

Report on the doctoral dissertation  
'Boundary problems for nonlocal operators'  
of Damian Fafała

Perth, May 16, 2024

The thesis under review is an excellent piece of mathematics. Not only does it show a deep general theoretical knowledge of the candidate in mathematics, but it also highlights the ability of the candidate to independently conduct scientific work. In fact, the work performed and the results obtained are, in my opinion, original, brilliant, and of very high quality.

The subject of the doctoral dissertation is a creative solution to a set of scientific problems related to fractional operators. These problems are very topical since they have important consequences in mathematics and in applied sciences, including finance, genetics, physics, chemistry, etc. The thesis neatly explains these motivations and places the overall line of research into a broad context.

The thesis presents a variety of important results. Let me try to summarize some of them:

A direct, ad hoc construction of a new stochastic process is presented, and the corresponding properties (lifetime, asymptotic, associated bilinear forms) are studied in detail. This process is relevant also because it is related to a nonlocal equation with Neumann conditions. An interesting feature of this process is that the time spent on the complement of the domain depends on the position reached after the egress.

The construction of this process required technically advanced methods, such as concatenation of the right processes.

The analysis of the process is profound and detailed. It includes the study of the lifetime and the limit behaviors, which are deeply influenced by the Lévy exponent of the process. The analysis also involves the study of consecutive returns and the sum of time-increments between consecutive returns.

It is also established that for large Lévy exponents the process satisfies the Feller property. This is important also because, roughly speaking, this property is related to a suitable continuity with respect to the starting point and a strong Markov property.

A new Hardy inequality is also established, with an interesting conjecture on a case remained open.

A direct solution to the Neumann problem is presented using the notion of 0-potential.

In my opinion, for all the reasons mentioned above, the thesis under consideration deserves the **distinction of the doctoral dissertation** and I am very happy to propose it.

The thesis is also very well written, I just have a small number of stylistic suggestions:

Line 5, page 2, not sure "which" is the right pronoun here, maybe "and the"?

Before (1.4), not sure "more" is the right word here, maybe "extensively"?

Line 3, page 3, I think that "to nonlocal" should be "to the nonlocal".

Line 10, page 4, I think that "followed" should be "following".

End of page 4, I think that "mainly is" should be "is mainly".

In Figures 1.1 and 1.2, maybe one should specify that  $D = (0, +\infty)$  here?

I wonder if a comparison of Figures 1.1 and 1.2 with similar figures obtained from the other processes in the literature would visually highlight similarities and differences between the previous processes and the new one.

As a final remark, I would like to invite the candidate to post the thesis in arxiv, since this is a very nice and complete work which can be a solid reference for others.

Sincerely yours,

Enrico Valdinoci

A handwritten signature in blue ink, appearing to be 'E. Valdinoci', written in a cursive style.