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Properties of Lévy processes in smooth domains

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Introduction

The doctoral thesis is a collection of three publications written with or under supervision of Mateusz Kwaśnicki, PhD, DSc:

[1] T. Juszczyszyn, M. Kwaśnicki, *Hitting times of points for symmetric Lévy processes with completely monotone jumps.* Electronic Journal of Probability 20 (2015).

[2] T. Juszczyszyn, M. Kwaśnicki *Martin kernels for Markov processes with jumps.* Potential Anal. 47(3) (2017): 313–335.

[3] T. Juszczyszyn, *Decay rate of harmonic functions for non-symmetric strictly α-stable Lévy processes.* Studia Mathematica 260 (2021), 141-165

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Hitting times of points for symmetric Lévy processes with completely monotone jumps*

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Abstract

Small-space and large-time estimates and asymptotic expansion of the distribution function and (the derivatives of) the density function of hitting times of points for symmetric Lévy processes are studied. The Lévy measure is assumed to have completely monotone density function, and a scaling-type condition $\inf \xi\Psi''(\xi)/\Psi'(\xi) > 0$ is imposed on the Lévy–Khintchine exponent Ψ. Proofs are based on generalised eigenfunction expansion for processes killed upon hitting the origin.

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1 Introduction and statement of the results

Let X be a one-dimensional Lévy process, that is, a real-valued stochastic process with stationary and independent increments, càdlàg paths, and initial value $X_0 = 0$. The process X is completely characterised by its Lévy–Khintchine exponent Ψ, which is given by the Lévy–Khintchine formula:

$$\Psi(\xi) = -\log(\mathbb{E}e^{i\xi X_1}) = a\xi^2 - ib\xi + \int_{\mathbb{R}\setminus\{0\}} (1 - e^{i\xi z} + i\xi z \mathbb{1}_{(-1,1)}(z))\nu(dz)$$

for $\xi \in \mathbb{R}$, where $a \geq 0$ is the Gaussian component, $b \in \mathbb{R}$ is the drift coefficient and ν is a non-negative measure such that $\int_{\mathbb{R}\setminus\{0\}} \min(1, z^2)\nu(dz) < \infty$, called Lévy measure. The *first hitting time* of a point $x \in \mathbb{R}$ is defined by the formula

$$\tau_x = \inf\{t \geq 0 : X_t = x\}.$$

In this article estimates and asymptotic formulae, in terms of the Lévy–Khintchine exponent Ψ, for the tail and the density function of τ_x are derived, under a number of conditions on the process X.

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The distribution of τ_x plays an important role in various contexts: local times and excursion theory ([2, 7, 18, 29]), potential theory ([3]), penalisation problems ([19, 28, 30, 31]). The estimates of τ_x may also prove useful in the study of one-dimensional unimodal Lévy processes, developed recently in [5, 6, 9]. More precisely, description of τ_x is the limiting case of a more general problem of finding the time and place the process X first hits a (small) ball, see [6] and a recent preprint [10].

Surprisingly little is known about the properties of τ_x for general Lévy processes. By [23, Theorem 43.3 and Remark 43.6], if $1/|\Psi|$ is integrable at infinity, then

$$\int_{\mathbb{R}} e^{i\xi x} \mathbb{E} e^{-\lambda \tau_x} dx = \frac{c_\lambda}{\lambda + \Psi(\xi)} , \quad \text{with} \quad c_\lambda = \left(\int_{\mathbb{R}} \frac{1}{\lambda + \Psi(\xi)} \, d\xi \right)^{-1} . \tag{1.1}$$

The inversion of the Laplace and Fourier transforms in (1.1) is often problematic. An application of the inverse Fourier transform to both sides of (1.1) leads to an expression for $\mathbb{E} e^{-\lambda \tau_x}$ in terms of an oscillatory integral. In fact,

$$u_\lambda(x) = c_\lambda^{-1} \mathbb{E} e^{-\lambda \tau_x} \tag{1.2}$$

is a well-studied object, the λ-potential density of X. Nevertheless, a closed-form expression for u_λ is known only in some special cases, e.g. when X is stable and $\lambda = 0$, or when X is relativistic with $\beta = 2$ (with the notation of Example 1.4 below) and $\lambda = 1$. Therefore, in order to invert the Laplace transform in (1.2), one needs additional regularity of Ψ. This is the rough idea of the proof of the main result of [13], which is recalled as Theorem 1.9 below, and which is the starting point for our development.

There are essentially two classes of Lévy processes for which the description of τ_x simplifies dramatically and has been studied. When X is an α-stable process, τ_x is equal in distribution to $x^\alpha \tau_1$ (*scaling*), so the originally two-dimensional problem becomes one-dimensional. Numerous results are available in this case. In particular, a complete series expansion of the distribution function of τ_x is known (see [20] for processes with one-sided jumps, [4, 7, 21, 30] for the symmetric case, and [11] for the general result). Other closely related results for the stable case (unimodality, distributional identities, applications) can be found in [16, 26, 31].

The distribution of τ_x for $x > 0$ is rather well-studied also for Lévy processes with negative jumps only (also known as spectrally negative processes). Then τ_x is equal to the first passage time through the level x, $\tau_x = \inf\{t \ge 0 : X_t \ge x\}$, and fluctuation theory for Lévy processes can be used to study the properties of τ_x. We refer to [23, Chapter 9] for more information.

For non-stable Lévy processes with two-sided jumps, we are aware of no estimates or asymptotic formulae similar to the main results of this article.

Throughout the article, X is assumed to be symmetric, that is, $b = 0$ and $\nu(E) = \nu(-E)$ for all Borel $E \subseteq \mathbb{R}$. In this case Ψ is a real function with non-negative values. We impose two additional restrictions: we require X to have completely monotone jumps and satisfy a certain scaling-type condition. These notions are briefly discussed below.

Recall that a function $f : (0, \infty) \to \mathbb{R}$ is said to be *completely monotone* if it is infinitely differentiable and $(-1)^n f^{(n)}(\xi) \ge 0$ for all $\xi > 0$ and $n = 0, 1, 2, \ldots$ By Bernstein's theorem, this is equivalent to f being the Laplace transform of a non-negative Radon measure on $[0, \infty)$. Similarly, we say that a process X has *completely monotone jumps* if its Lévy measure ν is absolutely continuous with respect to the Lebesgue measure, and its density is a completely monotone function on $(0, \infty)$. Note that due to symmetry, the density of ν on $(-\infty, 0)$ is *absolutely monotone*: its derivatives of all orders are non-negative.

Lévy processes with completely monotone jumps (without the symmetry condition) were introduced by Rogers in [22], see also [14]. In the symmetric case, an equivalent condition can be given in terms of Ψ. Recall that ψ is a complete Bernstein function if and only if

$$\psi(\xi) = c_1 + c_2\xi + \int_{(0,\infty)} \frac{\xi}{s+\xi} \frac{\mu(ds)}{s}$$

for $\xi \geq 0$, where $c_1, c_2 \geq 0$ and μ is a non-negative measure for which the above integral converges (see [24]). A symmetric Lévy process X has completely monotone jumps if and only if $\Psi(\xi) = \psi(\xi^2)$ for a complete Bernstein function ψ (see [12, 22]). The most prominent examples of symmetric processes with completely monotone jumps are stable processes, with $\Psi(\xi) = c|\xi|^\alpha$ for some $c > 0$ and $\alpha \in (0, 2]$. This class includes also mixtures of stable processes and relativistic Lévy processes (discussed later in this section), as well as variance gamma process and geometric stable processes (which with probability one do not hit single points and thus are not considered here; see [25] for definitions and properties of these processes).

The aforementioned *scaling-type condition* of order α requires that

$$\frac{\xi\Psi''(\xi)}{\Psi'(\xi)} \geq \alpha - 1 \tag{1.3}$$

for all $\xi > 0$. Here α is an arbitrary real number, although in our main theorems we assume that $\alpha \in (1, 2]$. The scaling-type condition plays a crucial role in our development. By integration, (1.3) implies that (and in fact, it is equivalent to)

$$\frac{\Psi'(\xi_2)}{\Psi'(\xi_1)} \geq \left(\frac{\xi_2}{\xi_1}\right)^{\alpha-1}$$

for all $\xi_2 > \xi_1 > 0$. In Lemma 2.2 we will see that (1.3) also gives (but it is essentially stronger than)

$$\frac{\Psi(\xi_2)}{\Psi(\xi_1)} \geq \left(\frac{\xi_2}{\xi_1}\right)^{\alpha} \tag{1.4}$$

for all $\xi_2 > \xi_1 > 0$. This explains why we call (1.3) a scaling-type condition.

We note that the scaling-type condition of order $\alpha > 1$ implies that $\mathbb{P}(\tau_x < \infty) = 1$ for all $x \in \mathbb{R}$. Indeed, by (1.4), $1/|\Psi|$ is not integrable near 0, so X is recurrent by Chung–Fuchs criterion ([23, Theorem 37.5]). Furthermore, again by (1.4), $1/|\Psi|$ is integrable at infinity, so $\mathbb{P}(\tau_x < \infty) > 0$ by [23, Remark 43.6]. Now $\mathbb{P}(\tau_x < \infty) = 1$ follows by [23, Remark 43.12].

The scaling-type condition (1.3) with $\alpha \in (1, 2]$ is satisfied by the typical examples of symmetric Lévy processes with completely monotone jumps which hit single points with probability 1: stable, mixed stable (see Example 1.5) and relativistic (see Example 1.4). An equivalent form of (1.3), as well as a sufficient condition in terms of the Lévy measure, are given in Remark 1.8. Nevertheless, (1.3) is rather restrictive, see Example 1.7. We conjecture that the estimates of $\mathbb{P}(\tau_x > t)$ hold in greater generality, for example, with (1.3) replaced by $\Psi(\xi_2)/\Psi(\xi_1) \geq C(\xi_2/\xi_1)^\alpha$ for some $C > 0$ and $\alpha > 1$ (a more general version of (1.4), see [5, 6, 9]). However, with the present methods, we were unable to significantly relax the assumption that (1.3) holds with $\alpha > 1$.

For symmetric processes with completely monotone jumps, Ψ is an increasing function on $(0, \infty)$. Let Ψ^{-1} denote the inverse function of the restriction of Ψ to $(0, \infty)$. Our first main result provides large t and small x estimates of $\mathbb{P}(\tau_x > t)$ and its time derivatives. A corollary that follows extends the estimate of $\mathbb{P}(\tau_x > t)$ (with no time derivative) to the full range of $t > 0$ and $x \in \mathbb{R} \setminus \{0\}$. The constants in these estimates are given explicitly, see Remark 5.5.

Below we state the main results of the paper.

Theorem 1.1. *Suppose that X is a symmetric Lévy process with completely monotone jumps, which satisfies the scaling-type condition* (1.3) *for some $\alpha \in (1,2]$. Then there are positive constants $C_1(\alpha,n)$, $C_2(\alpha,n)$, $C_3(\alpha,n)$ such that*

$$\frac{C_1(\alpha,n)}{t^{n+1}|x|\Psi(1/|x|)\Psi^{-1}(1/t)} \le (-\tfrac{d}{dt})^n \mathbb{P}(\tau_x > t) \le \frac{C_2(\alpha,n)}{t^{n+1}|x|\Psi(1/|x|)\Psi^{-1}(1/t)} \tag{1.5}$$

for all $n \ge 0$, $t > 0$ and $x \in \mathbb{R} \setminus \{0\}$ such that $t\Psi(1/|x|) \ge C_3(\alpha,n)$.

Corollary 1.2. *For $n = 0$, the conclusion of Theorem 1.1 can be rewritten as follows: there are positive constants $\tilde{C}_1(\alpha)$ and $\tilde{C}_2(\alpha)$ such that*

$$\frac{\tilde{C}_1(\alpha)}{1 + t|x|\Psi(1/|x|)\Psi^{-1}(1/t)} \le \mathbb{P}(\tau_x > t) \le \frac{\tilde{C}_2(\alpha)}{1 + t|x|\Psi(1/|x|)\Psi^{-1}(1/t)} \tag{1.6}$$

for all $t > 0$ and $x \in \mathbb{R} \setminus \{0\}$.

Under an additional regularity condition, the above two-sided estimates can be turned into asymptotic formulae for $\mathbb{P}(\tau_x > t)$ as $t \to \infty$ or $x \to 0$. Recall that a function $\psi : (0,\infty) \to \mathbb{R}$ is *regularly varying at infinity* with index α if $\lim_{\xi\to\infty} \psi(k\xi)/\psi(\xi) = k^\alpha$ for all $k > 0$. If the same equation holds with the limit as $\xi \to 0^+$ instead of $\xi \to \infty$, ψ is said to be *regularly varying at zero* with index α. Observe that if Ψ satisfies the scaling-type condition (1.3) and it is regularly varying with index γ at infinity or at zero, then, by (1.4), we have $\gamma \ge \alpha$.

Theorem 1.3. *Suppose that X is a symmetric Lévy process with completely monotone jumps, which satisfies the scaling-type condition* (1.3) *for some $\alpha \in (1,2]$.*

(a) *If Ψ is regularly varying at infinity with index $\gamma \in (1,2]$, then the limit*

$$\lim_{x\to 0} \Bigl(|x|\Psi(\tfrac{1}{|x|})(-\tfrac{d}{dt})^n \mathbb{P}(\tau_x > t) \Bigr)$$

exists and belongs to $(0,\infty)$ for all $n \ge 0$ and $t > 0$.

(b) *If Ψ is regularly varying at zero with index $\delta \in (1,2]$, then the limit*

$$\lim_{t\to\infty} \bigl(t^{n+1}\Psi^{-1}(\tfrac{1}{t})(-\tfrac{d}{dt})^n \mathbb{P}(\tau_x > t) \bigr)$$

exists and belongs to $(0,\infty)$ for all $n \ge 0$ and $x \in \mathbb{R} \setminus \{0\}$.

The limits in the above theorem are given explicitly by rather complicated expressions, see Remark 5.6. In the following examples, application of Theorems 1.1 and 1.3 to three types of symmetric Lévy processes with completely monotone jumps is given. Technical details, such as verification of (1.3), are left to the reader.

Note that our main results for symmetric stable processes follow immediately from the full series expansion given in [11]: Theorem 1.3 gives the first term, and two-sided estimates of Theorem 1.1 follow easily by a scaling argument. On the other hand, Theorems 1.1 and 1.3 seem to be completely new for non-stable processes.

Example 1.4. Suppose that $1 < \alpha < \beta \le 2$ and let X be the Lévy process with Lévy–Khintchine exponent $\Psi(\xi) = (1 + |\xi|^\beta)^{\alpha/\beta} - 1$ (sometimes X is called the *relativistic Lévy process*). Then

$$\frac{c_1(\alpha,n)|x|^{\alpha-1}(1+|x|)^{\beta-\alpha}}{t^{n+1-1/\alpha}(1+t)^{1/\alpha-1/\beta}} \le \left(-\frac{d}{dt}\right)^n \mathbb{P}(\tau_x > t) \le \frac{c_2(\alpha,n)|x|^{\alpha-1}(1+|x|)^{\beta-\alpha}}{t^{n+1-1/\alpha}(1+t)^{1/\alpha-1/\beta}}$$

for all $n \ge 0$, $t > 0$ and $x \in \mathbb{R} \setminus \{0\}$ such that $t/\min(|x|^\alpha, |x|^\beta) \ge c_3(\alpha,n)$. Furthermore, finite and positive limits

$$\lim_{x\to 0} \bigl(|x|^{1-\alpha}(-\tfrac{d}{dt})^n \mathbb{P}(\tau_x > t) \bigr), \qquad \lim_{t\to\infty} \Bigl(t^{n+1-1/\beta}(-\tfrac{d}{dt})^n \mathbb{P}(\tau_x > t) \Bigr)$$

exist for all $n \geq 0$, $t > 0$ and $x \in \mathbb{R} \setminus \{0\}$. Note that the restriction $\alpha > 1$ is required by the scaling-type condition (1.3). Otherwise, if $\alpha \leq 1$, we have that $\mathbb{P}(\tau_x < \infty) = 0$.

Example 1.5. Suppose that $1 < \alpha < \beta \leq 2$, and let X be the Lévy process with Lévy–Khintchine exponent $\Psi(\xi) = |\xi|^\alpha + |\xi|^\beta$ (that is, X is the sum of independent stable Lévy processes). Then

$$\frac{c_1(\alpha,n)|x|^{\beta-1}(1+t)^{1/\alpha-1/\beta}}{t^{n+1-1/\beta}(1+|x|)^{\beta-\alpha}} \leq \left(-\frac{d}{dt}\right)^n \mathbb{P}(\tau_x > t) \leq \frac{c_2(\alpha,n)|x|^{\beta-1}(1+t)^{1/\alpha-1/\beta}}{t^{n+1-1/\beta}(1+|x|)^{\beta-\alpha}}$$

for all $n \geq 0$, $t > 0$ and $x \in \mathbb{R} \setminus \{0\}$ such that $t/\max(|x|^\alpha, |x|^\beta) \geq c_3(\alpha, n)$. Furthermore, finite and positive limits

$$\lim_{x\to 0} \left(|x|^{1-\beta}(-\tfrac{d}{dt})^n \mathbb{P}(\tau_x > t)\right), \qquad \lim_{t\to\infty} \left(t^{n+1-1/\alpha}(-\tfrac{d}{dt})^n \mathbb{P}(\tau_x > t)\right)$$

exist for all $n \geq 0$, $t > 0$ and $x \in \mathbb{R} \setminus \{0\}$. As in the previous example, the restriction $\alpha > 1$ is required by the scaling-type condition (1.3). If $\alpha \leq 1 < \beta$, then $0 < \mathbb{P}(\tau_x < \infty) < 1$ and the estimates of $\mathbb{P}(\tau_x < t)$ are unknown. When $\beta \leq 1$, then $\mathbb{P}(\tau_x < \infty) = 0$.

Example 1.6. Let X be the pure-jump Lévy process with Lévy–Khintchine exponent $\Psi(\xi) = \xi^2(\log(1+\xi^2))^{-1} - 1$ (see [17]). Since Ψ is regularly varying with index 2 both at 0 and at infinity, it can be checked that both large-time and small-time scaling limits:

$$\begin{aligned} &(k^{-1/2} X_{kt} : t \geq 0) && \text{as } k \to \infty, \\ &((2k)^{-1/2} X_{k\log(1/k)t} : t \geq 0) && \text{as } k \to 0^+, \end{aligned}$$

are standard Wiener processes (cf. [8]). Let $\varphi(t) = 1$ for $t \geq \frac{1}{e}$ and $\varphi(t) = (te^{-W_{-1}(-t)})^{1/2}$ when $0 < t < \frac{1}{e}$ (where W_{-1} is the lower branch of the Lambert W function). We have

$$\frac{c_1(n)|x|\log(2+\frac{1}{|x|})}{t^{n+1/2}\varphi(t)} \leq \left(-\frac{d}{dt}\right)^n \mathbb{P}(\tau_x > t) \leq \frac{c_2(n)|x|\log(2+\frac{1}{|x|})}{t^{n+1/2}\varphi(t)}$$

for all $n \geq 0$, $t > 0$ and $x \in \mathbb{R} \setminus \{0\}$ such that $t/(|x|^2 \log(2 + \frac{1}{|x|})) \geq c_3(n)$. Furthermore, finite and positive limits

$$\lim_{x\to 0} \frac{(-\frac{d}{dt})^n \mathbb{P}(\tau_x > t)}{|x|\log(2+\frac{1}{|x|})}, \qquad \lim_{t\to\infty} \left(t^{n+1/2}(-\tfrac{d}{dt})^n \mathbb{P}(\tau_x > t)\right)$$

exist for all $n \geq 0$, $t > 0$ and $x \in \mathbb{R} \setminus \{0\}$.

Example 1.7. Let X be the sum of a standard Wiener process and a compound Poisson process with Lévy measure $ce^{-|x|}dx$. Then X is symmetric, has completely monotone jumps and $\Psi(\xi) = \frac{1}{2}\xi^2 + c\xi^2/(1+\xi^2)$. By a direct calculation,

$$\frac{\xi\Psi''(\xi)}{\Psi'(\xi)} = 1 - \frac{8c\xi^2}{((1+\xi^2)^2+2c)(1+\xi^2)}\,.$$

The right-hand side decreases with $c \geq 0$, and for $c = 2$ we have

$$\inf\left\{\frac{\xi\Psi''(\xi)}{\Psi'(\xi)} : \xi \in (0,\infty)\right\} = \frac{\Psi''(1)}{\Psi'(1)} = 0.$$

It follows that X satisfies the scaling-type condition (1.3) with $\alpha \in (1,2]$ if and only if $c \in [0,2)$. We remark that the restriction $c < 2$ is apparently the limitation of our method, there is no reason to believe that for $c \geq 2$ the conclusions of Theorems 1.1 and 1.3 no longer hold.

Remark 1.8. The scaling-type condition (1.3) with $\alpha \in (1, 2]$ is easily shown to be equivalent to concavity of $\Psi(\xi^{1-\varepsilon})$ for some $\varepsilon \in (0, \frac{1}{2}]$ (with $\alpha - 1 = \frac{\varepsilon}{1-\varepsilon}$). A sufficient condition for (1.3) with $\alpha \in (1, 2]$ in terms of the Lévy measure of X is described below.

Let X be a symmetric Lévy process with completely monotone jumps, and denote the density function of the Lévy measure ν of X by the same symbol ν. Then

$$\Psi(\xi) = a\xi^2 + 2 \int_0^\infty (1 - \cos(\xi z))\nu(z)dz = a\xi^2 + 2 \int_0^\infty (1 - \cos s)\tfrac{1}{\xi}\nu(\tfrac{s}{\xi})ds.$$

Assuming that $\frac{d}{d\xi}(\frac{1}{\xi}\nu(\frac{s}{\xi})) \geq 0$ and $\frac{d^2}{d\xi^2}(\frac{1}{\xi}\nu(\frac{s}{\xi})) \geq 0$, differentiation in ξ under the integral sign is permitted. It follows that

$$\begin{aligned}\xi^2\Psi''(\xi) - (\alpha - 1)\xi\Psi'(\xi) &= 2a(2 - \alpha)\xi^2 \\ &\quad + 2 \int_0^\infty (1 - \cos s)\tfrac{1}{\xi}((\tfrac{s}{\xi})^2\nu''(\tfrac{s}{\xi}) + (3 + \alpha)\tfrac{s}{\xi}\nu'(\tfrac{s}{\xi}) + (1 + \alpha)\nu(\tfrac{s}{\xi}))ds.\end{aligned}$$

The right-hand side is non-negative if $z^2\nu''(z) + (3 + \alpha)z\nu'(z) + (1 + \alpha)\nu(z) \geq 0$ for all $z > 0$, which is equivalent to $\frac{d^2}{dz^2}(z^{-1/\alpha}\nu(z^{-1/\alpha})) \geq 0$. This condition alone implies that $\frac{d^2}{d\xi^2}(\frac{1}{\xi}\nu(\frac{s}{\xi})) \geq 0$, and if $z^{-1/\alpha}\nu(z^{-1/\alpha})$ is increasing, then also $\frac{d}{d\xi}(\frac{1}{\xi}\nu(\frac{s}{\xi})) \geq 0$.

