Report on the Ph.D. thesis

PROPERTIES OF LÉVY PROCESSES IN SMOOTH DOMAINS

by Tomasz Juszczyszyn

The Ph.D. thesis under review is the collection of three already published papers:

1. T.Juszczyszyn, M.Kwaśnicki, Hitting times of points for symmetric Lévy processes with completely monotone jumps, *Electron. J. Probab.* **20** (2015), no.48, 24 pp.

2. T.Juszczyszyn, M.Kwaśnicki, Martin kernels for Markov processes with jumps, *Potential Anal.* **47**(3) (2017), 313–335.

3. T.Juszczyszyn, Decay rate of harmonic functions for non-symmetric strictly α-stable Lévy processes, *Studia Mathematica* **260** (2021), 141-165.

The subject of all three papers belongs to the theory of stochastic processes, and more precisely, to probabilistic potential theory and boundary theory. The common theme is the study of a particular Markov process until its first exit from an open set.

The first paper deals with a one-dimensional symmetric Lévy process whose Lévy measure has a completely monotone density, and the characteristic exponent Ψ satisfies the following scaling-type condition:

$$\frac{\xi\Psi''(\xi)}{\Psi'(\xi)} \geq \alpha - 1 \quad \text{for all } \xi > 0.$$

The key object of the study is the function $t \mapsto \mathbb{P}(\tau_x > t)$, the tail of the distribution of the hitting time τ_x to the point x (or, equivalently, the first exit time from $\mathbb{R} \setminus \{x\}$). In case $\alpha > 1$, it holds that $\mathbb{P}(\tau_x < \infty) = 1$, i.e., points are not polar. The authors derive sharp large-time and small-space estimates of that function and its derivatives. At the time of writing the paper, very little was known about τ_x for processes more general than the stable ones. In this respect, the paper is an important breakthrough. The main ideas of the proof are similar to the ones in M. Kwaśnicki, J. Malecki, M. Ryznar, First passage times for subordinate Brownian motions, *Stoch. Proc. Appl.* **123** (2013), 1820–1850, while their implementation relies on a generalized eigenfunction expansion obtained in M. Kwaśnicki, Spectral theory for one-dimensional symmetric Lévy processes killed upon hitting the origin, *Electron. J. Probab.* **17** (2012), 1–29. The path to the full proofs is quite involved and technical and requires deep knowledge of the relevant theory, high technical skills, and a lot of ingenuity.

The purpose of the second paper is to study, in the context of metric measure spaces, boundary limits of ratios of positive functions thah are harmonic in an arbitrary open set with respect to a Markov process with jumps. The proof of the main result, the existence of boundary limits of ratios, is based on the general boundary Harnack inequality from K. Bogdan, T. Kumagai, M. Kwaśnicki, *Trans. Amer. Math. Soc.* **367** (2015), 477–517. As a corollary, the authors identify the Martin boundary with the topological boundary and prove the Martin representation theorem for non-negative harmonic functions. This is an important paper that, essentially, gives the definitive answer to the question of the Martin boundary under a very general setup. The proofs are based on the methods from the paper K. Bogdan, T. Kulczycki, M. Kwaśnicki, Estimates and structure of α-harmonic functions *Probab. Theory Relat. Fields* **140** 345–381 (2008), and the oscillation reduction method. The implementation of the latter is quite delicate and requires a very fine analysis.

The boundary decay rate of harmonic functions for rotationally symmetric Lévy processes in smooth open sets is a well-studied subject. The earliest result was obtained for the isotropic α-stable process, $0 < \alpha < 2$, in open $C^{1,1}$ set D. This has been subsequently extended to unimodal isotropic Lévy processes satisfying certain weak scaling conditions. The third paper comprising the Ph.D. thesis is the first one to study the same question for non-symmetric processes. Naturally, the starting point is a non-symmetric strictly α-stable process $X = (X_t, \mathbb{P}_x)_{t\geq 0, x\in\mathbb{R}^d}$ in $\mathbb{R}^d$, $d \geq 2$. Due to non-symmetry, the decay rate cannot be described as a single power of the distance to the boundary. The correct decay rate is $\delta_D(x)^{\beta(x)}$, where $\beta(x) = \alpha\mathbb{P}^0(\langle X_t, x - x_D\rangle > 0)$. Here $x \in D$ and $x_D \in \partial D$ is the point on the boundary closest to x. The factor $\mathbb{P}^0(\langle X_t, x - x_D\rangle > 0)$ can be explained as the positivity parameter of the 1-dimensional projection of X on the inner normal at x_D. This decay rate is proved in the main theorem under two assumptions: Assumption A is about the spherical part of the Lévy measure, and Assumption B is about the smoothness of D. The result of the paper is highly original. Proofs combine existing methods for symmetric processes and some new ideas needed for the non-symmetric case.

The submitted Ph.D. thesis solves three important mathematical problems: (i) Explain the behavior of the tail probability of the first hitting time to the point $x \in \mathbb{R}$ for symmetric Lévy process under reasonable assumptions; (ii) Identify the Martin boundary of a Markov process in a general context of a metric measure space; (iii) Find the boundary decay of nonnegative harmonic functions for isotropic Lévy process in smooth open sets. The thesis provides original and highly non-trivial solutions to these problems that have been published in very respectable mathematical journals. I am most impressed by the solution to the third problem – there are only a few papers dealing with the boundary behavior of the non-symmetric Lévy processes. The results of the papers comprising the thesis have been already recognized by the researchers in the field, and have been cited 18 times by MathSciNet (33 by Google Scholar, 17 by WoS). In solving these problems, the Ph.D. candidate has demonstrated a deep knowledge of the theoretical background including the state-of-the-art results from probability theory and theory of stochastic processes, potential theory, and functional analysis. This shows a thorough theoretical knowledge of the relevant fields. Originality of solutions and background theoretical knowledge guarantee that the candidate is capable to carry out independent mathematical research. Therefore, I strongly recommend to accept the submitted manuscript as a Ph.D. thesis.

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Zoran Vondraček

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