Cancelable Option in Lévy Models.* Journal of Risk and Financial Management **16(2)**.

[2] Palmowski, Z. and Stępniak, P. (2025) *Pricing time-capped American options using Least Squares Monte Carlo method.* Journal of Computational Finance, **28(3)**.