The above argument shows that if $\alpha \in (1, 2]$ and $z^{-1/\alpha}\nu(z^{-1/\alpha})$ is convex and nondecreasing in $z > 0$, then (1.3) holds.

Since the proofs of main theorems are rather technical, below we outline the main idea and briefly discuss the structure of the article. Our starting point is the following generalised eigenfunction expansion, proved in [13]. Note that in the original statement the condition $\xi\Psi''(\xi) \leq \Psi'(\xi)$ was erroneously given as $2\xi\Psi''(\xi) \leq \Psi'(\xi)$ (the proof, however, used the correct condition). In the statement, as well as in the remaining part of the article, by $\mathcal{F}f(\xi) = \int_{-\infty}^\infty f(s)e^{-is\xi}ds$ we denote the Fourier transform of an integrable function f. Occasionally, the distributional Fourier transform is used: if f is a Schwartz distribution, then $\mathcal{F}f$ is again a Schwartz distribution, defined by $\langle \mathcal{F}f, \varphi\rangle = \langle f, \mathcal{F}\varphi\rangle$ for all φ in the Schwartz class.

Theorem 1.9 ([13, Theorem 1.1 and Remark 1.2])**.** *Suppose that X is a symmetric Lévy process. If $1/\Psi$ is integrable at infinity and*

$$\Psi'(\xi) > 0, \quad \frac{\xi\Psi''(\xi)}{\Psi'(\xi)} \leq 1 \tag{1.7}$$

for all $\xi > 0$ (cf. (1.3)), then

$$(-\tfrac{d}{dt})^n\mathbb{P}(t < \tau_x < \infty) = \frac{1}{\pi} \int_0^\infty \cos\vartheta_\lambda e^{-t\Psi(\lambda)}\Psi'(\lambda)(\Psi(\lambda))^{n-1}F_\lambda(x)d\lambda \tag{1.8}$$

for all $n \geq 0$ and $t > 0$, and almost all $x \in \mathbb{R}$. Here F_λ is a bounded, continuous function, defined by

$$F_\lambda(x) = \sin(\lambda|x| + \vartheta_\lambda) - G_\lambda(x)$$

for all $x \in \mathbb{R}$, where

$$\vartheta_\lambda = \arctan\left(\frac{1}{\pi} \int_0^\infty \left(\frac{\Psi'(\lambda)}{\Psi(\xi) - \Psi(\lambda)} - \frac{2\lambda}{\xi^2 - \lambda^2}\right) d\xi\right) \tag{1.9}$$

and G_λ is an $L^2(\mathbb{R}) \cap C_0(\mathbb{R})$ function with (integrable) Fourier transform

$$\mathcal{F}G_\lambda(\xi) = \cos\vartheta_\lambda \left(\frac{\Psi'(\lambda)}{\Psi(\xi) - \Psi(\lambda)} - \frac{2\lambda}{\xi^2 - \lambda^2}\right)$$

for all $\xi \in \mathbb{R} \setminus \{-\lambda, \lambda\}$*. The distributional Fourier transform of* F_λ *is given by*

$$\langle \mathcal{F} F_\lambda, \varphi \rangle = \cos \vartheta_\lambda \, \mathrm{pv}\!\int_{-\infty}^{\infty} \frac{\Psi'(\lambda)\varphi(\xi)}{\Psi(\lambda) - \Psi(\xi)} \, d\xi + \pi \sin \vartheta_\lambda (\varphi(\lambda) + \varphi(-\lambda))$$

for φ *in the Schwartz class (here* $\mathrm{pv}\!\int$ *stands for the Cauchy principal value integral).*

As it is explained right after formula (1.11) below, symmetric Lévy processes with completely monotone jumps automatically satisfy (1.7), so Theorem 1.9 can be applied whenever $1/\Psi$ is integrable at infinity. The latter condition holds, for example, if the scaling-type condition (1.3) is satisfied with $\alpha \in (1, 2]$.

The main idea of the proof of Theorems 1.1 and 1.3 is taken from [15], where a similar problem for first passage times was studied. The *generalised eigenfunctions* $F_\lambda(x)$ are oscillatory due to the $\sin(\lambda|x| + \vartheta_\lambda)$ term, but $F_\lambda(x) > 0$ when $\lambda|x|$ is small enough, and two-sided estimates for $F_\lambda(x)$ can be given in this case. Thanks to the exponential term $e^{-t\Psi(\lambda)}$ in (1.8), the main contribution to the integral comes from $\lambda \in (0, \frac{c}{|x|})$, provided that t is large enough, or $|x|$ is small enough. This essentially gives Theorem 1.1. The proof of Theorem 1.3 requires in addition an asymptotic expression for $F_\lambda(x)$ as $x \to 0$ or $\lambda \to 0$.

We collect some simple technical results in Section 2, so that they do not distract attention of the reader at a later point. In Section 3 the properties of ϑ_λ are studied. In Lemma 3.1 it is proved that the scaling-type condition (1.3) implies $\vartheta_\lambda \le \frac{\pi}{\alpha} - \frac{\pi}{2}$ for all $\lambda > 0$. The asymptotic behaviour of ϑ_λ as $\lambda \to 0^+$ or $\lambda \to \infty$ is given in Lemma 3.2.

The estimates and asymptotic properties of F_λ are given in Section 4. Lemma 4.3 contains a rather general estimate, which is then simplified in Lemma 4.4 for processes satisfying the scaling-type condition (1.3). Asymptotic expansions of F_λ are given in Lemmas 4.5 and 4.6.

The final Section 5 contains proofs of main theorems, preceded by two propositions of more general nature and two technical lemmas. Proposition 5.2 extends (1.8) to all $x \in \mathbb{R} \setminus \{0\}$. Lemmas 5.4 and 5.3 contain estimates of the main part ($\lambda < \frac{c}{|x|}$) and the remainder part ($\lambda > \frac{c}{|x|}$) of the integral in (1.8).

Instead of using the Lévy–Khintchine exponent Ψ, it is convenient to work with $\psi(\xi) = \Psi(\sqrt{\xi})$. Recall that when X has completely monotone jumps, then ψ is a complete Bernstein function. In the remaining part of the article Ψ is virtually dropped from the notation. For reader's convenience, we note that

$$\Psi(\xi) = \psi(\xi^2), \quad \frac{\xi \Psi'(\xi)}{\Psi(\xi)} = 2\,\frac{\xi^2 \psi'(\xi^2)}{\psi(\xi^2)}\,, \quad \frac{\xi \Psi''(\xi)}{\Psi'(\xi)} = 1 + 2\,\frac{\xi^2 \psi''(\xi^2)}{\psi'(\xi^2)}\,, \tag{1.10}$$

so that the scaling-type condition (1.3) translates to

$$\frac{-\xi \psi''(\xi)}{\psi'(\xi)} \le \frac{2 - \alpha}{2}\,.$$

To facilitate extensions, all intermediate results are stated for rather general functions ψ. For this reason, statements of the results often contain assumptions, such as differentiability or monotonicity of ψ, which are automatically satisfied when ψ corresponds to a symmetric Lévy process with completely monotone jumps (that is, ψ is a complete Bernstein function). In particular, in this more general setting, a two-sided scaling-type condition

$$\frac{2 - \beta}{2} \le \frac{-\xi \psi''(\xi)}{\psi'(\xi)} \le \frac{2 - \alpha}{2} \tag{1.11}$$

is often imposed. When ψ is a complete Bernstein function, the lower bound in (1.11) always holds with $\beta = 2$ (see [12, Proposition 2.21]).

It should be pointed out that although we follow closely the approach of [15], there are essential differences between the present problem and the one considered therein. The overall form of the generalised eigenfunctions is similar (sine term plus completely monotone correction G_λ), but the expressions for ϑ_λ and G_λ are different, and thus require different methods. For example, the estimates of G_λ in [15] follow easily from the expression for the Laplace transform of G_λ. We were unable to follow the same approach and needed to use Fourier transform instead. Also the technical details of the arguments are different, so virtually no part of [15] can be re-used in our setting.

2 Preliminaries

Throughout the article, by c, c_1, c_2, etc. we denote positive constants. Dependence on a parameter α is always indicated by writing $c(\alpha)$, etc.

Following [15], for $\lambda > 0$ and a continuous function $\psi : (0,\infty) \to (0,\infty)$ such that $\psi(\xi) \neq \psi(\lambda^2)$ when $\xi \neq \lambda^2$, we define

$$\psi_\lambda(\xi) = \frac{1 - \frac{\xi}{\lambda^2}}{1 - \frac{\psi(\xi)}{\psi(\lambda^2)}}$$

for $\xi > 0$, $\xi \neq \lambda^2$. This definition is extended continuously at $\xi = \lambda^2$ by $\psi_\lambda(\lambda^2) = \psi(\lambda^2)/(\lambda^2\psi'(\lambda^2))$ whenever ψ is differentiable at λ^2 and $\psi'(\lambda^2) > 0$. In this case we say that ψ_λ is well-defined.

If for some $\lambda > 0$ the function ψ_λ is well-defined and $\psi_\lambda(\xi) \neq \psi_\lambda(\lambda^2)$ for $\xi \neq \lambda^2$, then $(\psi_\lambda)_\lambda$ can be defined, and

$$\frac{1}{(\psi_\lambda)_\lambda(\xi^2)} = \frac{\lambda^2\psi'(\lambda^2)}{\psi(\xi^2) - \psi(\lambda^2)} - \frac{\lambda^2}{\xi^2 - \lambda^2} \tag{2.1}$$

for $\xi > 0$, $\xi \neq \lambda^2$. Note that if $\psi : (0,\infty) \to (0,\infty)$ is twice differentiable and $\psi'(\xi) > 0$, $\psi''(\xi) < 0$ for all $\xi > 0$, then ψ_λ is strictly increasing for every $\lambda > 0$, and hence $(\psi_\lambda)_\lambda$ is well-defined and positive. Furthermore, if ψ is a complete Bernstein function (equivalently, if $\Psi(\xi) = \psi(\xi^2)$ is the Lévy–Khintchine exponent of a symmetric Lévy process with completely monotone jumps), then also ψ_λ and $(\psi_\lambda)_\lambda$ are complete Bernstein functions (see [13, 24]).

Below we list some rather elementary results used in the proofs of main results.

Lemma 2.1. *If $\psi, \tilde\psi : (0,\infty) \to (0,\infty)$ are twice differentiable, $\psi'(\xi), \tilde\psi'(\xi) > 0$ and $\psi''(\xi), \tilde\psi''(\xi) \le 0$ for all $\xi > 0$, and furthermore*

$$\frac{-\psi''(\xi)}{\psi'(\xi)} \le \frac{-\tilde\psi''(\xi)}{\tilde\psi'(\xi)} \tag{2.2}$$

for all $\xi > 0$, then

$$(\psi_\lambda)_\lambda(\xi^2) \ge (\tilde\psi_\lambda)_\lambda(\xi^2) \tag{2.3}$$

for all $\lambda, \xi > 0$.

Proof. Integration of (2.2) in ξ gives

$$\frac{\psi'(\zeta)}{\psi'(\xi_1)} \ge \frac{\tilde\psi'(\zeta)}{\tilde\psi'(\xi_1)}$$

when $0 < \xi_1 < \zeta$. By another integration in ζ,

$$\frac{\psi(\xi_2) - \psi(\xi_1)}{\psi'(\xi_1)} \ge \frac{\tilde\psi(\xi_2) - \tilde\psi(\xi_1)}{\tilde\psi'(\xi_1)}$$

when $0 < \xi_1 < \xi_2$. Substituting $\xi_1 = \lambda^2$ and $\xi_2 = \xi^2$, one gets

$$\frac{\lambda^2 \psi'(\lambda^2)}{\psi(\xi^2) - \psi(\lambda^2)} - \frac{\lambda^2}{\xi^2 - \lambda^2} \le \frac{\lambda^2 \tilde{\psi}'(\lambda^2)}{\tilde{\psi}(\xi^2) - \tilde{\psi}(\lambda^2)} - \frac{\lambda^2}{\xi^2 - \lambda^2} \,,$$

that is, (2.3), provided that $0 < \lambda < \xi$. A similar argument can be given when $0 < \xi < \lambda$. The case $\lambda = \xi > 0$ follows by continuity. □

Lemma 2.2. *If $\psi : (0, \infty) \to (0, \infty)$ is twice differentiable, $\psi'(\xi) > 0$ for all $\xi > 0$, and the scaling-type condition* (1.11) *holds for some $\alpha, \beta > 0$ and all $\xi > 0$, then*

$$\frac{\alpha}{2} \le \frac{\xi \psi'(\xi)}{\psi(\xi) - \psi(0^+)} \le \frac{\beta}{2} \tag{2.4}$$

for all $\xi > 0$, and

$$\left(\frac{\xi_1}{\xi_2}\right)^{1-\frac{\alpha}{2}} \le \frac{\psi'(\xi_2)}{\psi'(\xi_1)} \le \left(\frac{\xi_1}{\xi_2}\right)^{1-\frac{\beta}{2}}, \qquad \left(\frac{\xi_2}{\xi_1}\right)^{\frac{\alpha}{2}} \le \frac{\psi(\xi_2) - \psi(0^+)}{\psi(\xi_1) - \psi(0^+)} \le \left(\frac{\xi_2}{\xi_1}\right)^{\frac{\beta}{2}} \tag{2.5}$$

whenever $0 < \xi_1 < \xi_2$.

Proof. By (1.11), if $0 < \xi_1 < \xi_2$,

$$\log \left(\frac{\xi_2}{\xi_1}\right)^{1-\frac{\alpha}{2}} = \int_{\xi_1}^{\xi_2} \frac{1 - \frac{\alpha}{2}}{\zeta} \, d\zeta \ge \int_{\xi_1}^{\xi_2} \frac{-\psi''(\zeta)}{\psi'(\zeta)} \, d\zeta = \log \frac{\psi'(\xi_1)}{\psi'(\xi_2)} \,,$$

proving the lower bound in the first part of (2.5). Hence,

$$\frac{\xi_2}{\frac{\alpha}{2}} = \int_0^{\xi_2} \left(\frac{\xi_2}{\xi_1}\right)^{1-\frac{\alpha}{2}} d\xi_1 \ge \int_0^{\xi_2} \frac{\psi'(\xi_1)}{\psi'(\xi_2)} \, d\xi_1 = \frac{\psi(\xi_2) - \psi(0^+)}{\psi'(\xi_2)} \,,$$

which shows the lower bound in (2.4). Furthermore,

$$\log \left(\frac{\xi_2}{\xi_1}\right)^{\frac{\alpha}{2}} = \int_{\xi_1}^{\xi_2} \frac{\frac{\alpha}{2}}{\zeta} d\zeta \le \int_{\xi_1}^{\xi_2} \frac{\psi'(\zeta)}{\psi(\zeta) - \psi(0^+)} \, d\zeta = \log \frac{\psi(\xi_2) - \psi(0^+)}{\psi(\xi_1) - \psi(0^+)} \,,$$

proving the other lower bound in (2.5). The upper bounds are proved in the same way. □

When $\psi(\xi^2)$ is the Lévy–Khintchine exponent of a Lévy process, then $\psi(0^+) = 0$. Hence, the latter part of (2.5) takes the simpler form

$$\left(\frac{\xi_2}{\xi_1}\right)^{\frac{\alpha}{2}} \le \frac{\psi(\xi_2)}{\psi(\xi_1)} \le \left(\frac{\xi_2}{\xi_1}\right)^{\frac{\beta}{2}} .$$

Note that in this case

$$\left(\frac{t_2}{t_1}\right)^{\frac{2}{\beta}} \le \frac{\psi^{-1}(t_2)}{\psi^{-1}(t_1)} \le \left(\frac{t_2}{t_1}\right)^{\frac{2}{\alpha}} \tag{2.6}$$

for all $t_1, t_2 > 0$ such that $t_1 < t_2$.

Lemma 2.3. *If $g : (0, \infty) \to (0, \infty)$ is integrable and decreasing, then*

$$\lim_{\xi \to \infty} (\xi g(\xi)) = 0.$$

Proof. As an integrable and decreasing function, $g(\xi)$ converges to 0 as $\xi \to \infty$. Since $g(\xi) \mathbb{1}_{(0,\xi)}(\zeta) \le g(\zeta)$ for all $\xi, \zeta > 0$, by the Dominated Convergence Theorem,

$$\lim_{\xi \to \infty} (\xi g(\xi)) = \lim_{\xi \to \infty} \int_0^\infty g(\xi) \mathbb{1}_{(0,\xi)}(\zeta) d\zeta = 0.$$

□

Lemma 2.4. *If* $g : \mathbb{R} \to (0, \infty)$ *is integrable and decreasing on* $(0, \infty)$*, and* $g(\xi) = g(-\xi)$ *for* $\xi > 0$*, then*

$$\frac{1}{2} \int_0^\infty \min(\xi^2 x^2, 4) g(\xi) d\xi \le \mathcal{F}g(0) - \mathcal{F}g(x) \le \int_0^\infty \min(\xi^2 x^2, 4) g(\xi) d\xi \tag{2.7}$$

for all $x \in \mathbb{R}$*. Furthermore,*

$$|\mathcal{F}g(x_1) - \mathcal{F}g(x_2)| \le \int_0^\infty \min(\xi |x_1 - x_2|, 2) \min(\xi |x_1 + x_2|, 2) g(\xi) d\xi \tag{2.8}$$

for all $x_1, x_2 \in \mathbb{R}$*.*

Proof. Fix $x > 0$. By symmetry of g,

$$\mathcal{F}g(0) - \mathcal{F}g(x) = 2 \int_0^\infty (1 - \cos(\xi x)) g(\xi) d\xi.$$

Clearly, $1 - \cos(\xi x) \le 2$ and $1 - \cos(\xi x) = 2 \sin(\frac{\xi x}{2})^2 \le \frac{1}{2} \xi^2 x^2$. Therefore,

$$\mathcal{F}g(0) - \mathcal{F}g(x) \le \int_0^{\frac{2}{x}} \xi^2 x^2 g(\xi) d\xi + \int_{\frac{2}{x}}^\infty 4 g(\xi) d\xi.$$

For the lower bound, integration by parts gives

$$\mathcal{F}g(0) - \mathcal{F}g(x) = 2 \lim_{\xi \to \infty} \left((\xi - \tfrac{1}{x} \sin(\xi x)) g(\xi) \right) + 2 \int_0^\infty (\xi - \tfrac{1}{x} \sin(\xi x))(-dg(\xi)),$$

where the integral in the right-hand side is a Lebesgue–Stieltjes one (if g is differentiable, then $(-dg(\xi)) = (-g'(\xi)) d\xi$). By Lemma 2.3, the limit in the right-hand side is 0. Furthermore, $(-dg(\xi))$ is a non-negative measure on $(0, \infty)$, and one easily verifies that $\xi - \frac{1}{x} \sin(\xi x) \ge \frac{1}{8} \xi^3 x^2$ for $\xi \in (0, \frac{2}{x})$ and $\xi - \frac{1}{x} \sin(\xi x) \ge \xi - \frac{1}{x}$ for $\xi \in (\frac{2}{x}, \infty)$. Hence,

$$\mathcal{F}g(0) - \mathcal{F}g(x) \ge \int_0^{\frac{2}{x}} \frac{\xi^3 x^2}{4} (-dg(\xi)) + \int_{\frac{2}{x}}^\infty 2(\xi - \tfrac{1}{x})(-dg(\xi)).$$

The function $\frac{1}{4} \xi^3 x^2 \mathbb{1}_{(0, 2/x)}(\xi) + 2(\xi - \frac{1}{x}) \mathbb{1}_{[2/x, \infty)}(\xi)$ is continuous at $\xi = \frac{2}{x}$. Therefore, another integration by parts gives

$$\mathcal{F}g(0) - \mathcal{F}g(x) \ge \int_0^{\frac{2}{x}} \tfrac{3}{4} \xi^2 x^2 g(\xi) d\xi + \int_{\frac{2}{x}}^\infty 2 g(\xi) d\xi.$$

It follows that

$$\mathcal{F}g(0) - \mathcal{F}g(x) \ge \frac{1}{2} \left(\int_0^{\frac{2}{x}} \xi^2 x^2 g(\xi) d\xi + \int_{\frac{2}{x}}^\infty 4 g(\xi) d\xi \right),$$

as desired. The estimates (2.7) for $x < 0$ follow by symmetry.

In a similar manner, for $x_1, x_2 \in \mathbb{R}$,

$$\begin{aligned} |\mathcal{F}g(x_1) - \mathcal{F}g(x_2)| &\le 2 \int_0^\infty |\cos(\xi x_1) - \cos(\xi x_2)| g(\xi) d\xi \\ &= 4 \int_0^\infty |\sin \tfrac{\xi x_1 - \xi x_2}{2}| |\sin \tfrac{\xi x_1 + \xi x_2}{2}| g(\xi) d\xi, \end{aligned}$$

and (2.8) follows from $|\sin s| \le \min(s, 1)$ for $s > 0$. □

Lemma 2.5. *If $\psi : (0, \infty) \to (0, \infty)$ and $\xi/\psi(\xi)$ is increasing in $\xi > 0$, then*

$$\int_0^\xi \frac{\zeta^2}{\psi(\zeta^2)} \, d\zeta \le \xi^2 \int_\xi^\infty \frac{1}{\psi(\zeta^2)} \, d\zeta \tag{2.9}$$

for all $\xi > 0$.

