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Report on the Dissertation
'Boundary problems for nonlocal operators'
by Damian Fafuła

Recently, there is an increased research interest in boundary conditions for (elliptic and/or parabolic) equations involving nonlocal operators such as the fractional Laplacian $(-\Delta)^{\frac{\alpha}{2}}$ for $\alpha \in (0,2)$. Intimately related to this is a description of 'reflection mechanisms' that can be used to confine a stochastic process – in the case of the fractional Laplacian the isotropic α-stable process – to a domain $D \subset \mathbb{R}^d$. In contrast to diffusion processes, where the probabilistic interpretation of various boundary contions is well understood, this topic is largely open for Lévy processes as the isotropic α-stable process.

In his thesis, Damian Fafuła studies a (one-dimensional) non-local boundary value problem of the form

$$(-\Delta)^{\frac{\alpha}{2}} u = f \quad \text{in } (0,\infty) \tag{1}$$

$$\mathcal{N}_{\frac{\alpha}{2}} u = u \quad \text{in } (-\infty, 0], \tag{2}$$

first introduced by Dipierro, Ros-Oton and Valdinoci [29][1]. Here, the operators $\mathcal{N}_{\frac{\alpha}{2}}$ appearing in the boundary condition (2) is the *nonlocal normal derivative*

$$\mathcal{N}_{\frac{\alpha}{2}} u(x) := \int_D [u(x) - u(y)]\nu(x,y)\,dy \quad \text{where} \quad \nu(x,y) = c_{1,\alpha}\frac{1}{|x-y|^{\alpha+1}}.$$

Special emphasis is given to the probabilistic interpretation of the boundary condition (2) for the associated parabolic equation and thus the reflection mechanism used to confine the stochastic process to $(0,\infty)$. In [29] a possible interpretation of (2) was given but, as Vondraček [66] pointed out, this interpretation is somewhat ambiguous. He proceeded to give a probabilistic interpretation of the stochastic process associated to the quadratic form appearing in the variational formulation of (1)–(2) which is different from the one in [29]. However, a drawback of the approach of Vondraček is that the ambient Hilbert space in the form approach is *not* the classical L^2-space with respect to Lebesgue measure, but a weighted L^2-space.

This downside of Vondraček's approach is remedied in the present thesis. Mr. Fafuła constructs, by very different methods than in [29, 66], a Hunt process $(X_t)_{t\geq 0}$ and its associated transition semigroup $(K_t)_{t\geq 0}$. Studying these objects in detail, he shows that (for $\alpha \neq 1$) the boundary value problem (1)–(2) is solved by the Green operator of the process (Theorem 7.6). He also shows that the stochastic process is associated to a quadratic form on the classical L^2-space with respect to Lebesgue measure (Section 6).

[1] References follow the numbering in the thesis

Let me now discuss the chapters of the dissertation in more detail.

After an introcution and some preliminaries in Chapter 2, the main line of work is picked up in *Chapter 3*, where the stochastic process and its transition semigroup are constructed. The actual state-space of the process is $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ and the process behaves as follows. Starting at a point $x > 0$, the process behaves like the isotropic α-stable process until it jumps out of $(0, \infty)$, say to a point $z \in (-\infty, 0)$. It then remains at position z for an exponential time with parameter $\nu(z, D) = \int_D \nu(z, y)\, dy$ after which it jumps back to $(0, \infty)$ where the position is chosen according to the kernel $\nu(z, y)/\nu(z, D)$. In the name of this chapter, the process is attributed to Servadei and Valdinoci but, alas, this nomenclature is not explained further in the main text.

The structure of this chapter is rather interesting, as Mr. Fafuła combines probabilistic and analytic results. First, in Section 3.1, a stochastic process as described above is constructed using a concatenation procedure due to Werner [67]. In Section 3.2 a transition semigroup is constructed by means of perturbation of the preliminary semigroup $(\widehat{P}_t)_{t \geq 0}$ and it is shown in Theorem 3.7, that this semigroup is associated to the process constructed in Section 3.1. Subsequently, additional properties of the semigroup are established in Theorem 3.14 (the semigroup extends to a symmetric, strongly continuous, positive contraction semigroup on $L^2(\mathbb{R})$ and in Theorem 3.16 (strong Feller property for the semigroup).