Proof. When $0 < \zeta < \xi$, then $\zeta^2/\psi(\zeta^2) \le \xi^2/\psi(\xi^2)$, and so

$$\int_0^\xi \frac{\zeta^2}{\psi(\zeta^2)} \, d\zeta \le \int_0^\xi \frac{\xi^2}{\psi(\xi^2)} \, d\zeta = \frac{\xi^3}{\psi(\xi^2)} \, .$$

When $0 < \xi < \zeta$, then $\zeta/\psi(\zeta^2) \ge \xi^2/\psi(\xi^2)$, so that

$$\int_\xi^\infty \frac{1}{\psi(\zeta^2)} \, d\zeta \ge \int_\xi^\infty \frac{\xi^2}{\zeta^2 \psi(\xi^2)} \, d\zeta = \frac{\xi}{\psi(\xi^2)} \, .$$

Formula (2.9) follows. □

Lemma 2.6. *If $g : \mathbb{R} \to \mathbb{R}$ is integrable and regularly varying at infinity with index $-\gamma$ for $\gamma \in (1, 3)$, and $g(x) = g(-x)$ for $x > 0$, then*

$$\lim_{x \to 0^+} \left(\frac{x}{g(1/x)} \left(\mathcal{F}g(0) - \mathcal{F}g(x) \right) \right) = \frac{\pi}{\Gamma(\gamma) |\cos \frac{\gamma\pi}{2}|} \, .$$

Proof. Clearly, $\mathcal{F}g(x) = 2 \int_0^\infty g(\xi) \cos(\xi x) d\xi$. By [27, Theorem 5],

$$\lim_{x \to 0^+} \left(\frac{x}{g(1/x)} \left(\mathcal{F}g(x) - \mathcal{F}g(0) \right) \right) = 2\Gamma(1 - \gamma) \sin \tfrac{\gamma\pi}{2},$$

where for $\gamma = 2$ it is understood that the right-hand side is equal to π. Furthermore, $\Gamma(1 - \gamma)\Gamma(\gamma) = \pi / \sin(\gamma\pi)$. □

3 Estimates of ϑ_λ

Recall that

$$\vartheta_\lambda = \arctan \left(\frac{1}{\pi} \int_0^\infty \frac{2}{\lambda} \frac{1}{(\psi_\lambda)_\lambda(\xi^2)} \, d\xi \right) \tag{3.1}$$

for $\lambda > 0$.

Lemma 3.1. *If $\psi(\xi^2)$ is the Lévy–Khintchine exponent of a symmetric Lévy process, $\psi'(\xi) > 0$ for all $\xi > 0$ and the scaling-type condition (1.11) holds for some $\alpha, \beta \in [1, 2]$ and all $\xi > 0$, then*

$$\frac{\pi}{\beta} - \frac{\pi}{2} \le \vartheta_\lambda \le \frac{\pi}{\alpha} - \frac{\pi}{2}$$

for all $\lambda > 0$.

Proof. If $\tilde{\psi}(\xi) = \xi^{\alpha/2}$, then $-\xi\tilde{\psi}''(\xi)/\tilde{\psi}'(\xi) = 1 - \frac{\alpha}{2}$. Hence, by Lemma 2.1,

$$(\psi_\lambda)_\lambda(\xi^2) \ge (\tilde{\psi}_\lambda)_\lambda(\xi^2)$$

for all $\lambda, \xi > 0$. By (3.1), it follows that $\vartheta_\lambda \le \tilde{\vartheta}_\lambda$, where $\tilde{\vartheta}_\lambda$ is defined as ϑ_λ, but using $\tilde{\psi}$ instead of ψ. By [15, Example 5.1], $\tilde{\vartheta}_\lambda = \frac{\pi}{\alpha} - \frac{\pi}{2}$. This proves the upper bound. The lower one is obtained in a similar manner. □

Lemma 3.2. *Suppose that $\psi(\xi^2)$ is the Lévy–Khintchine exponent of a symmetric Lévy process, $\psi'(\xi) > 0$ for all $\xi > 0$, and the scaling-type condition* (1.11) *holds for some $\alpha, \beta \in [1, 2]$ and all $\xi > 0$. If ψ' is regularly varying at zero with index $\frac{\delta}{2} - 1$ for some $\delta \in [1, 2]$, then*

$$\lim_{\lambda \to 0^+} \vartheta_\lambda = \frac{\pi}{\delta} - \frac{\pi}{2} .$$

Similarly, if ψ' is regularly varying at infinity with index $\frac{\gamma}{2} - 1$ for some $\gamma \in [1, 2]$, then

$$\lim_{\lambda \to \infty} \vartheta_\lambda = \frac{\pi}{\gamma} - \frac{\pi}{2} .$$

Proof. Suppose that ψ' is regularly varying at zero with index $\frac{\delta}{2} - 1$ and let $\tilde{\psi}(\xi) = \xi^{\alpha/2}$, so that $-\xi\tilde{\psi}''(\xi)/\tilde{\psi}'(\xi) = 1 - \frac{\alpha}{2}$. By Karamata's theorem [1, Theorem 1.5.11], $\lim_{\lambda \to 0^+}(\lambda^2\psi'(\lambda^2)/\psi(\lambda^2)) = \frac{\delta}{2}$ and ψ is regularly varying at zero with index $\frac{\delta}{2}$.

By a substitution $\xi = \lambda s$,

$$\begin{aligned} \lim_{\lambda \to 0^+} \vartheta_\lambda &= \arctan\left(\frac{1}{\pi} \lim_{\lambda \to 0^+} \int_0^\infty \frac{2}{(\psi_\lambda)_\lambda(\lambda^2 s^2)} \, ds\right) \\ &= \arctan\left(\frac{1}{\pi} \lim_{\lambda \to 0^+} \int_0^\infty \left(\frac{2\lambda^2\psi'(\lambda^2)/\psi(\lambda^2)}{\psi(\lambda^2 s^2)/\psi(\lambda^2) - 1} - \frac{2}{s^2 - 1}\right) ds\right). \end{aligned}$$

As $\lambda \to 0^+$, the integrand converges pointwise to $\delta/(s^\delta - 1) - 2/(s^2 - 1)$. Furthermore, it is positive and bounded above by $2/(\tilde{\psi}_\lambda)_\lambda(\lambda^2 s^2) = \alpha/(s^\alpha - 1) - 2/(s^2 - 1)$ by Lemma 2.1. Note that this upper bound does not depend on $\lambda > 0$ and it is integrable in $s \in (0, \infty)$. Hence, by the Dominated Convergence Theorem and [13, Example 5.1],

$$\lim_{\lambda \to 0^+} \vartheta_\lambda = \arctan\left(\frac{1}{\pi} \int_0^\infty \left(\frac{\delta}{s^\delta - 1} - \frac{2}{s^2 - 1}\right) ds\right) = \frac{\pi}{\delta} - \frac{\pi}{2} .$$

The other statement is proved in an analogous way. □

4 Estimates of $F_\lambda(x)$

Recall that

$$\mathcal{F}G_\lambda(\xi) = \frac{2\cos\vartheta_\lambda}{\lambda} \frac{1}{(\psi_\lambda)_\lambda(\xi^2)} \qquad \text{with} \qquad \frac{1}{(\psi_\lambda)_\lambda(\xi^2)} = \frac{\lambda^2\psi'(\lambda^2)}{\psi(\xi^2) - \psi(\lambda^2)} - \frac{\lambda^2}{\xi^2 - \lambda^2}$$

for $\lambda > 0$, $\xi \in \mathbb{R}$, and

$$F_\lambda(x) = \sin(\lambda|x| + \vartheta_\lambda) - G_\lambda(x)$$

for $\lambda > 0$, $x \in \mathbb{R}$.

Lemma 4.1. *If $\psi(\xi^2)$ is the Lévy–Khintchine exponent of a symmetric Lévy process, $1/(1 + \psi(\xi^2))$ is integrable, $\lambda > 0$ and $(\psi_\lambda)_\lambda(\xi)$ is well-defined and increasing in $\xi > 0$, then*

$$\frac{1}{4\pi} \int_0^\infty \min(\xi^2 x^2, 4)\mathcal{F}G_\lambda(\xi) d\xi \le G_\lambda(0) - G_\lambda(x) \le \frac{1}{2\pi} \int_0^\infty \min(\xi^2 x^2, 4)\mathcal{F}G_\lambda(\xi) d\xi$$

for all $x \in \mathbb{R}$. Furthermore,

$$|G_\lambda(x_1) - G_\lambda(x_2)| \le \frac{1}{2\pi} \int_0^\infty \min(\xi|x_1 - x_2|, 2) \min(\xi|x_1 + x_2|, 2)\mathcal{F}G_\lambda(\xi) d\xi$$

for all $x_1, x_2 \in \mathbb{R}$.

Proof. Due to symmetry of G_λ, $\mathcal{F}(\mathcal{F}G_\lambda) = 2\pi G_\lambda$. Furthermore, $\mathcal{F}G_\lambda$ is differentiable and decreasing on $(0,\infty)$. Hence, the result follows by Lemma 2.4. □

Lemma 4.2. *If $\psi(\xi^2)$ is the Lévy–Khintchine exponent of a symmetric Lévy process, $1/(1+\psi(\xi^2))$ is integrable, $\lambda > 0$, $(\psi_\lambda)_\lambda(\xi)$ is well-defined and $(\psi_\lambda)_\lambda(\xi)$ and $\xi/(\psi_\lambda)_\lambda(\xi)$ are increasing in $\xi > 0$, then*

$$\frac{1}{\pi}\int_{\frac{2}{|x|}}^{\infty} \mathcal{F}G_\lambda(\xi)d\xi \le G_\lambda(0) - G_\lambda(x) \le \frac{4}{\pi}\int_{\frac{2}{|x|}}^{\infty} \mathcal{F}G_\lambda(\xi)d\xi$$

for all $x \in \mathbb{R}$.

Proof. Since $\xi/(\psi_\lambda)_\lambda(\xi)$ is increasing in $\xi > 0$, by Lemma 2.5,

$$\int_0^{\frac{2}{|x|}} \xi^2 x^2 \mathcal{F}G_\lambda(\xi)d\xi \le \int_{\frac{2}{|x|}}^{\infty} 4\mathcal{F}G_\lambda(\xi)d\xi.$$

The result follows now from Lemma 4.1. □

Lemma 4.3. *If $\psi(\xi^2)$ is the Lévy–Khintchine exponent of a symmetric Lévy process, $1/(1+\psi(\xi^2))$ is integrable, $\lambda > 0$, $(\psi_\lambda)_\lambda(\xi)$ is well-defined and $(\psi_\lambda)_\lambda(\xi)$ and $\xi/(\psi_\lambda)_\lambda(\xi)$ are increasing in $\xi > 0$, then*

$$\frac{\cos\vartheta_\lambda}{\pi}\int_{\frac{2}{x}}^{\infty} \frac{2\lambda\psi'(\lambda^2)}{\psi(\xi^2)-\psi(\lambda^2)}\,d\xi \le F_\lambda(x) \le \frac{4}{\pi}\int_{\frac{2}{x}}^{\infty} \frac{2\lambda\psi'(\lambda^2)}{\psi(\xi^2)-\psi(\lambda^2)}\,d\xi$$

for $\lambda, x > 0$ satisfying $\lambda x < \frac{\pi}{2} - \vartheta_\lambda$. The upper bound holds when $\lambda x < 2$.

Proof. Suppose that $\lambda, x > 0$ and write

$$F_\lambda(x) = (\sin(\lambda x + \vartheta_\lambda) - \sin(\vartheta_\lambda)) + (G_\lambda(0) - G_\lambda(x)). \tag{4.1}$$

By Lemma 4.2, $G_\lambda(0) - G_\lambda(x)$ is bounded below and above by a constant times (see (2.1))

$$\begin{aligned}\int_{\frac{2}{x}}^{\infty} \mathcal{F}G_\lambda(\xi)d\xi &= \cos\vartheta_\lambda \int_{\frac{2}{x}}^{\infty} \frac{2}{\lambda(\psi_\lambda)_\lambda(\xi^2)}\,d\xi \\ &= \cos\vartheta_\lambda \int_{\frac{2}{x}}^{\infty} \left(\frac{2\lambda\psi'(\lambda^2)}{\psi(\xi^2)-\psi(\lambda^2)} - \frac{2\lambda}{\xi^2-\lambda^2}\right)d\xi\end{aligned}$$

Observe that $\frac{d}{ds}(\log(1+s) - \log(1-s)) \ge 2$ for $s \in (0,1)$. Therefore, if $\lambda x < 2$, then

$$\int_{\frac{2}{x}}^{\infty} \frac{2\lambda}{\xi^2-\lambda^2}d\xi = \log(1+\tfrac{\lambda x}{2}) - \log(1-\tfrac{\lambda x}{2}) \ge \lambda x \ge \sin(\lambda x + \vartheta_\lambda) - \sin(\vartheta_\lambda).$$

Hence,

$$\begin{aligned}F_\lambda(x) &\le \int_{\frac{2}{x}}^{\infty} \frac{2\lambda}{\xi^2-\lambda^2}d\xi + \frac{4\cos\vartheta_\lambda}{\pi}\int_{\frac{2}{x}}^{\infty} \frac{2}{\lambda(\psi_\lambda)_\lambda(\xi^2)}\,d\xi \\ &\le \frac{4}{\pi}\int_{\frac{2}{x}}^{\infty} \frac{2\lambda}{\xi^2-\lambda^2}d\xi + \frac{4}{\pi}\int_{\frac{2}{x}}^{\infty} \frac{2}{\lambda(\psi_\lambda)_\lambda(\xi^2)}\,d\xi = \frac{4}{\pi}\int_{\frac{2}{x}}^{\infty} \frac{2\lambda\psi'(\lambda^2)}{\psi(\xi^2)-\psi(\lambda^2)}\,d\xi.\end{aligned}$$

The lower bound is found in a similar manner. Observe that $\log(1+s) - \log(1-s)$ is convex on $(0,1)$. Hence, if $\frac{\lambda x}{2} < \frac{\pi}{4}$, then

$$\begin{aligned}\int_{\frac{2}{x}}^{\infty} \frac{2\lambda}{\xi^2-\lambda^2}d\xi &= \log(1+\tfrac{\lambda x}{2}) - \log(1-\tfrac{\lambda x}{2}) \\ &\le (\log(1+\tfrac{\pi}{4}) - \log(1-\tfrac{\pi}{4}))\tfrac{4}{\pi}\,\tfrac{\lambda x}{2} = \tfrac{2}{\pi}(\log\tfrac{4+\pi}{4-\pi})\lambda x.\end{aligned}$$

Furthermore, by concavity, $\sin(s+\vartheta_\lambda) - \sin\vartheta_\lambda \ge s(1-\sin\vartheta_\lambda)/(\frac{\pi}{2}-\vartheta_\lambda)$ for $s \in (0, \frac{\pi}{2}-\vartheta_\lambda)$. It follows that if $\lambda x < \frac{\pi}{2} - \vartheta_\lambda$, then

$$\begin{aligned}\int_{\frac{2}{x}}^\infty \frac{2\lambda}{\xi^2-\lambda^2} d\xi &\le \tfrac{2}{\pi}(\log\tfrac{4+\pi}{4-\pi})\,\frac{\frac{\pi}{2}-\vartheta_\lambda}{1-\sin\vartheta_\lambda}\,(\sin(\lambda x+\vartheta_\lambda)-\sin(\vartheta_\lambda)) \\ &\le \frac{4\log\frac{4+\pi}{4-\pi}}{\pi\cos\vartheta_\lambda}\,(\sin(\lambda x+\vartheta_\lambda)-\sin(\vartheta_\lambda));\end{aligned}$$

the last inequality follows from the inequality $1-\cos s \ge \frac{1}{2} s\sin s$ for $s \in (0, \frac{\pi}{2})$ (which is easily proved by differentiation) with $s = \frac{\pi}{2}-\vartheta_\lambda$. This gives the desired lower bound,

$$\begin{aligned}F_\lambda(x) &\ge \frac{\pi\cos\vartheta_\lambda}{4\log\frac{4+\pi}{4-\pi}} \int_{\frac{2}{x}}^\infty \frac{2\lambda}{\xi^2-\lambda^2} d\xi + \frac{\cos\vartheta_\lambda}{\pi}\int_{\frac{2}{x}}^\infty \frac{2}{\lambda(\psi_\lambda)_\lambda(\xi^2)}\,d\xi \\ &\ge \frac{\cos\vartheta_\lambda}{\pi}\int_{\frac{2}{x}}^\infty \frac{2\lambda}{\xi^2-\lambda^2} d\xi + \frac{\cos\vartheta_\lambda}{\pi}\int_{\frac{2}{x}}^\infty \frac{2}{\lambda(\psi_\lambda)_\lambda(\xi^2)}\,d\xi \\ &= \frac{\cos\vartheta_\lambda}{\pi}\int_{\frac{2}{x}}^\infty \frac{2\lambda\psi'(\lambda^2)}{\psi(\xi^2)-\psi(\lambda^2)}\,d\xi.\end{aligned}$$

□

Lemma 4.4. *If $\psi(\xi^2)$ is the Lévy–Khintchine exponent of a symmetric Lévy process, $\lambda > 0$, $(\psi_\lambda)_\lambda(\xi)$ and $\xi/(\psi_\lambda)_\lambda(\xi)$ are increasing in $\xi > 0$, and the scaling-type condition* (1.11) *holds for some $\alpha, \beta \in (1, 2]$ and all $\xi > 0$, then*

$$\frac{\alpha-1}{\pi}\,\frac{\lambda\psi'(\lambda^2)}{x\psi(1/x^2)} \le F_\lambda(x) \le \frac{40}{\pi(\alpha-1)}\,\frac{\lambda\psi'(\lambda^2)}{x\psi(1/x^2)} \tag{4.2}$$

for $\lambda, x > 0$ satisfying $\lambda x < \pi - \frac{\pi}{\alpha}$. The upper bound holds when $\lambda x < 2$. Furthermore,

$$|F_\lambda(x_1) - F_\lambda(x_2)| \le 3\lambda|x_1-x_2| + \frac{2\lambda\psi'(\lambda^2)}{\pi}\int_{2\lambda}^\infty \frac{\min(\xi|x_1-x_2|, 2)\min(\xi|x_1+x_2|, 2)}{\psi(\xi^2)}\,d\xi \tag{4.3}$$

for $\lambda > 0$ and $x_1, x_2 \in \mathbb{R}$.

Note that the scaling-type condition (1.11) implies that $1/(1+\psi(\xi^2))$ is integrable (by Lemma 2.2) and that $(\psi_\lambda)_\lambda(\xi)$ is well-defined.

Proof. By Lemma 3.1, $\vartheta_\lambda \le \frac{\pi}{\alpha} - \frac{\pi}{2}$. Hence, by Lemma 4.3,

$$\frac{\sin\frac{\pi}{\alpha}}{\pi}\int_{\frac{2}{x}}^\infty \frac{2\lambda\psi'(\lambda^2)}{\psi(\xi^2)-\psi(\lambda^2)}\,d\xi \le F_\lambda(x) \le \frac{4}{\pi}\int_{\frac{2}{x}}^\infty \frac{2\lambda\psi'(\lambda^2)}{\psi(\xi^2)-\psi(\lambda^2)}\,d\xi$$

for $\lambda, x > 0$ such that $\lambda x < \pi - \frac{\pi}{\alpha}$. In this case $\xi > \frac{2}{x}$ implies $\xi > 2(\pi - \frac{\pi}{\alpha})^{-1}\lambda > \frac{4}{\pi}\lambda$, and hence, by Lemma 2.2, $\psi(\lambda^2) \le (\frac{\pi}{4})^\alpha \psi(\xi^2)$. Therefore,

$$\int_{\frac{2}{x}}^\infty \frac{2\lambda\psi'(\lambda^2)}{\psi(\xi^2)}\,d\xi \le \int_{\frac{2}{x}}^\infty \frac{2\lambda\psi'(\lambda^2)}{\psi(\xi^2)-\psi(\lambda^2)}\,d\xi \le \frac{1}{1-(\frac{\pi}{4})^\alpha}\int_{\frac{2}{x}}^\infty \frac{2\lambda\psi'(\lambda^2)}{\psi(\xi^2)}\,d\xi.$$

Finally, again by Lemma 2.2,

$$\int_{\frac{2}{x}}^\infty \frac{1}{\psi(\xi^2)}\,d\xi \le \frac{1}{\psi(1/x^2)}\int_{\frac{2}{x}}^\infty \frac{1}{(\xi x)^\alpha}\,d\xi = \frac{1}{(\alpha-1)2^{\alpha-1}x\psi(1/x^2)}\,,$$

and a similar lower bound is valid with α replaced by β. By combining the above estimates, one obtains

$$\frac{\sin\frac{\pi}{\alpha}}{\pi}\,\frac{2\lambda\psi'(\lambda^2)}{(\beta-1)2^{\beta-1}x\psi(1/x^2)} \le F_\lambda(x) \le \frac{4}{\pi}\,\frac{1}{1-(\frac{\pi}{4})^\alpha}\,\frac{2\lambda\psi'(\lambda^2)}{(\alpha-1)2^{\alpha-1}x\psi(1/x^2)}\,,$$

and (4.2) follows by elementary estimates: $\sin \frac{\pi}{\alpha} \geq (\alpha - 1)$, $(\beta - 1)2^{\beta-1} \leq 2$, $2^{\alpha-1} \geq 1$, $1 - (\frac{\pi}{4})^\alpha \geq 1 - \frac{\pi}{4} \geq \frac{1}{5}$.

Formula (4.3) is proved in a similar way. By Lemma 4.1 and (4.1), for $\lambda > 0$ and $x_1, x_2 \in \mathbb{R}$,

$$|F_\lambda(x_1) - F_\lambda(x_2)| \leq \lambda x + \frac{1}{2\pi} \int_0^\infty \frac{2}{\lambda (\psi_\lambda)_\lambda(\xi^2)} \min(\xi x, 2) \min(\xi y, 2) d\xi$$

where for brevity $x = |x_1 - x_2|$ and $y = |x_1 + x_2|$. Since $(\psi_\lambda)_\lambda(\xi) \geq (\psi_\lambda)_\lambda(0) = 1$ for $\xi \in (0, 2\lambda)$, and $1/(\psi_\lambda)_\lambda(\xi) \leq \lambda^2/(\psi(\xi^2) - \psi(\lambda^2))$ for $\xi > 2\lambda$,

$$\begin{aligned} |F_\lambda(x_1) - F_\lambda(x_2)| &\leq \lambda x + \int_0^{2\lambda} \frac{2\xi x}{\pi \lambda} d\xi + \frac{1}{\pi} \int_{2\lambda}^\infty \frac{\lambda \psi'(\lambda^2)}{\psi(\xi^2) - \psi(\lambda^2)} \min(\xi x, 2) \min(\xi y, 2) d\xi \\ &\leq 3\lambda x + \frac{\lambda \psi'(\lambda^2)}{(1 - (\frac{1}{2})^\alpha)\pi} \int_{2\lambda}^\infty \frac{\min(\xi x, 2) \min(\xi y, 2)}{\psi(\xi^2)} d\xi; \end{aligned}$$

here the last inequality follows by Lemma 2.2. □

Lemma 4.5. *If $\psi(\xi^2)$ is the Lévy–Khintchine exponent of a symmetric Lévy process, $\lambda > 0$, $(\psi_\lambda)_\lambda(\xi)$ is well-defined, and ψ is regularly varying at infinity with index $\frac{\gamma}{2}$ for some $\gamma \in (1, 2]$, then*

$$\lim_{x \to 0^+} (x \psi(1/x^2) F_\lambda(x)) = \frac{\lambda \psi'(\lambda^2) \cos \vartheta_\lambda}{\Gamma(\gamma) |\cos \frac{\gamma\pi}{2}|} .$$

Note that $1/(1 + \psi(\xi^2))$ is integrable, because it is regularly varying at infinity with index $-\gamma$.

Proof. Recall that

$$\mathcal{F} G_\lambda(\xi) = \frac{2 \cos \vartheta_\lambda}{\lambda} \frac{1}{(\psi_\lambda)_\lambda(\xi^2)} = \frac{2\lambda \psi'(\lambda^2) \cos \vartheta_\lambda}{\psi(\xi^2) - \psi(\lambda^2)} - \frac{2\lambda \cos \vartheta_\lambda}{\xi^2 - \lambda^2} ,$$

and that $G_\lambda = \frac{1}{2\pi} \mathcal{F}(\mathcal{F} G_\lambda)$. Let $a = \lim_{\xi \to \infty} (\psi(\xi^2)/\xi^2)$; necessarily $a \in [0, \infty)$. Then

$$\lim_{\xi \to \infty} (\psi(\xi^2) \mathcal{F} G_\lambda(\xi)) = 2\lambda (\psi'(\lambda^2) - a) \cos \vartheta_\lambda.$$

Therefore, $\mathcal{F} G_\lambda(\xi)$ is regularly varying at infinity with index $-\gamma$, and by Lemma 2.6,

$$\begin{aligned} \lim_{x \to 0^+} (x \psi(1/x^2)(G_\lambda(0) - G_\lambda(x))) &= 2\lambda (\psi'(\lambda^2) - a) \cos \vartheta_\lambda \lim_{x \to 0^+} \frac{x (G_\lambda(0) - G_\lambda(x))}{\mathcal{F} G_\lambda(1/x)} \\ &= \frac{\lambda (\psi'(\lambda^2) - a) \cos \vartheta_\lambda}{\Gamma(\gamma) |\cos \frac{\gamma\pi}{2}|} . \end{aligned}$$

Furthermore,

$$\lim_{x \to 0^+} (x \psi(1/x^2)(\sin(\lambda x + \vartheta_\lambda) - \sin \vartheta_\lambda)) = \lambda a \cos \vartheta_\lambda,$$

and $F_\lambda(x) = (\sin(\lambda x + \vartheta_\lambda) - \sin \vartheta_\lambda) + (G_\lambda(0) - G_\lambda(x))$. The result clearly follows when $a = 0$. If $a > 0$, then necessarily $\gamma = 2$, and hence $\Gamma(\gamma) |\cos \frac{\gamma\pi}{2}| = 1$. □

Recall that the compensated potential kernel v of X is defined by

$$v(x) = \int_0^\infty (p_t(0) - p_t(x)) dt,$$

where $p_t(x)$ is the density function of the distribution of X_t. Since $\mathcal{F}p_t(\xi) = e^{-t\psi(\xi^2)}$, the distributional Fourier transform of v satisfies

$$\begin{aligned}\langle \mathcal{F}v, \varphi \rangle &= \int_0^\infty \int_{-\infty}^\infty e^{-t\psi(\xi^2)}(\varphi(0) - \varphi(\xi))d\xi dt \\ &= \int_0^\infty \int_0^\infty e^{-t\psi(\xi^2)}(2\varphi(0) - \varphi(\xi) - \varphi(-\xi))d\xi dt \\ &= \int_0^\infty \frac{2\varphi(0) - \varphi(\xi) - \varphi(-\xi)}{\psi(\xi^2)}\, d\xi\end{aligned}$$

for φ in the Schwartz class (the Fubini theorem is used in the last equality).

Lemma 4.6. *If $\psi(\xi^2)$ is the Lévy–Khintchine exponent of a symmetric Lévy process, $\lambda > 0$, $(\psi_\lambda)_\lambda(\xi)$ and $\xi/(\psi_\lambda)_\lambda(\xi)$ are increasing in $\xi > 0$, and the scaling-type condition* (1.11) *holds for some $\alpha, \beta \in (1, 2]$ and all $\xi > 0$, then*

$$\lim_{\lambda \to 0^+} \frac{F_\lambda(x)}{2\lambda\psi'(\lambda^2)\cos\vartheta_\lambda} = v(x)$$

locally uniformly in $x \in \mathbb{R}$, where $v(x)$ is the compensated potential kernel of X.

Noteworthy, convergence in the space of tempered distributions holds in full generality, that is, with the hypotheses of Theorem 1.9. Under the assumptions of the lemma, one also has $\vartheta_\lambda \to \frac{\pi}{\gamma} - \frac{\pi}{2}$ as $\lambda \to 0^+$ by Lemma 3.2. As before, the scaling-type condition (1.11) implies that $1/(1 + \psi(\xi^2))$ is integrable (by Lemma 2.2) and that $(\psi_\lambda)_\lambda(\xi)$ is well-defined.

Proof. By Theorem 1.9, for φ in the Schwartz class,

$$\begin{aligned}\left\langle \frac{\mathcal{F}F_\lambda}{\lambda\psi'(\lambda^2)\cos\vartheta_\lambda}, \varphi \right\rangle &= 2\,\mathrm{pv}\!\int_{-\infty}^\infty \frac{\varphi(\xi)}{\psi(\lambda^2) - \psi(\xi^2)} d\xi + \frac{\pi\tan\vartheta_\lambda}{\lambda\psi'(\lambda^2)}(\varphi(\lambda) + \varphi(-\lambda)) \\ &= 2\,\mathrm{pv}\!\int_{-\infty}^\infty \frac{\varphi(\xi)}{\psi(\lambda^2) - \psi(\xi^2)} d\xi - 2\,\mathrm{pv}\!\int_0^\infty \frac{\varphi(\lambda) + \varphi(-\lambda)}{\psi(\lambda^2) - \psi(\xi^2)}\, d\xi \\ &= 2\int_0^\infty \frac{\varphi(\xi) - \varphi(\lambda) + \varphi(-\xi) - \varphi(-\lambda)}{\psi(\lambda^2) - \psi(\xi^2)}\, d\xi.\end{aligned}$$

As $\lambda \to 0^+$, the integrand converges pointwise to $(2\varphi(0) - \varphi(\xi) - \varphi(-\xi))/\psi(\xi^2)$. We claim that the Dominated Convergence Theorem applies to the above limit. Indeed,

$$\begin{aligned}|\varphi(\xi) - \varphi(\lambda) + \varphi(-\xi) - \varphi(-\lambda)| &\le |\xi - \lambda| \sup\{|\varphi'(s) - \varphi'(-s)| : 0 < s < \xi + \lambda\} \\ &\le |\xi - \lambda|(\xi + \lambda)\|\varphi''\|_\infty = |\xi^2 - \lambda^2|\|\varphi''\|_\infty,\end{aligned}$$

for all $\lambda, \xi > 0$, and since ψ' is decreasing,

$$|\psi(\lambda^2) - \psi(\xi^2)| \ge |\xi^2 - \lambda^2|\psi'(2)$$

for all $\lambda \in (0, 1)$ and $\xi \in (0, 2)$. Hence,

$$\left| \frac{\varphi(\xi) - \varphi(\lambda) + \varphi(-\xi) - \varphi(-\lambda)}{\psi(\lambda^2) - \psi(\xi^2)} \right| \le \frac{\|\varphi''\|_\infty}{\psi'(2)}$$

for all $\lambda \in (0, 1)$ and $\xi \in (0, 2)$. On the other hand,

$$\left| \frac{\varphi(\xi) - \varphi(\lambda) + \varphi(-\xi) - \varphi(-\lambda)}{\psi(\lambda^2) - \psi(\xi^2)} \right| \le \frac{4\|\varphi\|_\infty}{\psi(\xi^2) - \psi(1)}$$

for all $\lambda \in (0, 1)$ and $\xi \geq 2$. The upper bound found above is integrable in $\xi \in (0, \infty)$, and the claim is proved. It follows that

$$\lim_{\lambda \to 0^+} \left\langle \frac{\mathcal{F}F_\lambda}{\lambda \psi'(\lambda^2) \cos \vartheta_\lambda}, \varphi \right\rangle = 2 \langle \mathcal{F}v, \varphi \rangle$$

for every φ in the Schwartz class. This proves the desired result, but with locally uniform convergence replaced by convergence in the space of tempered distributions.

By Lemmas 4.4 and 3.1, for all $\lambda > 0$ and $x_1, x_2 \in \mathbb{R}$,

$$\frac{|F_\lambda(x_1) - F_\lambda(x_2)|}{\lambda \psi'(\lambda^2) \cos \vartheta_\lambda} \leq \frac{3|x_1 - x_2|}{\psi'(\lambda) \sin \frac{\pi}{\alpha}} + \frac{2}{\pi \sin \frac{\pi}{\alpha}} \int_{2\lambda}^{\infty} \frac{\min(\xi|x_1 - x_2|, 2) \min(\xi|x_1 + x_2|, 2)}{\psi(\xi^2)} \, d\xi.$$

Hence, if $\lambda \in (0, \lambda_0)$ and $x_1, x_2 \in [-x_0, x_0]$, then

$$\frac{|F_\lambda(x_1) - F_\lambda(x_2)|}{\lambda \psi'(\lambda^2) \cos \vartheta_\lambda} \leq \frac{3|x_1 - x_2|}{\psi'(\lambda_0) \sin \frac{\pi}{\alpha}} + \frac{2}{\pi \sin \frac{\pi}{\alpha}} \int_{0}^{\infty} \frac{\min(\xi|x_1 - x_2|, 2) \min(2\xi x_0, 2)}{\psi(\xi^2)} \, d\xi.$$

The right-hand side is finite and converges to 0 as $|x_2 - x_1| \to 0^+$ by the Dominated Convergence Theorem. Hence, the functions $F_\lambda(x)/(\lambda\psi'(\lambda^2)\cos\vartheta_\lambda)$ are equicontinuous in $x \in [-x_0, x_0]$ for $\lambda \in (0, \lambda_0)$. It remains to note that on a bounded interval, distributional convergence and equicontinuity imply uniform convergence. □

5 Estimates of hitting times

We begin with two technical results.

Proposition 5.1. *If X is a symmetric Lévy process with Lévy–Khintchine exponent Ψ, and $1/(1 + \Psi(\xi))$ is integrable, then $\mathbb{P}(t < \tau_x < \infty)$ is jointly continuous in $t > 0$ and $x \in \mathbb{R}$.*

Proof. By [23, Theorem 43.5 and Remark 43.6], $\mathbb{E}e^{-\lambda\tau_x}$ is a continuous function of $x \in \mathbb{R}$ for every $\lambda > 0$. Therefore, the distributions of τ_x are continuous in x with respect to vague convergence of measures. It follows that the function $\mathbb{P}(t < \tau_x < \infty)$ is continuous in x at every point (t, x) at which it is continuous in t.

Since $\mathbb{P}(\tau_x = t) \leq \mathbb{P}(X_t = x) = 0$, the function $\mathbb{P}(t < \tau_x < \infty)$ is continuous and non-increasing in $t > 0$ for every $x \in \mathbb{R}$. This implies that it is in fact jointly continuous in $t > 0$ and $x \in \mathbb{R}$. □

Proposition 5.2. *If $\psi(\xi^2)$ is the Lévy–Khintchine exponent of a symmetric Lévy process, $1/(1 + \psi(\xi^2))$ is integrable, $(\psi_\lambda)_\lambda$ is well-defined and $(\psi_\lambda)_\lambda(\xi)$ and $\xi/(\psi_\lambda)_\lambda(\xi)$ are increasing in $\xi > 0$ for all $\lambda > 0$, then equation* (1.8) *in Theorem 1.9 holds for all $x \in \mathbb{R}$ (and not just for almost all $x \in \mathbb{R}$).*

Proof. It suffices to consider $n = 0$, the result for $n > 0$ follows then by differentiation, see [13, Remark 1.2]. Let $t > 0$. By Proposition 5.1, the left-hand side of (1.8) is a continuous function of $x \in \mathbb{R} \setminus \{0\}$. For each $t > 0$, the integrand in the right-hand side of (1.8) is continuous in $x \in \mathbb{R} \setminus \{0\}$. Therefore, it remains to show that the Dominated Convergence Theorem can be applied to prove continuity of the right-hand side of (1.8) in $x > 0$ (equality for $x = 0$ is trivial, and the result for $x < 0$ follows by symmetry).

Fix $[a, b] \subseteq (0, \infty)$. By Lemma 4.3, for $x \in [a, b]$ and $\lambda \in (0, \frac{1}{b})$,

$$\left| \cos\vartheta_\lambda e^{-t\psi(\lambda^2)} 2\lambda\psi'(\lambda^2)(\psi(\lambda^2))^{-1} F_\lambda(x) \right| \leq \frac{4}{\pi} \frac{(2\lambda\psi'(\lambda^2))^2}{\psi(\lambda^2)} \int_{\frac{2}{b}}^{\infty} \frac{1}{\psi(\xi^2) - \psi(1/b^2)} \, d\xi,$$

while for $x \in [a, b]$ and $\lambda \geq \frac{1}{b}$,

$$\left| \cos \vartheta_\lambda e^{-t\psi(\lambda^2)} 2\lambda\psi'(\lambda^2)(\psi(\lambda^2))^{-1} F_\lambda(x) \right| \leq \frac{4\lambda\psi'(\lambda^2)}{\psi(\lambda^2)} \, e^{-t\psi(\lambda^2)},$$

because $|F_\lambda(x)| \leq 2$ (indeed, $|F_\lambda(x)| \leq 1 + |G_\lambda(x)|$; since $\mathcal{F}G_\lambda(\xi) \geq 0$ for all $\xi \in \mathbb{R}$, one has $|G_\lambda(x)| \leq G_\lambda(0)$; finally, $G_\lambda(0) = \sin \vartheta_\lambda \leq 1$; see [13, Theorem 1.9(a)]). Clearly,

$$\int_{\frac{2}{b}}^{\infty} \frac{4\lambda\psi'(\lambda^2)}{\psi(\lambda^2)} \, e^{-t\psi(\lambda^2)} d\lambda = \int_{\psi(4/b^2)}^{\infty} \frac{2e^{-ts}}{s} \, ds < \infty.$$

Furthermore, $\lambda^2\psi'(\lambda^2)/\psi(\lambda^2) = 1/\psi_\lambda(\lambda^2) \leq 1/\psi_\lambda(0) = 1$, and therefore

$$\begin{aligned} &\int_0^{\frac{2}{b}} \left(\frac{4}{\pi} \, \frac{(2\lambda\psi'(\lambda^2))^2}{\psi(\lambda^2)} \int_{\frac{2}{b}}^{\infty} \frac{1}{\psi(\xi^2) - \psi(1/b^2)} \, d\xi \right) d\lambda \\ &\qquad \leq \frac{16}{\pi} \int_0^{\frac{2}{b}} \psi'(\lambda^2) d\lambda \int_{\frac{2}{b}}^{\infty} \frac{1}{\psi(\xi^2) - \psi(1/b^2)} \, d\xi < \infty, \end{aligned}$$

which completes the proof. □

By Proposition 5.2, under appropriate assumptions, for $n \geq 0$, $t > 0$ and $x \in \mathbb{R} \setminus \{0\}$,

$$\left(-\frac{d}{dt} \right)^n \mathbb{P}(t < \tau_x < \infty) = \frac{2}{\pi} \int_0^{\infty} \cos \vartheta_\lambda e^{-t\psi(\lambda^2)} \lambda\psi'(\lambda^2)(\psi(\lambda^2))^{n-1} F_\lambda(x) d\lambda.$$

Throughout this section we denote

$$\begin{aligned} I_n(t, x, a) &= \frac{2}{\pi} \int_a^{\infty} \cos \vartheta_\lambda e^{-t\psi(\lambda^2)} \lambda\psi'(\lambda^2)(\psi(\lambda^2))^{n-1} F_\lambda(x) d\lambda, \\ J_n(t, x, a) &= \frac{2}{\pi} \int_0^{a} \cos \vartheta_\lambda e^{-t\psi(\lambda^2)} \lambda\psi'(\lambda^2)(\psi(\lambda^2))^{n-1} F_\lambda(x) d\lambda. \end{aligned} \tag{5.1}$$

In the remaining part of the article, $\gamma(k; z)$ and $\Gamma(k; z)$ denote the lower and the upper incomplete gamma functions, respectively.

Lemma 5.3. *If $\psi(\xi^2)$ is the Lévy–Khintchine exponent of a symmetric Lévy process, $1/(1 + \psi(\xi^2))$ is integrable, $(\psi_\lambda)_\lambda(\xi)$ is well-defined and $(\psi_\lambda)_\lambda(\xi)$ and $\xi/(\psi_\lambda)_\lambda(\xi)$ are increasing in $\xi > 0$ for all $\lambda > 0$, and the scaling-type condition* (1.11) *holds for some $\alpha, \beta \in (1, 2]$ and all $\xi > 0$, then*

$$|I_n(t, x, a)| \leq \tfrac{2}{\pi} t^{-n} \Gamma(n; (\alpha - 1)^\beta t\psi(1/x^2))$$

for all $n \geq 0$, $t, x > 0$ and $a \geq (\pi - \frac{\pi}{\alpha})/x$.

Proof. Fix $t, x > 0$ and let $a_0 = (\pi - \frac{\pi}{\alpha})/x$ and $b_0 = t\psi(a_0^2)$. Using $|F_\lambda(x)| \leq 2$ (see the proof of Proposition 5.2) and a substitution $s = t\psi(\lambda^2)$, one finds that

$$|I_n(t, x, a)| \leq \frac{4}{\pi} \int_{a_0}^{\infty} e^{-t\psi(\lambda^2)} \lambda\psi'(\lambda^2)(\psi(\lambda^2))^{n-1} d\lambda = \frac{2}{\pi t^n} \int_{b_0}^{\infty} e^{-s} s^{n-1} ds = \frac{2\Gamma(n; b_0)}{\pi t^n} \, .$$

Furthermore, $b_0 = t\psi(a_0^2) \geq t\psi(1/x^2)$ if $\alpha > \frac{\pi}{\pi - 1}$, and $b_0 = t\psi(a_0^2) \geq (\pi - \frac{\pi}{\alpha})^\beta t\psi(1/x^2)$ otherwise (by Lemma 2.2). In either case, $b_0 \geq (\alpha - 1)^\beta t\psi(1/x^2)$. □

Lemma 5.4. *If $\psi(\xi^2)$ is the Lévy–Khintchine exponent of a symmetric Lévy process, $(\psi_\lambda)_\lambda(\xi)$ and $\xi/(\psi_\lambda)_\lambda(\xi)$ are increasing in $\xi > 0$ for all $\lambda > 0$, and the scaling-type* *condition* (1.11) *holds for some* $\alpha, \beta \in (1, 2]$ *and all* $\xi > 0$*, then there are constants* $c_1(\alpha, \beta, n), c_2(\alpha, \beta, n) > 0$ *such that*

$$\frac{c_1(\alpha,\beta,n)}{t^{n+1} x \psi(1/x^2) \sqrt{\psi^{-1}(1/t)}} \le J_n(t, x, (\pi - \tfrac{\pi}{\alpha})/x) \le \frac{c_2(\alpha,\beta,n)}{t^{n+1} x \psi(1/x^2) \sqrt{\psi^{-1}(1/t)}}$$

for $n \ge 0$ *and* $t, x > 0$ *such that* $t\psi(1/x^2) \ge 1$*. Here*

$$\begin{aligned} c_1(\alpha,\beta,n) &= \frac{(\alpha-1)^2 \gamma(n+1-\frac{1}{\beta}; (\alpha-1)^\beta)}{2\pi^2}\,, \\ c_2(\alpha,\beta,n) &= \frac{40(\gamma(n+1-\frac{1}{\alpha};1) + \Gamma(n+1-\frac{1}{\beta};1))}{\pi^2(\alpha-1)}\,. \end{aligned} \tag{5.2}$$

As before, the scaling-type condition (1.11) implies that $1/(1+\psi(\xi^2))$ is integrable (by Lemma 2.2) and that $(\psi_\lambda)_\lambda(\xi)$ is well-defined.

Proof. Fix $t, x > 0$ and let $a = (\pi - \frac{\pi}{\alpha})/x$ and $b = t\psi(a^2)$. Denote $J = J_n(t, x, a)$. Observe that when $\lambda < a$, then $\lambda x < \pi - \frac{\pi}{\alpha}$ and Lemma 4.4 applies. Hence,

$$J \ge \frac{2(\alpha-1)}{\pi^2} \int_0^a \cos\vartheta_\lambda e^{-t\psi(\lambda^2)} \lambda\psi'(\lambda^2)(\psi(\lambda^2))^{n-1} \frac{\lambda\psi'(\lambda^2)}{x\psi(1/x^2)}\, d\lambda.$$

Using $\cos\vartheta_\lambda \ge \cos(\frac{\pi}{\alpha} - \frac{\pi}{2}) \ge (\alpha-1)$, $\lambda^2\psi'(\lambda^2) \ge \frac{\alpha}{2}\psi(\lambda^2)$ (by Lemma 2.2) and a substitution $s = t\psi(\lambda^2)$,

$$J \ge \frac{\alpha(\alpha-1)^2}{\pi^2 x\psi(1/x^2)} \int_0^a e^{-t\psi(\lambda^2)} \psi'(\lambda^2)(\psi(\lambda^2))^n d\lambda = \frac{\alpha(\alpha-1)^2}{\pi^2 t^{n+1} x\psi(1/x^2)} \int_0^b \frac{e^{-s}s^n}{2\sqrt{\psi^{-1}(s/t)}}\, ds.$$

By Lemma 2.2 and (2.6),

$$\begin{aligned} J &\ge \frac{\alpha(\alpha-1)^2}{2\pi^2 t^{n+1} x\psi(1/x^2)} \int_0^b \frac{e^{-s}s^n}{\sqrt{\max(s^{2/\beta}, s^{2/\alpha})\psi^{-1}(1/t)}}\, ds \\ &\ge \frac{\alpha(\alpha-1)^2 \gamma(n+1-\frac{1}{\beta}; \min(b,1))}{2\pi^2 t^{n+1} x\psi(1/x^2)\sqrt{\psi^{-1}(1/t)}}\,. \end{aligned}$$

Finally, as in the proof of Lemma 5.3, $b \ge (\alpha-1)^\beta t\psi(1/x^2)$. This proves the desired lower bound. The upper bound is shown in a similar manner,

$$\begin{aligned} J &\le \frac{80}{\pi^2(\alpha-1)} \int_0^a \cos\vartheta_\lambda e^{-t\psi(\lambda^2)} \lambda\psi'(\lambda^2)(\psi(\lambda^2))^{n-1} \frac{\lambda\psi'(\lambda^2)}{x\psi(1/x^2)}\, d\lambda \\ &\le \frac{40\beta}{\pi^2(\alpha-1)x\psi(1/x^2)} \int_0^a e^{-t\psi(\lambda^2)} \psi'(\lambda^2)(\psi(\lambda^2))^n d\lambda \\ &= \frac{40\beta}{\pi^2(\alpha-1)t^{n+1}x\psi(1/x^2)} \int_0^b \frac{e^{-s}s^n}{2\sqrt{\psi^{-1}(s/t)}}\, ds \\ &\le \frac{20\beta}{\pi^2(\alpha-1)t^{n+1}x\psi(1/x^2)} \int_0^b \frac{e^{-s}s^n}{\sqrt{\min(s^{2/\beta}, s^{2/\alpha})\psi^{-1}(1/t)}}\, ds \\ &\le \frac{20\beta(\gamma(n+1-\frac{1}{\alpha};1) + \Gamma(n+1-\frac{1}{\beta};1))}{\pi^2(\alpha-1)t^{n+1}x\psi(1/x^2)\sqrt{\psi^{-1}(1/t)}}\,. \end{aligned}$$ □

As observed in the introduction, with the hypotheses of Theorems 1.1 and 1.3, $\psi(\xi) = \Psi(\sqrt{\xi})$ is a complete Bernstein function (see [24]), and hence ψ_λ and $(\psi_\lambda)_\lambda$ are complete Bernstein functions (see [13]). In particular, $(\psi_\lambda)_\lambda$ is well-defined and $(\psi_\lambda)_\lambda(\xi)$

and $\xi/(\psi_\lambda)_\lambda(\xi)$ are increasing in $\xi > 0$ for all $\lambda > 0$. Furthermore, if ψ is regularly varying at zero or at infinity with index $\frac{\alpha}{2}$, then ψ' is regularly varying at the same point with index $\frac{\alpha}{2} - 1$. The scaling-type condition (1.3) implies the upper bound of (1.11), and the lower bound is automatically satisfied with $\beta = 2$. Finally, $\Psi^{-1}(t) = (\psi^{-1}(t))^{1/2}$, and the relation between the derivatives of ψ and Ψ is given in (1.10).