One of the main differences between the process considered in this thesis and that from the article of Vondraček is that the time the process spends at a point $z \in (-\infty, 0)$ depends upon the point z itself. Indeed, the closer z is to the boundary point 0, the shorter the time is on average. This makes it possible for the process to be reflected to and from $(0, \infty)$ infinitely often in finite time after which – by construction – is killed. This behaviour is studied in *Chapter 4*, which is, in my opinion, the most interesting part of this thesis. In the main result of this chapter, Theorem 4.11, it is proved that if $\alpha \in (0, 1)$, then the lifetime of the process is almost surely $+\infty$ and $\lim_{t\to\infty} |X_t| = \infty$. On the other hand, for $\alpha \in (1, 2)$, the lifetime ζ is almost surely finite and $\lim_{t \to \zeta} X_t = 0$ almost surely. In the borderline case $\alpha = 1$, the lifetime is almost surely infinite, but no limit exists. The basic strategy to obtain these results is to prove that – properly rescaled – the consecutive return times and positions are independent and identically distributed, which allows to use the Borel–Cantelli lemma to study the finiteness of ζ. In Section 4.5, it is proved that for $\alpha \in (1, 2)$, the semigroup is a Feller semigroup on $C_0(\mathbb{R}^*)$. The main point here is to prove the invariance of $C_0(\mathbb{R}^*)$ under $(K_t)_{t \geq 0}$ (Lemma 4.13), while the other necessary properties follow easily given the results of Chapter 3. Unfortunately, no information in the case $\alpha \in (0, 1]$ is provided.

The rather technical *Chapter 5* establishes the fact that for certain excessive functions h_β, introduced in Section 3.3, the pointwise derivative of the semigroup at zero exists. While this is reminiscent of the definition of the pointwise generator of a semigroup, it should be pointed out that the functions h_β are unbounded and therefore do not belong to the domain of the pointwise generator in the sense of semigroup theory. The motivation to study these functions rather stems from Chapter 7, where the results of Chapter 5 are used to establish boundedness of the Green potential Gf for functions f that have a similar behaviour as the excessive functions (see Lemma 7.3).

As we noted above, a the starting point for the investigations in [29, 66] was a certain Dirichlet form. It actually follows already from Theorem 3.14 that also the process considered here is associated to a Dirichlet form $\mathcal{E}$. This form is studied in more detail in *Chapter 6*. It is proved that for smooth functions with compact support, the form coincides with that considered in [66] (Theorem 6.4; let us point out, however, that the ambient Hilbert space is different). Making clever use of a version of Hardy's inequality (Section 6.1), Mr. Fafuła is able to identify the form domain

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with a weighted L^2-space.

The concluding *Chapter 7* returns to the initial question of solvability of the boundary value problem (1)–(2). In Theorem 7.6 it is proved that for $\alpha \neq 1$ and $f \in C_c(\mathbb{R}^*)$, the Green potential $u = Gf$ gives a solution of the Neumann problem

$$\begin{aligned} (-\Delta)^{\frac{\alpha}{2}} u &= f \quad \text{on } (0,\infty) \\ \mathcal{N}_{\frac{\alpha}{2}} u &= f \quad \text{on } (-\infty,0]. \end{aligned}$$

By considering f with $f|_{(-\infty,0)} \equiv 0$, also the initial problem is solved.

Overall assessment

Damian Fafuła has demonstrated profund knowledge in several areas of mathematics, including probability theory, potential theory and analysis as well as the ability to apply this knowledge to obtain new and independent results. In my opinion, this thesis constitutes a significant contribution to current mathematical research. I wholeheartedly suggest to accept this thesis and – pending successful oral defense – award Mr. Fafuła the degree ‘doctor of philosophy’.

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