Observe that the distributions of τ_x and τ_{-x} are equal, and F_λ are even functions. Hence, only $x > 0$ needs to be considered in the proofs of main theorems.

Proof of Theorem 1.1. Let $\beta = 2$. Choose $c > 1$ large enough, so that for $s \ge c$,

$$\tfrac{2}{\pi} s\, \Gamma(n; (\alpha-1)^\beta s) \le \tfrac{1}{2} c_1(\alpha, \beta, n),$$

where $c_1(\alpha, \beta, n)$ is defined in (5.2) in Lemma 5.4 (this is possible, because $\Gamma(n; (\alpha-1)^\beta s)$ decays exponentially fast with s at infinity). Fix $t, x > 0$ such that $t\psi(1/x^2) \ge c$, and let $a = (\pi - \frac{\pi}{\alpha})/x$. Observe that

$$x^2 \psi^{-1}(\tfrac{1}{t}) = \frac{\psi^{-1}(1/t)}{\psi^{-1}(\psi(1/x^2))} \le \frac{\psi^{-1}(1/t)}{\psi^{-1}(c/t)} \le 1.$$

Hence, by Lemmas 5.3 and 5.4, if $t\psi(1/x^2) \ge c$, then

$$t^n |I_n(t, x, a)| \le \tfrac{2}{\pi} \Gamma(n; (\alpha-1)^\beta t\psi(1/x^2)) \le \frac{c_1(\alpha, \beta, n)}{2t\psi(1/x^2)}\,,$$
$$t^n J_n(t, x, a) \ge \frac{c_1(\alpha, \beta, n)}{tx\psi(1/x^2)\sqrt{\psi^{-1}(1/t)}} \ge \frac{c_1(\alpha, \beta, n)}{t\psi(1/x^2)}\,,$$

so that $|I_n(t, x, a)| \le \frac{1}{2} J_n(t, x, a)$. It follows that

$$\tfrac{1}{2} J_n(t, x, a) \le \left(-\frac{d}{dt}\right)^n \mathbb{P}(t < \tau_x < \infty) \le \tfrac{3}{2} J_n(t, x, a),$$

and the theorem follows now directly from Lemma 5.4, with $C_1(\alpha, n) = \frac{1}{2} c_1(\alpha, \beta, n)$, $C_2(\alpha, n) = \frac{3}{2} c_2(\alpha, \beta, n)$ (see (5.2) in Lemma 5.4) and $C_3(\alpha, n) = c$. □

Proof of Corollary 1.2. For brevity, denote the constants of Theorem 1.1 by $C_j = C_j(\alpha, 0)$ for $j = 1, 2, 3$; recall that $C_3 \ge 1$. Suppose first that $t\psi(1/x^2) \ge C_3$. By (2.6),

$$tx\psi(1/x^2)\sqrt{\psi^{-1}(1/t)} = t\psi(1/x^2)\sqrt{\frac{\psi^{-1}(1/t)}{\psi^{-1}(\psi(1/x^2))}} \ge \frac{t\psi(1/x^2)}{(t\psi(1/x^2))^{1/\alpha}} \ge C_3^{1-1/\alpha} \ge 1.$$

Hence, estimate (1.6) follows from (1.5) with arbitrary $\tilde{C}_1(\alpha) \le C_1$ and $\tilde{C}_2(\alpha) \ge 2C_2$. Consider now the case $t\psi(1/x^2) \le C_3$. Again by (2.6),

$$tx\psi(1/x^2)\sqrt{\psi^{-1}(1/t)} \le t\psi(1/x^2)\sqrt{\frac{\psi^{-1}(C_3/t)}{\psi^{-1}(\psi(1/x^2))}} \le \frac{t\psi(1/x^2)}{(t\psi(1/x^2)/C_3)^{1/\alpha}} \le C_3.$$

Hence,

$$\mathbb{P}(\tau_x > t) \le 1 \le \frac{2C_3}{1 + tx\psi(1/x^2)\sqrt{\psi^{-1}(1/t)}}\,.$$

Finally, by (1.5),

$$\mathbb{P}(\tau_x > t) \ge \mathbb{P}(\tau_x > C_3/\psi(1/x^2)) \ge \frac{C_1}{C_3 x\sqrt{\psi^{-1}(\psi(1/x^2)/C_3)}}$$
$$= \frac{C_1}{C_3}\sqrt{\frac{\psi^{-1}(1/x^2)}{\psi^{-1}(\psi(1/x^2)/C_3)}} \ge \frac{C_1}{C_3} \ge \frac{C_1}{C_3}\, \frac{1}{1 + tx\psi(1/x^2)\sqrt{\psi^{-1}(1/t)}}\,.$$

Therefore, (1.5) holds with arbitrary $\tilde{C}_1(\alpha) \le C_1/C_3$ and $\tilde{C}_2(\alpha) \ge 2C_3$. □

Remark 5.5. From the proof of Theorem 1.1 it follows that the constants in this result are given by

$$C_1(\alpha,n) = \frac{(\alpha-1)^2\gamma(n+\frac{1}{2};(\alpha-1)^2)}{4\pi^2}\,,$$
$$C_2(\alpha,n) = \frac{60(\gamma(n+1-\frac{1}{\alpha};1)+\Gamma(n+\frac{1}{2};1))}{\pi^2(\alpha-1)}\,,$$

and $C_3(\alpha,n) > 1$ is large enough, so that for $s \ge C_3(\alpha,n)$,

$$\tfrac{2}{\pi}s\,\Gamma(n;(\alpha-1)^2 s) \le C_1(\alpha,n).$$

In a similar way, in Corollary 1.2,

$$\tilde{C}_1(\alpha) = \frac{C_1(\alpha,0)}{C_3(\alpha,0)}, \qquad\qquad \tilde{C}_2(\alpha) = 2C_2(\alpha,0)+2C_3(\alpha,0).$$

Proof of Theorem 1.3. Part (a). As before, let $\beta = 2$. We claim that by the Dominated Convergence Theorem,

$$\lim_{x\to 0^+}(x\psi(1/x^2)J_n(t,x,\tfrac{2}{x})) = \frac{2}{\pi\Gamma(\gamma)|\cos\frac{\gamma\pi}{2}|}\int_0^\infty(\cos\vartheta_\lambda)^2 e^{-t\psi(\lambda^2)}\lambda^2(\psi'(\lambda^2))^2(\psi(\lambda^2))^{n-1}d\lambda$$

for all $n \ge 0$ and $t > 0$. Indeed, the left-hand side is the limit of integrals (see (5.1)), with integrands convergent pointwise to the integrand in the right-hand side by Lemma 4.5. Furthermore, by Lemma 4.4, the integrands in the left-hand side are bounded by

$$\frac{80}{\pi(\alpha-1)}\cos\vartheta_\lambda e^{-t\psi(\lambda^2)}\lambda^2(\psi'(\lambda^2))^2(\psi(\lambda^2))^{n-1},$$

which is easily shown to be integrable in $\lambda \in (0,\infty)$, because $\lambda^2\psi'(\lambda^2) \le \frac{\beta}{2}\psi(\lambda^2)$. The claim is proved.

On the other hand, by Lemma 5.3, for $x \in (0,1)$,

$$x\psi(1/x^2)|I_n(t,x,\tfrac{2}{x})| \le \frac{2x\psi(1/x^2)\Gamma(n;(\alpha-1)^\beta t\psi(1/x^2))}{\pi t^n} \le c(\alpha,\beta,n,t)x.$$

Part (b). Again let $\beta = 2$. Fix $x > 0$ and $a = \frac{2}{x}$. Observe that

$$t^{n+1}\sqrt{\psi^{-1}(1/t)}J_n(t,x,a) = \frac{4}{\pi}\,t^{n+1}\sqrt{\psi^{-1}(1/t)}\int_0^a e^{-t\psi(\lambda^2)}\psi'(\lambda^2)(\psi(\lambda^2))^n\,\frac{F_\lambda(x)}{2\lambda\psi'(\lambda^2)\cos\vartheta_\lambda}\,(\cos\vartheta_\lambda)^2\,\frac{\lambda^2\psi'(\lambda^2)}{\psi(\lambda^2)}\,d\lambda$$

for all $n \ge 0$ and $t > 0$. By Lemmas 4.6 and 3.2, and Karamata's theorem [1, Theorem 1.5.11],

$$\lim_{\lambda\to 0^+}\frac{F_\lambda(x)}{2\lambda\psi'(\lambda^2)\cos\vartheta_\lambda} = v(x), \qquad \lim_{\lambda\to 0^+}\vartheta_\lambda = \frac{\pi}{\delta}-\frac{\pi}{2}, \qquad \lim_{\lambda\to 0^+}\frac{\lambda^2\psi'(\lambda^2)}{\psi(\lambda^2)} = \frac{\delta}{2}\,.$$

We claim that

$$\lim_{t\to\infty}\left(\frac{4}{\pi}\,t^{n+1}\sqrt{\psi^{-1}(1/t)}e^{-t\psi(\lambda^2)}\psi'(\lambda^2)(\psi(\lambda^2))^n\mathbb{1}_{(0,a)}(\lambda)d\lambda\right) = \frac{2\Gamma(n+1-\frac{1}{\delta})}{\pi}\,\delta_0(d\lambda),$$

with the vague limit of measures in the left-hand side. Indeed, the density function converges to 0 uniformly on $[\varepsilon, a)$ for every $\varepsilon > 0$. Furthermore, by a substitution $s = t\psi(\lambda^2)$,

$$\lim_{t\to\infty} \left(\frac{4}{\pi}\, t^{n+1} \sqrt{\psi^{-1}(1/t)} \int_0^a e^{-t\psi(\lambda^2)} \psi'(\lambda^2) (\psi(\lambda^2))^n d\lambda \right)$$
$$= \lim_{t\to\infty} \left(\frac{2}{\pi} \int_0^{t\psi(a^2)} \sqrt{\frac{\psi^{-1}(1/t)}{\psi^{-1}(s/t)}}\, e^{-s} s^n ds \right) = \frac{2}{\pi} \int_0^\infty e^{-s} s^{n-1/\delta} ds = \frac{2\Gamma(n+1-\frac{1}{\delta})}{\pi} \,;$$

the second equality follows by the Dominated Convergence Theorem, because ψ^{-1} is regularly varying at zero with index $\frac{2}{\delta}$, and $\psi^{-1}(1/t)/\psi^{-1}(s/t) \le \max(s^{-2/\alpha}, s^{-2/\beta})$ for $s, t > 0$ by Lemma 2.2 and (2.6). The claim is proved.

It follows that

$$\lim_{t\to\infty} \Big(t^{n+1} \sqrt{\psi^{-1}(1/t)} J_n(t,x,a) \Big) = \tfrac{2}{\pi} \Gamma(n+1-\tfrac{1}{\delta}) v(x) (\cos(\tfrac{\pi}{\delta} - \tfrac{\pi}{2}))^2 \tfrac{\delta}{2}.$$

Finally, by Lemma 5.3,

$$t^{n+1} \sqrt{\psi^{-1}(1/t)} |I_n(t,x,a)| \le \tfrac{2}{\pi} t \sqrt{\psi^{-1}(1/t)}\, \Gamma(n; (\alpha-1)^\beta t\psi(1/x^2)),$$

and the right-hand side converges to 0 as $t \to \infty$. □

Remark 5.6. From the proof Theorem 1.3 it follows that in part (a),

$$\lim_{x\to 0} \Big(|x| \Psi(\tfrac{1}{|x|}) (-\tfrac{d}{dt})^n \mathbb{P}(\tau_x > t) \Big)$$
$$= \frac{1}{2\pi\Gamma(\gamma) |\cos\frac{\gamma\pi}{2}|} \int_0^\infty (\cos\vartheta_\lambda)^2 e^{-t\Psi(\lambda)} (\Psi'(\lambda))^2 (\Psi(\lambda))^{n-1} d\lambda,$$

with ϑ_λ given by (1.9). Also, in part (b),

$$\lim_{t\to\infty} \big(t^{n+1} \Psi^{-1}(\tfrac{1}{t}) (-\tfrac{d}{dt})^n \mathbb{P}(\tau_x > t) \big) = \frac{\delta\Gamma(n+1-\frac{1}{\delta})(\sin\frac{\pi}{\delta})^2}{\pi}\, v(x),$$

for all $n \ge 0$ and $x \in \mathbb{R} \setminus \{0\}$, where $v(x)$ is the compensated potential kernel of X.

Remark 5.7. The proofs clearly indicate that the hypotheses of Theorem 1.1 can be slightly relaxed to the following: $\psi(\xi^2)$ is the Lévy–Khintchine exponent of a symmetric Lévy process; $1/(1+\psi(\xi^2))$ is integrable; $(\psi_\lambda)_\lambda(\xi)$ is well-defined and $(\psi_\lambda)_\lambda(\xi)$ and $\xi/(\psi_\lambda)_\lambda(\xi)$ are increasing in $\xi > 0$ for all $\lambda > 0$; scaling-type condition (1.3) holds for some $\alpha \in (1, 2]$ and all $\xi > 0$, and a similar upper bound $\xi\Psi''(\xi)/\Psi'(\xi) \le \beta - 1$ holds for some $\beta \in (1, 2]$ and all $\xi > 0$ (the upper bound is now non-trivial also for $\beta = 2$). Apparently, these conditions can be further weakened at the price of more technical arguments. Since many important examples already belong to the class considered in this article, we decided to focus on simplicity rather than complete generality.

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Martin Kernels for Markov Processes with Jumps

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Abstract We prove the existence of boundary limits of ratios of positive harmonic functions for a wide class of Markov processes with jumps and irregular (possibly disconnected) domains of harmonicity, in the context of general metric measure spaces. As a corollary, we prove the uniqueness of the Martin kernel at each boundary point, that is, we identify the Martin boundary with the topological boundary. We also prove a Martin representation theorem for harmonic functions. Examples covered by our results include: strictly stable Lévy processes in $\mathbf{R}^d$ with positive continuous density of the Lévy measure; stable-like processes in $\mathbf{R}^d$ and in domains; and stable-like subordinate diffusions in metric measure spaces.

Keywords Markov process · Jump process · Killed process · Boundary Harnack inequality · Boundary limit · Martin kernel · Martin boundary · Martin representation

Mathematics Subject Classification (2010) 31C35 · 60J45 · 60J50 · 60J75

1 Introduction

The purpose of this article is to study boundary limits of ratios of positive functions which are harmonic in an arbitrary open set with respect to a Markov process with jumps. The proof of our main result, Theorem 2, relies on the *boundary Harnack inequality* for Markov

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processes with jumps, proved recently in [12], and the oscillation reduction argument, developed in [6] and [11]. As an application, we obtain Martin representation of harmonic functions in Theorem 3.

To explain the motivation for our research, we begin with a discussion of the classical case, where harmonicity has its usual meaning: f is harmonic in an open set D if $\Delta f = 0$ in D. The boundary Harnack inequality is a statement about positive harmonic functions in an open set, which are equal to zero on a part of the boundary. The result states that if D is regular enough (for example, a Lipschitz domain), x_0 is a boundary point of D, f and g are positive and harmonic in $D \cap B(x_0, R)$, and both f and g converge to 0 on $\partial D \cap B(x_0, R)$, then for every $r \in (0, R)$ the ratio f/g has bounded *relative oscillation* in $D \cap B(x_0, r)$:

$$\sup_{x \in D \cap B(x_0, r)} \frac{f(x)}{g(x)} \le c \inf_{x \in D \cap B(x_0, r)} \frac{f(x)}{g(x)} . \tag{1}$$

Here $c = c(D, x_0, r, R) - 1$ is a constant that depends only on the local geometric properties of D near x_0, and $B(x_0, r)$ denotes the ball of radius r, centred at x_0. The boundary Harnack inequality was first proved independently by A. Ancona ([5]), B. Dahlberg ([17]) and J.-M. Wu ([35]) for Lipschitz domains, and then extended by numerous authors to a wider class of domains and elliptic operators. We refer to [1–4, 31] for further discussion and references.

Under appropriate assumptions on the regularity of D, the estimate (1) turns out to be self-improving as $r \to 0^+$, in the sense that the constant c in Eq. 1 converges to 1 as $r \to 0^+$. Equivalently, the boundary limit

$$\lim_{\substack{x \to x_0 \\ x \in D}} \frac{f(x)}{g(x)} \tag{2}$$

exists. When D is a Lipschitz domain, then in fact $c(D, x_0, r, R)$ is of order r^β as $r \to 0^+$ for some $\beta > 0$, which means that f/g extends to a Hölder continuous function at x_0.

A closely related concept of *Martin representation* of positive harmonic functions was introduced by R. S. Martin in his beautiful article [32], more than three decades before the boundary Harnack inequality became available. Given the existence of limits (2) (for example, if D is a Lipschitz domain), Martin's result asserts that there is a one-to-one correspondence between positive harmonic functions f in D and finite positive measures μ on the boundary of D. The two objects are linked by the formula

$$f(x) = \int_{\partial D} M_D(x, z) \mu(dz),$$

where the *Martin kernel* is defined as the boundary limit of the ratio of *Green functions*:

$$M_D(x, z) = \lim_{\substack{y \to z \\ x \in D}} \frac{G_D(x, y)}{G_D(\tilde{x}, y)} . \tag{3}$$

Here $\tilde{x} \in D$ is an arbitrarily fixed reference point.

One of numerous equivalent definitions of harmonicity links harmonic functions with the Brownian motion: f is harmonic in D if and only if f has the *mean-value property* with respect to the distributions of the Brownian motion X_t at first exit times:

$$f(x) = \mathbf{E}_x f(X(\tau_U)) \tag{4}$$

for all bounded open sets U such that the closure of U is contained in D. Here $\mathbf{E}_x$ denotes the expectation (and $\mathbf{P}_x$ will denote the probability) corresponding to the Brownian motion process X_t that starts at x, and τ_U is the time of first exit from U:

$$\tau_U = \inf\{t \geq 0 : X_t \notin U\}.$$

This probabilistic definition has a number of advantages: it extends immediately to general Markov processes X_t, and it captures easily boundary conditions imposed on harmonic functions. More precisely, in the general statement of the boundary Harnack inequality one requires that positive harmonic functions f and g converge to zero at each boundary point in $\partial D \cap B(x_0, R)$ that is *regular for the Dirichlet problem*. This condition translates to requiring that Eq. 4 holds for all bounded open sets U such that $\overline{U} \subseteq D \cup (\partial D \cap B(x_0, R))$, with no reference to the notion of regular boundary points. Here we understand that $f = g = 0$ in $\partial D \cap B(x_0, R)$.

In this article we are interested in Markov processes with jumps, and from now on by saying that a function is harmonic we understand that it has the mean-value property (4) with respect to a Markov process X_t with jumps. In this case in order to evaluate $f(X(\tau_U))$ in Eq. 4 the function f needs to be defined everywhere, not just in D. For this reason one needs to replace the *boundary* condition $f = g = 0$ in $\partial D \cap B(x_0, R)$ in the statement of the boundary Harnack inequality with the *exterior* condition $f = g = 0$ in $D^c \cap B(x_0, R)$.

The history of the boundary Harnack inequality for Markov processes with jumps starts with the article by K. Bogdan ([6]), where he proved the result for the isotropic stable Lévy process (equivalently: for the fractional Laplace operator $-(-\Delta)^{\alpha/2}$) and Lipschitz domains. Later this was extended to more general sets ([11, 34]) and processes ([8, 13, 21–26]). Recently, a rather general result for Markov processes with jumps was proved in [12], and this is our starting point in the study of boundary limits (2).

The existence of the boundary limit (2) in this context was first proved independently by K. Bogdan ([7]) and by Z.-Q. Chen and R. Song ([14]) for the isotropic stable Lévy process and Lipschitz domains. This required an appropriate modification of the classical reasoning due to the presence of jumps. Since then essentially every time the boundary Harnack inequality was established for a given Markov process with jumps in a given class of domains, the existence of boundary limits (2) followed; see [27] for the most recent result of this kind. With two exceptions, however, the class of open sets under consideration was always limited to certain disconnected analogues of non-tangentially accessible domains, typically called *fat sets*. The first more general result is proved in [11] for the isotropic stable Lévy process, where completely arbitrary open sets are allowed. An extension to more general Markov processes with jumps, which in fact further extends the results of the present article, was obtained independently by P. Kim, R. Song and Z. Vondraček ([28–30]) soon after the present article has been submitted.

For the existence of boundary limits, we follow the approach of [11] using the boundary Harnack inequality of [12], and prove in our main results, Theorems 2 and 3, the existence of boundary limits of ratios of harmonic functions for arbitrary open sets and rather general Markov processes with jumps, as well as Martin representation of such functions. The application of the method developed in [11] in the present setting requires significant modifications. Further changes are introduced in order to make the description of the proof more accessible; for example, we first give a simpler argument which does not assert uniform convergence with respect to the domain of harmonicity, and only then explain how one improves it to get a domain-uniform version.

The proof of the Martin representation theorem for the isotropic stable Lévy processes in [11] is self-contained. It is possible to extend the method of [11] to our general setting, but that would require rather lengthy and technical arguments. For this reason, unlike in [11], we refer to the general theory of Martin boundary. Our argument still requires extension of some elements of [11] for more general Markov processes, but the most involved part of the proof is avoided. For an excellent exposition of the general theory of Martin boundary, we refer to Chapter 14 of [16].

We conclude the introduction with a description of the structure of this article. The assumptions for the boundary Harnack inequality of [12] are briefly recalled in Section 2. We omit a detailed discussion of these conditions and refer the interested reader to the original paper. Instead, we present a number of examples right after the statement of Theorems 2 and 3 in Section 3. We also provide a counter-example, which shows that the boundary limits (2) typically fail to exist in irregular domains when the process X_t has a non-trivial diffusion part. Finally, in Section 4 we prove Theorems 2 and 3.

2 Fundamental Assumptions for the Boundary Harnack Inequality

The formal statement of the assumptions for Theorem 2 requires some effort. We assume that $(\mathfrak{X}, d, m)$ is a locally compact metric measure space in which all bounded closed sets are compact and m has full support, and that $R_0 > 0$ (possibly $R_0 = \infty$) is a localisation radius such that $\mathfrak{X} \setminus B(x, r) \neq \varnothing$ if $x \in \mathfrak{X}$ and $0 < r < 2R_0$.

In [12] the following four conditions are introduced. A detailed discussion of these assumptions is beyond the scope of the present article, we refer the reader to [12] for more information. Here we only state the conditions, without explaining in a formal way the notions of *semi-polar* and *polar* sets, *processes in duality* X_t and $\hat{X}_t$, their *generators* $\mathfrak{A}$ and $\hat{\mathfrak{A}}$, densities $\nu(x, y)$ and $\hat{\nu}(x, y)$ (with respect to the measure m) of the *Lévy kernels* of X_t and $\hat{X}_t$, as well as their *Green functions* $G_D(x, y) = \hat{G}_D(y, x)$. We note that $\nu(x, y)$ describes the intensity of jumps from x to y and it is commonly used throughout the article. The Green function $G_D(x, y)$ is required for Theorem 3 only; informally, $G_D(x, y)$ is the average amount of time spent near y by the process X_t, started at x, until τ_D.

Assumption 1 *The Hunt processes X_t and $\hat{X}_t$ are dual with respect to the measure m. The transition semigroups of X_t and $\hat{X}_t$ are both Feller and strong Feller. Every semi-polar set of X_t is polar.*

Assumption 2 *There is a linear subspace $\mathfrak{D}$ of $\mathfrak{D}(\mathfrak{A}) \cap \mathfrak{D}(\hat{\mathfrak{A}})$ satisfying the following condition. If K is compact, D is open, and $K \subseteq D \subseteq \mathfrak{X}$, then there is $f \in \mathfrak{D}$ such that $f(x) = 1$ for $x \in K$, $f(x) = 0$ for $x \in \mathfrak{X} \setminus D$, $0 \le f(x) \le 1$ for $x \in \mathfrak{X}$, and the boundary of the set $\{x : f(x) > 0\}$ has measure m zero.*

Assumption 3 *We have $\nu(x, y) = \hat{\nu}(y, x) > 0$ for all $x, y \in \mathfrak{X}$, $x \neq y$. If $x_0 \in \mathfrak{X}$, $0 < r < R < R_0$, $x \in B(x_0, r)$ and $y \in \mathfrak{X} \setminus B(x_0, R)$, then*

$$C_{\text{Lévy}}^{-1}\nu(x_0, y) \le \nu(x, y) \le C_{\text{Lévy}}\nu(x_0, y), \; C_{\text{Lévy}}^{-1}\hat{\nu}(x_0, y) \le \hat{\nu}(x, y) \le C_{\text{Lévy}}\hat{\nu}(x_0, y), \quad (5)$$

with $C_{\text{Lévy}} = C_{\text{Lévy}}(x_0, r, R)$.

Assumption 4 *If $x_0 \in \mathfrak{X}$, $0 < r < s < R < R_0$ and $B = B(x_0, R)$, then*

$$C_{\text{Green}} = C_{\text{Green}}(x_0, r, s, R) = \sup_{x \in B(x_0, r)} \sup_{y \in \mathfrak{X} \setminus B(x_0, s)} \max(G_B(x, y), \hat{G}_B(x, y)) < \infty. \quad (6)$$

We denote

$$\rho(K, D) = \inf_f \sup_{x\in\mathfrak{X}} \max(\mathfrak{A}f(x), \hat{\mathfrak{A}}f(x)), \tag{7}$$

where the infimum is taken over all functions f described by the Assumption 2. If $x_0 \in \mathfrak{X}$ and $0 < r < R < R_0$, then we denote

$$C_{\text{Lévy-inf}}(x_0, r, R) = \inf_{y\in\overline{B}(x_0,R)\setminus B(x_0,r)} \min(\nu(x_0, y), \hat{\nu}(x_0, y)),$$

and

$$C_{\text{exit}}(x_0, r) = \sup_{x\in B(x_0,r)} \max(\mathbf{E}_x\tau_{B(x_0,r)}, \hat{\mathbf{E}}_x\hat{\tau}_{B(x_0,r)}).$$

Note that by Proposition 2.1 in [12], under Assumptions 1 through 3, $C_{\text{exit}}(x_0, r)$ is finite.

Following [6], we say that f is a *regular harmonic function* in an open set D if the mean-value property (4) holds with $U = D$. By the strong Markov property, this implies that Eq. 4 holds for arbitrary open $U \subseteq D$, so in particular f is harmonic in D.

We use the short-hand notation $f \approx cg$ for the two inequalities $c^{-1}g \le f \le cg$, where $c > 0$ is a positive constant. The following theorem is a reformulation of the main result of [12].

Theorem 1 (Lemma 3.2 and Theorems 3.4 and 3.5 in [12]) *Suppose that $x_0 \in \mathfrak{X}$, $0 < r_1 < r_2 < r_3 < r_6 < R_0$ and a non-negative function f is a regular harmonic function in $D \cap B(x_0, r_6)$, which is equal to zero in $B(x_0, r_6) \setminus D$. Then*

$$f(x) \approx C_{\text{BHI}}\mathbf{E}_x\tau_{D\cap B(x_0,r_2)} \int_{\mathfrak{X}\setminus B(x_0,r_3)} f(y)\nu(x_0, y)m(dy)$$

for $x \in D \cap B(x_0, r_1)$, where $C_{\text{BHI}} = C_{\text{BHI}}(x_0, r_1, r_2, r_3, r_6)$ is defined as

$$\begin{aligned} C_{\text{BHI}} = {} & C_{\text{Lévy}}(x_0, r_2, r_3) + 2\rho(\overline{B}(x_0, r_3) \setminus B(x_0, r_2), B(x_0, r_8) \setminus \overline{B}(x_0, r_1)) \\ & \times \left(C_{\text{Green}}(x_0, r_3, r_4, r_6) + \frac{C_{\text{exit}}(x_0, r_6)(C_{\text{Lévy}}(x_0, r_4, r_5))^2}{m(B(x_0, r_4))} \right) \\ & \times \left(\frac{\rho(\overline{B}(x_0, r_5), B(x_0, r_6))}{C_{\text{Lévy-inf}}(x_0, r_5, r_7)} + C_{\text{Lévy}}(x_0, r_6, r_7)m(B(x_0, r_6)) \right) \end{aligned}$$

for some r_4, r_5, r_7, r_8 such that $0 < r_1 < r_2 < r_3 < r_4 < r_5 < r_6 < r_7 < r_8$.

Note that it is important that f is non-negative everywhere, not just in D. Theorem 1 implies the more classical statement of the boundary Harnack inequality (Theorem 3.5 in [12]): if f and g satisfy the assumptions of Theorem 1, then

$$\sup_{x\in D\cap B(x_0,r_1)} \frac{f(x)}{g(x)} \le C_{\text{BHI}}^4 \inf_{x\in D\cap B(x_0,r_1)} \frac{f(x)}{g(x)}, \tag{8}$$

as in Eq. 1. We remark that although the original statement allows for an arbitrary sequence of radii, it will be sufficient for us to consider $r_1 = r$, $r_2 = 2r$, $r_3 = 3r$ and $r_6 = 4r$, and we will commonly write $C_{\text{BHI}} = C_{\text{BHI}}(x_0, r) = C_{\text{BHI}}(x_0, r, 2r, 3r, 4r)$ in this case.

3 Main Results and Examples

For the existence of limits, we introduce one more definition. If $x_0 \in \mathfrak{X}$ and $0 < r < R < R_0$, we let

$$C_{\text{Lévy-int}} = C_{\text{Lévy-int}}(x_0, r, R) = \frac{\int_{\mathfrak{X} \setminus B(x_0,r)} \nu(x_0, y) m(dy)}{\int_{\mathfrak{X} \setminus B(x_0,R)} \nu(x_0, y) m(dy)} . \tag{9}$$

Theorem 2 *Let $D \subseteq \mathfrak{X}$ be open, $x_0 \in \partial D$ and $R > 0$. Suppose that:*

(i) *X_t satisfies Assumptions 1 through 4;*
(ii) $\lim_{r \to 0^+} C_{\text{Lévy}}(x_0, r, R) = 1$*;*
(iii) *the constant $C_{\text{Lévy}}(x_0, r, 2r)$ is bounded in r, $0 < 2r < R_0$;*
(iv) *the constant $C_{\text{Lévy-int}}(x_0, r, 2r)$ is bounded in r, $0 < 2r < R_0$;*
(v) *the constant $C_{\text{BHI}}(x_0, r, 2r, 3r, 4r)$ is bounded in r, $0 < 4r < R_0$.*

Suppose furthermore that non-negative functions f and g are regular harmonic functions in $D \cap B(x_0, R)$ and are equal to zero in $B(x_0, R) \setminus D$. Then either one of f and g is zero everywhere in D, or the finite, positive boundary limit of $f(x)/g(x)$ exists as $x \to x_0$, $x \in D$. Furthermore,

$$\lim_{\substack{x \to x_0 \\ x \in D}} \frac{f(x)}{g(x)} = \lim_{r \to 0^+} \frac{\int_{\mathfrak{X} \setminus B(x_0,r)} \nu(x_0, y) f(y) m(dy)}{\int_{\mathfrak{X} \setminus B(x_0,r)} \nu(x_0, y) g(y) m(dy)} . \tag{10}$$

Remark 1 Condition (ii) is required only for *inaccessible* boundary points x_0, characterised by the property $\int_{D \cap B(x_0,R)} \mathbf{E}_y \tau_{D \cap B(x_0,R)} m(dy) < \infty$. The result for *accessible* boundary points x_0, for which the integral is infinite, holds under conditions (i) and (iii) through (v).

Remark 2 Theorem 2 also holds with $g(x) = \mathbf{E}_x \tau_{D \cap B(x_0,R)}$. This is formally shown in Section 4.4, but the informal explanation is rather straightforward: g is *essentially* a regular harmonic function in $D \cap B(x_0, R)$ (in sharp contrast with the case of continuous Markov processes).

Indeed, suppose that $\mathfrak{X}$ is unbounded, D is a bounded open set and that $C_{\text{Lévy}}(x_0, r, R)$ converges to 1 as $R \to \infty$. By Dynkin's formula (see Lemma 2 and estimate (14) below),

$$\mathbf{E}_x \tau_D = \lim_{R \to \infty} \frac{\mathbf{P}_x(X(\tau_D) \in \mathfrak{X} \setminus B(x_0, R))}{\int_{\mathfrak{X} \setminus B(x_0,R)} \nu(x_0, y) m(dy)}$$

is the limit of regular harmonic functions in D. Since the estimates in Theorem 2 are uniform in f and g, we obtain the desired result. (Note that the formal argument is completely different and requires no further assumptions on $\mathfrak{X}$ and X_t.)

Remark 3 As remarked in the introduction, the limit in Eq. 10 exists if and only if the relative oscillation of f and g converges to one, that is,

$$\lim_{r \to 0^+} \frac{\sup_{x \in D \cap B(x_0,r)} (f(x)/g(x))}{\inf_{x \in D \cap B(x_0,r)} (f(x)/g(x))} = 1.$$

By inspecting the proof of Theorem 2, one immediately sees that, given D and x_0, the boundary limits exist uniformly in f and g, in the sense that

$$\lim_{r \to 0^+} \sup_{f,g} \frac{\sup_{x \in D \cap B(x_0,r)} (f(x)/g(x))}{\inf_{x \in D \cap B(x_0,r)} (f(x)/g(x))} = 1,$$

with the supremum taken over all f and g satisfying the assumptions of the theorem. We remark that in fact one can prove uniformity also in D, just as in [11], by appropriately modifying the final part of the proof. More formally,

$$\lim_{r\to 0^+} \sup_{D,f,g} \frac{\sup_{x\in D\cap B(x_0,r)}(f(x)/g(x))}{\inf_{x\in D\cap B(x_0,r)}(f(x)/g(x))} = 1, \tag{11}$$

where the supremum is taken over all open sets D and f and g satisfying the assumptions of the theorem (here we let the ratio sup / inf be equal to 1 if $D \cap B(x_0, r)$ is empty). The proof of this result is sketched in Section 4.4.

Remark 4 It is not necessary to assume that $x_0 \in \partial D$ in Theorem 2. For $x_0 \notin \overline{D}$ the statement is void, but for $x_0 \in D$ we obtain *relative continuity* of positive harmonic functions: if f and g are positive harmonic functions in D, then f/g is continuous in D. By Remark 3, the family of functions f/g is in fact *relatively equicontinuous* at x_0, in the sense that the functions $\log(f/g)$ are equicontinuous at x_0.

If the process is conservative, then the constant $g(x) = 1$ is harmonic. In the general case, $\mathbf{P}^x(X(\tau_D) = \partial)$ is continuous (this is proved as in [15]; with the notation of that article, $\mathbf{P}^x(X(\tau_D) = \partial) = \mathbf{P}^x(T_{\mathfrak{X}\setminus D} = \infty)$), and so the harmonic function $g(x) = \mathbf{P}^x(X(\tau_D) \in \mathfrak{X}) = 1 - \mathbf{P}^x(X(\tau_D) = \partial)$ is positive, continuous and harmonic in D.

Consequently, positive harmonic functions are relatively equicontinuous at x_0. If in addition the characteristics of the process (that is, the constants in conditions (ii) through (v)) do not depend on x_0, then positive functions harmonic in D are in fact uniformly relatively equicontinuous in every compact subset of D.

Before we discuss examples, we provide one application. Recall that the Green function $G_D(x, y)$ is the density of the mean occupation measure of X_t up to τ_D, that is,

$$\int_A G_D(x, y)m(dy) = \mathbf{E}_x \int_0^{\tau_D} \mathbf{1}_A(X_s)ds.$$

Under Assumptions 1 and 4, there is a version of $G_D(x, y)$ which is a harmonic function of $x \in D \setminus \{y\}$, and a co-harmonic (that is, harmonic for the dual process) function of $y \in D \setminus \{x\}$. Hence, Theorem 2 (or, more precisely, its version for the dual process) immediately implies the existence of the Martin kernel

$$M_D(x, z) = \lim_{\substack{y\to z\\ x\in D}} \frac{G_D(x, y)}{G_D(\tilde{x}, y)}.$$

for $z = x_0$ (this is exactly the same as the classical definition (3)). Informally, the Martin boundary $\partial_M D$ of a set D is the set of all possible ways a point $y \in D$ approaches the boundary in such a way that the ratio $G_D(x, y)/G_D(\tilde{x}, y)$ converges for every $x \in D$ (with arbitrarily fixed $\tilde{x} \in D$). More formally, $D \cup \partial_M D$ is the *Constantinescu–Cornea compactification* of D with respect to the family of functions $\{G_D(x, \cdot)/G_D(\tilde{x}, \cdot) : x \in D\}$: the smallest compact space which contains D and on which these functions have continuous extensions.

Theorem 3 *Let $D \subseteq \mathfrak{X}$ be bounded and open, and if $\mathfrak{X}$ is compact, then assume in addition that $\mathbf{E}_x\tau_D$ and $\hat{\mathbf{E}}_x\hat{\tau}_D$ are finite and bounded in $x \in D$. Suppose that the assumptions of Theorem 2 are satisfied uniformly for all $x_0 \in \overline{D}$. Then the following assertions hold.*

(a) *The Martin boundary $\partial_M D$ coincides with the topological boundary ∂D.*

(b) *The Martin kernel* $M_D(x,z)$ *is a harmonic function in* D *with respect to* x *if and only if* z *is an accessible boundary point:* $\int_{D\cap B(x_0,R)} \mathbf{E}_y \tau_{D\cap B(x_0,R)} m(dy) = \infty$.
(c) *If* z *is an accessible boundary point, then* $M_D(x,z)$ *is a minimal harmonic function: if* f *is a harmonic function in* D *and* $0 \le f(x) \le M_D(x,z)$ *for all* $x \in \mathfrak{X}$*, then* $f(x)$ *is a multiple of* $M_D(x,z)$.
(d) *Every non-negative function* f *which is a harmonic function in* D *has a unique representation*

$$f(x) = \int_{\mathfrak{X}\setminus(D\cup\partial_m D)} \left(\int_D G_D(x,y)\nu(y,z)m(dy)\right) f(z)m(dz) + \int_{\partial_m D} M_D(x,z)\mu(dz), \tag{12}$$

where μ *is a measure on* $\partial_m D$*, the set of accessible boundary points of* D.
(e) *Conversely, given any non-negative function* f *and any measure* μ *on* $\partial_m D$*, the right-hand side of* Eq. 12 *is either a harmonic function in* D *or infinity everywhere in* D.

Remark 5 The terms *accessible* and *inaccessible* correspond to the probabilistic theory of Martin boundary. To be specific, the process X_t killed at the time of first exit from D and conditioned in the sense of Doob by the Martin kernel $M_D(\cdot,z)$ converges at its lifetime to z when z is accessible, and dies out in D when z is inaccessible. We refer to [16] for more information.

Remark 6 Unlike in the case of isotropic stable Lévy processes in [11], description of the infinite part of the Martin boundary of D for unbounded open sets is a completely different problem. This issue is addressed in a recent work of P. Kim, R. Song and Z. Vondraček ([28, 30]).

Remark 7 In order to apply the results of [16] about general theory of Martin representation, one requires the dual of the Green operator $\hat{G}_D$ to map bounded functions into bounded continuous ones (a strong Feller property for the Green operator, Hypothesis 13.42 in [16]). In particular, $\hat{\mathbf{E}}_x \hat{\tau}_D = \hat{G}_D \mathbf{1}(x)$ needs to be bounded in D. If $\mathfrak{X}$ is unbounded, then $\hat{\mathbf{E}}_x \hat{\tau}_D$ is bounded (this follows, for example, by the argument used in the proof of Proposition 2.1 in [12]). If, however, $\mathfrak{X}$ is bounded (and hence compact), then one needs to assume boundedness of $\hat{\mathbf{E}}_x \hat{\tau}_D$ explicitly (indeed, when X_t is conservative and $D = \mathfrak{X}$, then clearly $\hat{\tau}_D = \infty$ with probability one).

Boundedness of $\mathbf{E}_x \tau_D$ is assumed in order to keep perfect symmetry between X_t and $\hat{X}_t$ (which makes the proof easier to follow). Note, however, that this is a rather mild assumption. Indeed, it is rather easy to see that if $\mathfrak{X}$ is compact and $\mathfrak{X} \setminus D$ is not a polar set, then there is $\varepsilon > 0$ such that $\mathbf{P}_x(\tau_D < 1) > \varepsilon$ and $\hat{\mathbf{P}}_x(\hat{\tau}_D < 1) > \varepsilon$ for all $x \in \mathfrak{X}$, and therefore $\mathbf{E}_x \tau_D$ and $\hat{\mathbf{E}}_x \hat{\tau}_D$ are bounded.

The boundary Harnack inequality stated in Theorem 1 was applied to a variety of Markov processes in Section 5 of [12]. The *scale-invariant* version of Theorem 1 under α*-stable-like scaling* discussed therein already asserts conditions (i), (iii) and (v) in Theorem 2. Verification of the remaining conditions (ii) and (iv) is typically straightforward, and we obtain several classes of processes for which Theorems 2 and 3 apply.

In our first example, we use the result of Example 5.5 in [12], where the boundary Harnack inequality for Lévy processes is considered. In the asymmetric case, equality of the notions of semi-polar and polar sets (in Assumption 1) is not trivial, and this was apparently overlooked in [12]. Fortunately, for all asymmetric Lévy processes listed therein, this condition is satisfied by Theorem 2 in [33].

Example 1 (Strictly stable Lévy processes) Let m be the Lebesgue measure in $\mathbf{R}^d$, $R_0 = \infty$. Suppose that X_t is a strictly α-stable Lévy process in $\mathbf{R}^d$, where $d \geq 1$ and $0 < \alpha < 2$. Suppose, furthermore, that the Lévy measure of X_t has a density function of the form $\nu(z) = \varphi(z/|z|)|z|^{-d-\alpha}$, with φ continuous and positive on the unit sphere (for Lévy processes, $\nu(x, y) = \nu(y-x)$). It is easy to see that $C_{\text{Lévy}}(x_0, r, R)$ converges to 1 as $r \to 0^+$ and that $C_{\text{Lévy-int}}(x_0, r, R) = (R/r)^\alpha$. By Example 5.5 in [12], X_t satisfies the other assumptions of Theorem 2, and so we may use Theorems 2 and 3.

We remark that the above example can be extended to more general Lévy processes, including many subordinate Brownian motions and, more generally, unimodal isotropic Lévy processes. This is based on estimates obtained recently in [9, 10, 18, 20] and will be studied in detail in [19]. Other extensions can be obtained by allowing the Lévy kernel to depend on x or restricting it to a domain, as described in the following two examples.

Example 2 (Stable-like processes) Let m be the Lebesgue measure in $\mathbf{R}^d$, $R_0 = \infty$. Suppose that $0 < \alpha < 2$ and

$$\nu(x, y) = \varphi(x, y)|x - y|^{-d-\alpha},$$

where φ is symmetric (that is, $\varphi(x, y) = \varphi(y, x)$), bounded by positive constants, smooth, and has bounded partial derivatives of all orders. As in Example 5.6 in [12], in this case there is a pure-jump process X_t with the Lévy kernel $\nu(x, y)m(dy)$, and the assumptions of Theorem 2 are satisfied.

Example 3 (Reflected stable processes) Let $0 < \alpha < 2$. Let $\mathfrak{X}$ be the closure of either a Lipschitz domain in $\mathbf{R}^d$ if $\alpha < 1$ or a $C^{1,\alpha+\varepsilon}$ domain in $\mathbf{R}^d$ if $\alpha \geq 1$ (with some $\varepsilon > 0$). Let m be the Lebesgue measure on $\mathfrak{X}$, and $\nu(x, y) = c|x - y|^{-d-\alpha}$ for some $c > 0$. Again as in Example 5.6 in [12], there is a pure-jump process X_t with the Lévy kernel $\nu(x, y)m(dy)$, and the assumptions of Theorem 2 are satisfied for some $R_0 > 0$.

The state space $\mathfrak{X}$ need not be Euclidean.

Example 4 (Stable-like subordinate diffusions) Let $\mathfrak{X}$ be a sufficiently regular metric measure space in which there exists a diffusion process. For a rigorous definition, we refer to Example 5.7 in [12]; examples include Riemannian manifolds, Sierpiński gaskets or the Sierpiński carpet. Suppose that $0 < \alpha < d_{\text{w}}$, where d_{w} is the *walk dimension* of $\mathfrak{X}$ (that is, an approximate scaling exponent for the diffusion process). Finally, let X_t be a process subordinate to the diffusion process, corresponding to the (α/d_{w})-stable subordinator. In Example 5.7 in [12] it is shown that X_t satisfies conditions (i), (iii) and (v) of Theorem 2, and one easily proves that $C_{\text{Lévy-int}}(x_0, r, R) \leq c(R/r)^\alpha$ for some $c > 0$. Verification of (ii) requires some work, especially when $\mathfrak{X}$ is unbounded. For this reason, we only sketch the argument for compact $\mathfrak{X}$. For some $c > 0$ we have

$$\nu(x, y) = c \int_0^\infty t^{-1-\alpha/d_{\text{w}}} q_t(x, y) dt,$$

where $q_t(x, y)$ is the transition density of the diffusion process. Since for each $t > 0$, q_t is Hölder continuous, it is easy to see that $\nu(x, y)$ is positive and uniformly continuous in $x \in \overline{B}(x_0, r)$, $y \in \mathfrak{X} \setminus B(x_0, R)$, which clearly implies condition (ii). It follows that Theorems 2 and 3 apply to stable-like subordinate diffusions in compact metric measure spaces.

Surprisingly, Theorem 2 is not influenced by killing.

Example 5 (Processes with a multiplicative functional) Let M_t be a strong continuous multiplicative functional such that $M_0 = 1$ with probability one for all starting points $x \in \mathfrak{X}$. Such a functional describes gradual killing of the process X_t, and is typically obtained as the Feynman–Kac functional $M_t = \exp(-\int_0^t V(X_s)ds)$ for some non-negative function V. A function f is said to be harmonic with respect to the pair (X_t, M_t) if it has the mean-value property

$$f(x) = \mathbf{E}_x(f(X(\tau_U))M(\tau_U))$$

instead of Eq. 4. As in Theorem 5.10 in [12], if the assumptions of Theorem 2 are satisfied by the process X_t, then the conclusion also holds for functions harmonic with respect to the pair (X_t, M_t).

Our final example shows that when X_t has non-vanishing diffusion part, one cannot expect the existence of boundary limits (2) unless some geometric restrictions on D are imposed. For corresponding positive results in smooth domains, see [24].

Example 6 (Mixture of Brownian motion and stable process) Let $\mathfrak{X} = \mathbf{R}$ and let m be the Lebesgue measure. Let X_t be a one-dimensional Lévy process which is the sum of two independent Lévy processes: the Brownian motion and the symmetric α-stable Lévy process for some $\alpha \in (1, 2)$. That is, the characteristic exponent of X_t is given by $c_1\xi^2 + c_2|\xi|^\alpha$ for some $c_1, c_2 > 0$. Denote $D = (-1, 1) \setminus \{0\}$. Let $p_t(y - x)$ be the continuous version of the transition density of X_t. Then the three functions

$$u(x) = x, v(x) = \int_0^\infty (p_t(0) - p_t(x))dt, w(x) = \mathbf{E}_x|X(\tau_D)|$$

are regular harmonic in D: for u this is just the martingale property of X_t, for v (the *compensated potential kernel* of X_t) this is proved, for example, in [36], while for w it follows directly from the definition. Furthermore, $u(0) = v(0) = w(0) = 0$ and $v(x) = v(-x)$, $w(x) = w(-x)$. It is known that

$$v(x) \approx c_3 \min(|x|, |x|^{\alpha-1})$$

for $x \in \mathbf{R}$, with $c_3 = c_3(c_1, c_2, \alpha)$ (see, for example, Lemma 2.14 in [20]). In particular, $v(x) \approx c_3|x|$ for $x \in D$. Finally, by the boundary Harnack inequality given in Theorem 1 (see Examples 5.5 and 5.13 in [12] for a detailed discussion), we have

$$w(x) \approx c_4 v(x) \approx c_3 c_4 |x|$$

for $x \in \left(-\frac{1}{2}, \frac{1}{2}\right)$, with $c_4 = c_4(c_1, c_2, \alpha)$. Let us define

$$\begin{aligned} f(x) &= w(x) + u(x) = 2\mathbf{E}_x(|X(\tau_D)|\mathbf{1}_{[1,\infty)}(X(\tau_D))), \\ g(x) &= w(x) - u(x) = 2\mathbf{E}_x(|X(\tau_D)|\mathbf{1}_{(-\infty,-1]}(X(\tau_D))). \end{aligned}$$

Then f and g are non-negative, regular harmonic in D and equal to zero in $(-1, 1) \setminus D = \{0\}$, so that they satisfy the assumptions of Theorem 2. On the other hand,

$$\frac{f(x)}{g(x)} - \frac{f(-x)}{g(-x)} = \frac{w(x)+x}{w(x)-x} - \frac{w(x)-x}{w(x)+x} = \frac{4xw(x)}{(w(x))^2 - x^2}$$

for $x \in D$. Since $t/(t^2 - x^2)$ is decreasing in $t \in (x, \infty)$, and $w(x) \le c_3 c_4 x$ for $x \in (0, \frac{1}{2})$, we obtain

$$\frac{f(x)}{g(x)} - \frac{f(-x)}{g(-x)} \ge \frac{4c_3c_4}{(c_3c_4)^2 - 1}$$

for $x \in \left(0, \frac{1}{2}\right)$. In particular, the limit of $f(x)/g(x)$ as $x \to 0$ does not exist.

4 Proofs of Main Results

In this section we prove Theorem 2. We will always assume that x_0, R and D are fixed, where $x_0 \in \mathfrak{X}$, $0 < 2R < R_0$ and $D \subseteq B(x_0, R)$ is an open set. It is also understood that $x_0 \in \partial D$, although, at least formally, the argument extends also to $x_0 \in D$ and $x_0 \notin \overline{D}$. Recall that the notation $f \approx cg$ stands for $c^{-1}g \le f \le cg$ with $c > 0$.

We denote $B_r = B(x_0, r)$, $B_{r,s} = B_s \setminus B_r$, $D_r = D \cap B_r$ and $D_{r,s} = D_s \setminus D_r$ when $0 \le r \le s \le R$. We furthermore define $D_{r,\infty} = D_{r,R} \cup (\mathfrak{X} \setminus B_R)$. For a non-negative function f we let

$$M_{r,\infty}(f) = \int_{\mathfrak{X}\setminus B_r} f(y)\nu(x_0, y)m(dy),\ M_{r,s}(f) = \int_{B_{r,s}} f(y)\nu(x_0, y)m(dy).$$

Finally, we let $s_D(x) = \mathbf{E}_x \tau_D$.

To simplify the notation, we drop D from the notation in subscripts whenever possible, and we write $\tau_r = \tau_{D_r}$, $\tau_{r,s} = \tau_{D_{r,s}}$, $s_r(x) = s_{D_r}(x)$, $\mathbf{1}_{r,s}(x) = \mathbf{1}_{D_{r,s}}(x)$ etc.

Our argument is based on the boundary Harnack inequality of [12], stated in Theorem 1. Under the assumptions of Theorem 2, the constant $C_{\mathrm{BHI}}(x_0, r, 2r, 3r, 4r)$ can be chosen so that it does not depend on r, as long as $0 < 4r \le R$, and it will be denoted simply by C_{BHI} (recall that x_0 and R are fixed). In a similar way, we denote $C_{\text{Lévy}} = C_{\text{Lévy}}(x_0, r, 2r)$ (with $0 < 2r < R_0$) and $C_{\text{Lévy-int}} = C_{\text{Lévy-int}}(x_0, r, 2r)$ (with $0 < 2r < R_0$), chosen independently of r. With one exception, we will only use constants C_{BHI}, $C_{\text{Lévy}}$ and $C_{\text{Lévy-int}}$ with these parameters.

We prove Theorem 2 by considering separately two types of boundary points, which are called *accessible* and *inaccessible* in [11]. First, however, we introduce some further notation and prove preliminary estimates.

4.1 Decomposition of Harmonic Functions

From now on f and g are functions satisfying the assumptions of Theorem 2, and we assume that neither f nor g is equal to zero almost everywhere. Note that this implies that f and g are strictly positive in D. Whenever $0 < r < s \le R$, we decompose f into the sum of two functions, $f_{r,s}$ and $\tilde{f}_{r,s}$, which correspond to the process X_t exiting D_r near its boundary (into $D_{r,s}$) and away of its boundary (into $D_{s,\infty}$):

$$f_{r,s}(x) = \mathbf{E}_x((f\mathbf{1}_{r,s})(X(\tau_r))), \qquad \tilde{f}_{r,s}(x) = \mathbf{E}_x((f\mathbf{1}_{s,\infty})(X(\tau_r))).$$

Not unexpectedly, a similar notation is used for the function g. Clearly, $f = f_{r,s} + \tilde{f}_{r,s}$, and both $f_{r,s}$ and $\tilde{f}_{r,s}$ are non-negative regular harmonic functions in D_r which are equal to zero in $B_r \setminus D_r$. Therefore, we can apply Theorem 1 to $f_{4r,s}$ and $\tilde{f}_{4r,s}$ whenever $0 < 4r < s \le R$.

Note that by Theorem 1 $\left(\text{with } r = \frac{R}{4}\right)$, we have

$$f(x) \approx C_{\mathrm{BHI}} M_{3R/4,\infty}(f) \mathbf{E}_x \tau_{2R/4}$$

for $x \in D_{R/4}$. Therefore,

$$M_{r,s}(f) \approx C_{\mathrm{BHI}} M_{3R/4,\infty}(f) M_{r,s}(s_{R/2}) \tag{13}$$

whenever $0 \le r \le s \le \frac{R}{4}$. The next result states, in particular, that there is little difference whether we write $s_{R/2}$ or s_R in the above estimate.

Lemma 1 *If* $0 < 8r \le R$, *then*

$$\mathbf{E}_x \tau_{4r} \le \mathbf{E}_x \tau_{8r} \le (1 + C_{\mathrm{BHI}} C_{\text{Lévy}} C_{\text{Lévy-int}}^3) \mathbf{E}_x \tau_{4r}$$

for $x \in D_r$.

Proof The first inequality is clear. For the other one, we use the strong Markov property and Theorem 1:

$$\begin{aligned} \mathbf{E}_x \tau_{8r} - \mathbf{E}_x \tau_{4r} &= \mathbf{E}_x s_{8r}(X(\tau_{4r})) \\ &\le C_{\mathrm{BHI}} \mathbf{E}_x \tau_{2r} \int_{\mathfrak{X} \setminus B_{3r}} \mathbf{E}_y s_{8r}(X(\tau_{4r})) \nu(x_0, y) m(dy) \\ &\le C_{\mathrm{BHI}} \mathbf{E}_x \tau_{4r} \int_{\mathfrak{X} \setminus B_{2r}} \mathbf{E}_y \tau_{8r} \nu(x_0, y) m(dy). \end{aligned}$$

Furthermore, by Proposition 2.1 in [12] (combined with the last displayed formula in the proof of this result),

$$\begin{aligned} \int_{\mathfrak{X} \setminus B_{2r}} \mathbf{E}_y \tau_{8r} \nu(x_0, y) m(dy) &\le \left(\sup_{x \in \mathfrak{X}} \mathbf{E}_x \tau_{B_{8r}} \right) \int_{\mathfrak{X} \setminus B_{2r}} \nu(x_0, y) m(dy) \\ &\le C_{\text{Lévy}} \frac{\int_{\mathfrak{X} \setminus B_{2r}} \nu(x_0, y) m(dy)}{\int_{\mathfrak{X} \setminus B_{16r}} \nu(x_0, y) m(dy)}. \end{aligned}$$

It remains to use (9). □

For convenience, we denote

$$C_\tau = 1 + C_{\mathrm{BHI}} C_{\text{Lévy}} C_{\text{Lévy-int}}^3,$$

so that $s_{4r}(x) \approx C_\tau s_{8r}(x)$ if $0 < 8r \le R$ and $x \in D_r$.

Our next result compares $f_{8r,s}$ with $\tilde{f}_{8r,s}$. For $f_{8r,s}$, we will use Theorem 1, which states that in D_{2r} we have $f_{8r,s} \approx C_{\mathrm{BHI}} M_{6r,\infty}(f_{8r,s}) \mathbf{E}_x \tau_{4r}$. The same estimate can be written down for $\tilde{f}_{8r,s}$. However, $M_{6r,\infty}(\tilde{f}_{8r,s})$ involves an integral of $\tilde{f}_{8r,s}$ over $D_{6r,8r}$, which is often problematic. A much better estimate for $\tilde{f}_{8r,s}$ can be easily obtained from the following corollary of Dynkin's formula for X_t.

Lemma 2 (formula (2.12) in [12]) *Let $D \subseteq \mathfrak{X}$ be open and bounded, and let f be a non-negative function equal to zero in $\overline{D}$. Then*

$$\mathbf{E}_x f(X(\tau_D)) = \mathbf{E}_x \int_0^{\tau_D} \int_{\mathfrak{X} \setminus D} \nu(X_t, y) f(y) m(dy) dt \tag{14}$$

for $x \in D$.

Using the definition of $\tilde{f}_{8r,s}$ and Eq. 5 to substitute $\nu(x_0, y)$ for $\nu(X_t, y)$ in Eq. 14, we have

$$\tilde{f}_{8r,s}(x) \approx C_{\text{Lévy}}(x_0, 8r, s) M_{s,\infty}(f) \mathbf{E}_x \tau_{8r}. \tag{15}$$

Note that not only we have $M_{s,\infty}(f)$ instead of $M_{6r,\infty}(\tilde{f}_{8r,s})$, but also the constant $C_{\text{Lévy}}(x_0, 8r, s)$ tends to 1 as $r \to 0^+$.

Lemma 3 *If $0 < 8r \le s \le \frac{R}{4}$, then*

$$\frac{f_{8r,s}(x)}{\tilde{f}_{8r,s}(x)} \le C_{\text{BHI}}^4 \frac{M_{6r,s}(s_{R/2})}{1 + M_{s,R/4}(s_{R/2})}$$

for $x \in D_{2r}$. If $0 < 16r \le s \le \frac{R}{24}$, then

$$\frac{f_{8r,s}(x)}{\tilde{f}_{8r,s}(x)} \ge C_{\text{BHI}}^{-3} C_{\text{Lévy}}^{-1} C_\tau^{-3} \frac{M_{8r,s}(s_{R/2})}{1 + M_{s,R/4}(s_{R/2})}$$

for $x \in D_r$.

Proof By Theorem 1,

$$\begin{aligned} f_{8r,s}(x) &\le C_{\text{BHI}} M_{6r,\infty}(f_{8r,s}) \mathbf{E}_x \tau_{4r}, \\ \tilde{f}_{8r,s}(x) &\ge C_{\text{BHI}}^{-1} M_{6r,\infty}(\tilde{f}_{8r,s}) \mathbf{E}_x \tau_{4r}. \end{aligned}$$

Furthermore,

$$\begin{aligned} M_{6r,\infty}(f_{8r,s}) &= M_{6r,s}(f_{8r,s}) \le M_{6r,s}(f), \\ M_{6r,\infty}(\tilde{f}_{8r,s}) &\ge M_{s,\infty}(\tilde{f}_{8r,s}) = M_{s,\infty}(f) \ge M_{3R/4,\infty}(f) + M_{s,R/4}(f). \end{aligned}$$

Finally, by Eq. 13,

$$\begin{aligned} M_{6r,s}(f) &\le C_{\text{BHI}} M_{3R/4,\infty}(f) M_{6r,s}(s_{R/2}), \\ M_{s,R/4}(f) &\ge C_{\text{BHI}}^{-1} M_{3R/4,\infty}(f) M_{s,R/4}(s_{R/2}). \end{aligned}$$

We conclude that

$$\frac{f_{8r,s}(x)}{\tilde{f}_{8r,s}(x)} \le C_{\text{BHI}}^4 \frac{M_{6r,s}(s_{R/2})}{1 + M_{s,R/4}(s_{R/2})},$$

which is the desired upper bound. The lower bound is proved in a somewhat more complicated way. By Theorem 1 and estimate (15),

$$\begin{aligned} f_{8r,s}(x) &\ge C_{\text{BHI}}^{-1} M_{6r,\infty}(f_{8r,s}) \mathbf{E}_x \tau_{4r}, \\ \tilde{f}_{8r,s}(x) &\le C_{\text{Lévy}} M_{s,\infty}(f) \mathbf{E}_x \tau_{8r} \end{aligned}$$

(we can write $C_{\text{Lévy}} = C_{\text{Lévy}}(x_0, 8r, 16r)$ in the second inequality because $s \geq 16r$). By Lemma 1, $\mathbf{E}_x \tau_{8r} \leq C_\tau \mathbf{E}_x \tau_{4r}$. Furthermore, by Theorem 1 (as in Eq. 13, but with R replaced by $R/3$) and again Lemma 1,

$$\begin{aligned} M_{6r,\infty}(f_{8r,s}) &= M_{6r,s}(f_{8r,s}) \geq M_{8r,s}(f_{8r,s}) = M_{8r,s}(f) \\ &\geq C_{\text{BHI}}^{-1} M_{R/4,\infty}(f) M_{8r,s}(s_{R/6}) \\ &\geq C_{\text{BHI}}^{-1} C_\tau^{-2} M_{R/4,\infty}(f) M_{8r,s}(s_{2R/3}). \end{aligned}$$

On the other hand, by Eq. 13,

$$\begin{aligned} M_{s,\infty}(f) &= M_{s,R/4}(f) + M_{R/4,\infty}(f) \\ &\leq C_{\text{BHI}} M_{3R/4,\infty}(f) M_{s,R/4}(s_{R/2}) + M_{R/4,\infty}(f) \\ &\leq M_{R/4,\infty}(f)(1 + C_{\text{BHI}} M_{s,R/4}(s_{R/2})). \end{aligned}$$

We conclude that

$$\frac{f_{8r,s}(x)}{\tilde{f}_{8r,s}(x)} \geq C_{\text{BHI}}^{-3} C_{\text{Lévy}}^{-1} C_\tau^{-3} \frac{M_{8r,s}(s_{R/2})}{1 + M_{s,R/4}(s_{R/2})},$$

as desired. □

4.2 Inaccessible Boundary Points

Throughout this part we assume that x_0 is *inaccessible*, that is,

$$M_{0,\infty}(s_R) = \int_{D_R} \mathbf{E}_y \tau_R \, \nu(x_0, y) m(dy) < \infty.$$

In this case $f_{8r,s}$ and $g_{8r,s}$ turn out to be negligible compared to $\tilde{f}_{8r,s}$ and $\tilde{g}_{8r,s}$ for sufficiently small r and s.

Clearly, $M_{0,\infty}(s_{R/2}) \leq M_{0,\infty}(s_R) < \infty$. We remark that by Eq. 13,

$$\begin{aligned} M_{0,\infty}(f) &= M_{0,R/4}(f) + M_{R/4,\infty}(f) \\ &\leq C_{\text{BHI}} M_{3R/4,\infty}(f) M_{0,R/4}(s_{R/2}) + M_{R/4,\infty}(f) < \infty, \end{aligned}$$

and $M_{0,\infty}(g) < \infty$ by the same argument, and hence one can pass to the limit separately in the numerator and the denominator of Eq. 10.

Let $0 < \varepsilon < 1$. By the upper bound in Lemma 3, there is $s = s(\varepsilon) \leq \varepsilon R$ such that if $0 < 8r \leq s$, then

$$f_{8r,s}(x) \leq \varepsilon \tilde{f}_{8r,s}(x), \qquad g_{8r,s}(x) \leq \varepsilon \tilde{g}_{8r,s}(x) \tag{16}$$

for $x \in D_{2r}$. Furthermore, estimate (15) and the assumption $\lim_{r \to 0^+} C_{\text{Lévy}}(x_0, r, R) = 1$ imply that there is $r = r(\varepsilon) \leq s/8$ such that

$$\tilde{f}_{8r,s}(x) \approx (1 + \varepsilon) \mathbf{E}_x \tau_{8r} M_{s,\infty}(f), \qquad \tilde{g}_{8r,s}(x) \approx (1 + \varepsilon) \mathbf{E}_x \tau_{8r} M_{s,\infty}(g) \tag{17}$$

for $x \in D_{8r}$. It follows that

$$\frac{f(x)}{g(x)} \leq \frac{(1+\varepsilon)\tilde{f}_{8r,s}(x)}{\tilde{g}_{8r,s}(x)} \leq (1+\varepsilon)^3 \frac{M_{s,\infty}(f)}{M_{s,\infty}(g)}$$

for $x \in D_{2r}$. The lower bound is proved in a similar manner, and we obtain

$$(1+\varepsilon)^{-3} \frac{M_{s,\infty}(f)}{M_{s,\infty}(g)} \leq \frac{f(x)}{g(x)} \leq (1+\varepsilon)^3 \frac{M_{s,\infty}(f)}{M_{s,\infty}(g)} \tag{18}$$

for $x \in D_{2r}$. Since ε was arbitrary and s converges to 0 as $\varepsilon \to 0^+$, we have

$$\lim_{\substack{x \to x_0 \\ x \in D}} \frac{f(x)}{g(x)} = \lim_{s \to 0^+} \frac{M_{s,\infty}(f)}{M_{s,\infty}(g)},$$

and Theorem 2 for inaccessible boundary points is proved.

4.3 Accessible Boundary Points

In the second part of the proof we assume that x_0 is *accessible*, that is,

$$M_{0,\infty}(s_R) = \int_{D_R} \mathbf{E}_y \tau_R \nu(x_0, y) m(dy) = \infty.$$

In this case $f_{8r,s}$ and $g_{8r,s}$ dominate $\tilde{f}_{8r,s}$ and $\tilde{g}_{8r,s}$ for all sufficiently small r.

We remark that by Eq. 13 and Lemma 1,

$$M_{0,\infty}(f) \geq M_{0,R/4}(f) \geq C_{\mathrm{BHI}}^{-1} M_{3R/4,\infty}(f) M_{0,R/4}(s_{R/2}) = \infty,$$

and $M_{0,\infty}(g) = \infty$ by the same argument. In other words, the numerator and the denominator of the right-hand side of Eq. 10 diverge to infinity as $r \to 0^+$. In particular, if the limit of $f(x)/g(x)$ in Eq. 10 exists, then it is automatically equal to the right-hand side.

Our argument is based on the following standard oscillation reduction lemma.

Lemma 4 *If* $0 < 8r < s < R_0$, *then*

$$\left(\sup_{y \in D_{2r}} - \inf_{y \in D_{2r}}\right) \frac{f_{8r,s}(y)}{g_{8r,s}(y)} \leq \frac{C_{\mathrm{BHI}}^4 - 1}{C_{\mathrm{BHI}}^4 + 1} \left(\sup_{y \in D_s} - \inf_{y \in D_s}\right) \frac{f(y)}{g(y)}.$$

Proof For simplicity, we denote

$$A = \sup_{y \in D_s} \frac{f(y)}{g(y)}, \qquad B = \sup_{y \in D_{2r}} \frac{f_{8r,s}(y)}{g_{8r,s}(y)},$$
$$a = \inf_{y \in D_s} \frac{f(y)}{g(y)}, \qquad b = \inf_{y \in D_{2r}} \frac{f_{8r,s}(y)}{g_{8r,s}(y)}.$$

Since

$$ag\mathbf{1}_{8r,s} \leq f\mathbf{1}_{8r,s} \leq Ag\mathbf{1}_{8r,s},$$

we clearly have

$$ag_{8r,s} \leq f_{8r,s} \leq Ag_{8r,s}. \tag{19}$$

In particular, $a \leq b \leq B \leq A$, and Theorem 1 applies to *everywhere* non-negative functions $f_{8r,s} - ag_{8r,s}$, $Ag_{8r,s} - f_{8r,s}$ and $g_{8r,s}$ (note that $f - ag$ and $Ag - f$ typically fail to be non-negative everywhere). By Eq. 8,

$$\sup_{y \in D_{2r}} \frac{f_{8r,s}(y) - ag_{8r,s}(y)}{g_{8r,s}(y)} \leq C_{\mathrm{BHI}}^4 \inf_{y \in D_{2r}} \frac{f_{8r,s}(y) - ag_{8r,s}(y)}{g_{8r,s}(y)},$$
$$\sup_{y \in D_{2r}} \frac{Ag_{8r,s}(y) - f_{8r,s}(y)}{g_{8r,s}(y)} \leq C_{\mathrm{BHI}}^4 \inf_{y \in D_{2r}} \frac{Ag_{8r,s}(y) - f_{8r,s}(y)}{g_{8r,s}(y)}.$$

This translates to $B - a \le C_{\mathrm{BHI}}^4 (b - a)$ and $A - b \le C_{\mathrm{BHI}}^4 (A - B)$, and adding the sides of these inequalities leads to the desired inequality

$$\left(C_{\mathrm{BHI}}^4 + 1\right)(B - b) \le \left(C_{\mathrm{BHI}}^4 - 1\right)(A - a).$$

□

For continuous processes (in sufficiently regular domains), the above lemma easily yields the assertion of Theorem 2. For processes with jumps one needs to incorporate the non-local parts $\tilde{f}_{8r,s}$ and $\tilde{g}_{8r,s}$ using Lemma 3. As it was remarked in the introduction, this modification was developed in [7], and extended in [11].

Let $0 < \varepsilon < 1$ and $0 < s < \frac{R}{24}$. By the lower bound in Lemma 3, there is $r = r(\varepsilon, s) \le \frac{s}{8}$ such that

$$\tilde{f}_{8r,s}(x) \le \varepsilon f_{8r,s}(x), \qquad \tilde{g}_{8r,s}(x) \le \varepsilon g_{8r,s}(x) \tag{20}$$

for $x \in D_r$. It follows that

$$\left(\sup_{x \in D_r} - \inf_{x \in D_r}\right) \frac{f(x)}{g(x)} \le (1 + \varepsilon) \sup_{x \in D_r} \frac{f_{8r,s}(x)}{g_{8r,s}(x)} - \frac{1}{1 + \varepsilon} \inf_{x \in D_r} \frac{f_{8r,s}(x)}{g_{8r,s}(x)}.$$

By Lemma 4 and the inequality $1 - (1 + \varepsilon)^{-1} \le \varepsilon$,

$$\left(\sup_{x \in D_r} - \inf_{x \in D_r}\right) \frac{f(x)}{g(x)} \le \frac{C_{\mathrm{BHI}}^4 - 1}{C_{\mathrm{BHI}}^4 + 1} \left(\sup_{x \in D_s} - \inf_{x \in D_s}\right) \frac{f(x)}{g(x)} + \varepsilon \left(\sup_{x \in D_r} + \inf_{x \in D_r}\right) \frac{f_{8r,s}(x)}{g_{8r,s}(x)}. \tag{21}$$

Denote by Q the upper limit of the expression in the left-hand side as $r \to 0^+$. Using Eq. 19 and taking the upper limit of both sides as $s \to 0^+$ leads to

$$Q \le \frac{C_{\mathrm{BHI}}^4 - 1}{C_{\mathrm{BHI}}^4 + 1} Q + 2\varepsilon \sup_{x \in D_{R/4}} \frac{f(x)}{g(x)},$$

that is,

$$Q \le \varepsilon \left(1 + C_{\mathrm{BHI}}^4\right) \sup_{x \in D_{R/4}} \frac{f(x)}{g(x)}$$

Since ε is arbitrary, we conclude that $Q = 0$, and the proof of Theorem 2 is complete.

4.4 Extensions

We first prove the statement contained in Remark 2. Denote $g(x) = \mathbf{E}_x \tau_R$. Then g is not a regular harmonic function in D_R, but for every open $U \subseteq D_R$,

$$g(x) = \mathbf{E}_x \tau_U + \mathbf{E}_x g(X(\tau_U)).$$

We interpret $\mathbf{E}_x \tau_U$ as if it originated from a jump to a distant point (a point at infinity), and we define

$$M_{r,\infty}(g) = 1 + \int_{\mathfrak{X} \setminus B_r} g(y) \nu(x_0, y) m(dy), \qquad \tilde{g}_{r,s}(x) = \mathbf{E}_x \tau_r + \mathbf{E}_x((g \mathbf{1}_{s,\infty})(X(\tau_r)));$$

the definitions of $M_{r,s}(g)$ and $g_{r,s}(x)$ for finite s remain unaltered. One can then follow carefully the proof of Theorem 2 and see that no changes are required. This shows validity of Remark 2.

In the remaining part of this section we argue that an extension stated in Remark 3 is true: the limit in Theorem 2 converges uniformly in f and g, and also in D, in the sense of Eq. 11.

We claim that if $0 < q < R_0$ and $\eta > 0$, then there is p, which depends only on q, η and the characteristics of the process X_t, such that $0 < p < q$ and

$$\frac{\sup_{x \in D_p}(f(x)/g(x))}{\inf_{x \in D_p}(f(x)/g(x))} - 1 \le \eta + \frac{C_{\mathrm{BHI}}^4 - 1}{C_{\mathrm{BHI}}^4 + 1}\left(\frac{\sup_{x \in D_q}(f(x)/g(x))}{\inf_{x \in D_q}(f(x)/g(x))} - 1\right) \tag{22}$$

for all open sets D and all functions f and g as in Theorem 2 (this estimate is very similar to Eq. 21). By considering the supremum of both sides of Eq. 22 over all f, g and D, and then taking the upper limit as $q \to 0^+$, we obtain the desired result:

$$\limsup_{r \to 0^+} \sup_{D,f,g} \left(\frac{\sup_{x \in D_r}(f(x)/g(x))}{\inf_{x \in D_r}(f(x)/g(x))} - 1\right) \le \frac{\eta\left(1 + C_{\mathrm{BHI}}^4\right)}{2}$$

for arbitrary $\eta > 0$. Therefore, it remains to prove (22).

Let $0 < q < \frac{1}{24}R_0$ and $\eta > 0$. We consider two additional parameters $\delta, N > 0$; the actual values of δ (small real) and N (large integer) are to be specified at the end of the argument. By the assumption $\lim_{r \to 0^+} C_{\text{Lévy}}(x_0, r, R) = 1$ one can construct a decreasing sequence of radii $a_0, a_1, \ldots, a_N$ so that a_0 is the input radius q, $\frac{1}{8}a_N$ will be the output radius p, and we have $16a_{n+1} < a_n$ and $C_{\text{Lévy}}(x_0, 8a_{n+1}, a_n) \le 1 + \delta$ for all $n = 0, 1, \ldots, N-1$.

Following [11], we consider two scenarios. Suppose first that for some n we have

$$M_{a_{n+1},a_n}(s_{R/2}) \le \delta(1 + M_{a_n,R/4}(s_{R/2})). \tag{23}$$

Then the argument is fairly simple: as in Section 4.2, by Lemma 3 we have the inequality (16) with $r = a_{n+1}$, $s = a_n$ and $\varepsilon = C_{\mathrm{BHI}}^4\delta$. Since $C_{\text{Lévy}}(x_0, 8a_{n+1}, a_n) \le 1 + \delta$, the estimate (17) holds with $r = a_{n+1}$, $s = a_n$ and $\varepsilon = \delta$. This implies (18) (with $s = a_n$, $x \in D_{2a_{n+1}}$ and $\varepsilon = C_{\mathrm{BHI}}^4\delta$), and in particular the left-hand side of Eq. 22 does not exceed $(1+C_{\mathrm{BHI}}^4\delta)^6-1$. Estimate (22) follows with $p = a_{n+1}$, provided that $(1+C_{\mathrm{BHI}}^4\delta)^6-1 \le \eta$. We choose δ small enough, so that this inequality is satisfied.

In the other scenario, for each n the converse of Eq. 23 holds. Summing up these inequalities for $n = 0, 1, \ldots, N-1$ we obtain

$$M_{a_N,a_0}(s_{R/2}) \ge N\delta(1 + M_{a_0,R/4}(s_{R/2})),$$

and we argue as in Section 4.3. Again by Lemma 3, we have Eq. 20 with $r = \frac{1}{8}a_N$, $s = a_0$ and $\varepsilon = C_{\mathrm{BHI}}^3 C_{\text{Lévy}} C_\tau^3 (N\delta)^{-1}$. Inequality (21) follows. Dividing both sides of it by $\inf_{x \in D_r}(f(x)/g(x))$ and using monotonicity of this expression in r, we obtain (22) for $p = \frac{1}{8}a_N$, provided that $\varepsilon(C_{\mathrm{BHI}} + 1) \le \eta$. Since δ is now fixed, we may choose N large enough, so that this condition is satisfied. This completes the proof of the extension described in Remark 3.

4.5 Martin Representation

In this section we prove Theorem 3. We assume that the assumptions of Theorem 2 are satisfied in a uniform way for all $x_0 \in \overline{D}$.

We note one important property of the Green function: if U is an open subset of D and $y \notin \partial U$, then

$$G_D(x, y) = \mathbf{E}_x G_D(X(\tau_U), y) + G_U(x, y) \tag{24}$$

(where, as usual, we assume that $G_U(x, y) = 0$ whenever $x \notin U$ or $y \notin U$). In particular, $G_D(x, y)$ is a regular harmonic function in $D \setminus \overline{B}(y, r)$ for every $r > 0$. By a duality argument, $G_D(x, y)$ is a regular *co-harmonic* function in $D \setminus \overline{B}(x, r)$ for every $r > 0$.

Furthermore, by the strong Markov property,

$$\begin{aligned}\int_D G_D(x,y)f(y)m(dy) &= \mathbf{E}_x \int_0^{\tau_D} f(X_s)ds \\ &= \mathbf{E}_x \int_0^{\tau_D-\tau_U} f(X_{\tau_U+s})ds + \mathbf{E}_x \int_0^{\tau_U} f(X_s)ds \\ &= \mathbf{E}_x \int_D G_D(X(\tau_U),y) f(y)m(dy)y + \int_U G_U(x,y)f(y)m(dy) \quad (25)\end{aligned}$$

for any nonnegative function f. Note that if $m(\partial U) = 0$, then Eq. 25 follows from Eq. 24 and Fubini.

Proof of Theorem 3(a) The assumptions are completely symmetric under duality, and hence we may apply Theorem 2 to both harmonic and co-harmonic functions. In particular, as already remarked before the statement of Theorem 3, the Martin kernel, defined as the boundary limit of co-harmonic functions

$$M_D(x,z) = \lim_{\substack{y\to z \\ x\in D}} \frac{G_D(x,y)}{G_D(\tilde{x},y)},$$

exists for all boundary points $z \in D$ (here and below $\tilde{x} \in D$ is a fixed reference point). In other words, the Martin boundary coincides with the Euclidean boundary. □

The representation given in part (d) essentially follows now from the general theory of Martin boundary, together with some ideas developed in [11]. For simplicity, in the remaining part of the proof we simply write that a function is harmonic when we refer to harmonicity in D.

Proof of Theorem 3(b) Following the proof of Theorem 2 in [11], we find that $M_D(x,x_0)$ is a harmonic function with respect to x if and only if x_0 is accessible. Indeed, for an inaccessible boundary point x_0 we have, by Eq. 10 in Theorem 2,

$$M_D(x,x_0) = C \int_D \nu(y,x_0) G_D(x,y)m(dy)$$

for $C = (\int_D \nu(y,x_0)G_D(\tilde{x},y)m(dy))^{-1} > 0$, and so the Martin kernel is not harmonic (to see this, simply use (25)). On the other hand, if x_0 is accessible and $R > 0$, then

$$\mathbf{E}_x M_D(X(\tau_{D\setminus\overline{B}(x_0,R)}),x_0) = \mathbf{E}_x \lim_{\substack{y\to x_0 \\ y\in D}} \frac{G_D(X(\tau_{D\setminus\overline{B}(x_0,R)}),y)}{G_D(\tilde{x},y)}. \quad (26)$$

Recall that $G_D(x,y)$ is a regular harmonic function of $x \in D \setminus \overline{B}(x_0,R)$ when $y \in B(x_0,R)$. By Fatou's lemma,

$$\mathbf{E}_x M_D(X(\tau_{D\setminus\overline{B}(x_0,R)}),x_0) \le M_D(x,x_0), \quad (27)$$

and we claim that in fact equality holds, that is, we can exchange the limit with the expectation in Eq. 26. By Vitali's convergence theorem, it suffices to prove that that the ratio in the right-hand side of Eq. 26 is a uniformly integrable family of random variables for $y \in D \cap B(x_0,r)$ for some $r > 0$. The argument is exactly the same as in the proof of formula (77) in [11]; for the convenience of the reader, we repeat it below.

Assume that $0 < 8r < R$ and that $x, \tilde{x} \notin D \cap B(x_0, R)$. We will first prove that

$$\sup_{\substack{y \in D \cap B(x_0, r) \\ z \in D \setminus B(x_0, 4r)}} \frac{G_D(z, y)}{G_D(\tilde{x}, y)} < \infty. \tag{28}$$

By the boundary Harnack inequality (Theorem 1) applied to $G_D(z, \cdot)$ and $G_D(\tilde{x}, \cdot)$, it suffices to consider a fixed $y \in D \cap B(x_0, r)$, that is, to show that $G_D(\cdot, y)$ is bounded in $D \setminus B(x_0, 4r)$. This is relatively simple, but somewhat technical. Denote $D_1 = D \cap B(x_0, r)$, $D_2 = D \cap B(x_0, 2r)$, $D_4 = D \cap \overline{B}(x_0, 4r)$ and $D' = D \setminus \overline{B}(x_0, 4r)$. By Dynkin's formula (14),

$$\begin{aligned} &\mathbf{E}_z\big(G_D(X(\tau_{D'}), y)\mathbf{1}_{D_2}(X(\tau_{D'}))\big) \\ &\le \Bigg(\sup_{\substack{v \in D' \\ w \in D_2}} \nu(v, w)\Bigg) \int_{D'} G_{D'}(z, v) m(dv) \int_{D_2} G_D(w, y) m(dw). \end{aligned}$$

The supremum is finite by Assumption 3 and boundedness of D, and the integrals in the right-hand side are bounded by $\sup_{u \in D} \mathbf{E}_u \tau_D$ and $\sup_{u \in D} \hat{\mathbf{E}}_u \hat{\tau}_D$, respectively. Furthermore,

$$\mathbf{E}_z\big(G_D(X(\tau_{D'}), y)\mathbf{1}_{D_4 \setminus D_2}(X(\tau_{D'}))\big) \le \sup_{\substack{v \in D_4 \setminus D_2 \\ w \in D_1}} G_D(v, w),$$

and the right-hand side is finite by Assumption 4. By adding the sides of these two bounds and using harmonicity of the Green function, we complete the proof of Eq. 28.

On the other hand, if we denote $D'' = D \setminus \overline{B}(x_0, 8r)$ and $D''' = D \setminus \overline{B}(x_0, R)$, then, again by Lemma 2,

$$\begin{aligned} &\mathbf{E}_x\big(G_D(X(\tau_{D'''}), y)\mathbf{1}_{D_4}(X(\tau_{D'''}))\big) \\ &\le C_{\text{Lévy}} \left(\int_{D'''} \nu(x_0, v) G_{D'''}(x, v) m(dv)\right) \left(\int_{D_4} G_D(w, y) m(dw)\right), \end{aligned}$$

and, in a similar way,

$$\begin{aligned} G_D(\tilde{x}, y) &\ge \mathbf{E}_{\tilde{x}}\big(G_D(X(\tau_{D''}), y)\mathbf{1}_{D_4}(X(\tau_{D''}))\big) \\ &\ge C_{\text{Lévy}}^{-1} \left(\int_{D''} \nu(x_0, v) G_{D''}(\tilde{x}, v) m(dv)\right) \left(\int_{D_4} G_D(w, y) m(dw)\right). \end{aligned}$$

It follows that

$$\begin{aligned} &\mathbf{E}_x\left(\frac{G_D(X(\tau_{D \setminus \overline{B}(x_0, R)}), y)}{G_D(\tilde{x}, y)} \mathbf{1}_{D \cap B(x_0, 4r)}(X(\tau_{D \setminus B(x_0, R)}))\right) \\ &\le C \left(\int_{D \setminus B(x_0, 8r)} \nu(x_0, v) G_{D \setminus B(x_0, 8r)}(\tilde{x}, v) m(dv)\right)^{-1}, \end{aligned}$$

where C does not depend on (sufficiently small) $r > 0$ and $y \in D \cap B(x_0, r)$. Recall that $G_{D \setminus B(x_0, 8r)}(\tilde{x}, v)$ increases to $G_D(\tilde{x}, v)$ (because the corresponding exit times $\tau_{D \setminus B(x_0, 8r)}$ increase to τ_D). By monotone convergence, the right-hand side converges to zero as $r \to 0^+$. Together with Eq. 28, this completes the proof of uniform integrability of the right-hand side of Eq. 26.

Part (b) follows, and in addition we see that for accessible boundary points z, the Martin kernel $M_D(x, z)$ is a regular harmonic function in $D \setminus B(z, r)$ for every $r > 0$. □

In order to apply the general theory of Martin boundary, we need to prove that the Green operator, which maps a measurable function $f(x)$ to $G_D f(x) = \int_D G_D(x, y) f(y) m(dy)$, takes bounded functions into continuous ones. Let f be a bounded function on D, $x_0 \in D$ and $\varepsilon > 0$. Clearly, $|G_D f(x)| \le \|f\| \mathbf{E}_x \tau_D$ for $x \in D$, so that $G_D f$ is bounded. Let $r > 0$ be small enough, so that $\mathbf{E}_x \tau_{B(x_0,r)} < \varepsilon$ for $x \in B(x_0, r)$. By Eq. 25,

$$G_D f(x) = \mathbf{E}_x G_D f(X(\tau_{B(x_0,r)})) + G_{B(x_0,r)} f(x).$$

The first term is continuous in $B(x_0, r)$ by Theorem 2 (see Remark 4). The other one is bounded by $\varepsilon\|f\|$, an arbitrarily small number. Therefore, $G_D f$ is continuous at x_0.

The general theory of Martin boundary tells us now that if f satisfies the assumptions of Theorem 3 and f is equal to zero in the complement of D, then

$$f(x) = \int_{\partial_m D} M_D(x, z)\mu(dz) \tag{29}$$

for some measure μ on the set of accessible boundary points $\partial_m D$, see Theorem 14.8 in [16]. Furthermore, if we show that for every $z \in \partial_m D$, $M_D(x, z)$ is a *minimal* harmonic function with respect to x, then the measure μ in the above representation is unique. Minimality of $M_D(x, z)$ is proved as in the final part of the proof of Lemma 14 in [11].

Proof of Theorem 3(c) Suppose that f is harmonic, $0 \le f(x) \le M_D(x, x_0)$ for all $x \in \mathfrak{X}$ (in particular, $f(x) = 0$ for $x \in \mathfrak{X} \setminus D$) and that the measure μ in representation (29) is zero on $\partial_m D \cap B(x_0, 4r)$ for some $r > 0$. Our goal is to prove that f is identically zero. This will imply that if f is harmonic and $0 \le f(x) \le M_D(x, x_0)$ for all $x \in \mathfrak{X}$, then the measure μ in representation (29) is concentrated in $\{x_0\}$, and thus $M_D(x, x_0)$ is a minimal harmonic function.

For every $z \in \partial_m D \setminus B(x_0, 4r)$, $M_D(x, z)$ is a regular harmonic function in $D \cap B(x_0, 3r)$. Hence, by Fubini, f also has this property. Furthermore, by the boundary Harnack inequality (Theorem 1), f is bounded on $D \cap B(x_0, 2r)$.

On the other hand, since $f(x) \le M_D(x, x_0)$, one easily finds that f is also a regular harmonic function in $D \setminus B(x_0, r)$. This is exactly the same argument as in Lemma 9 in [11]; for the convenience of the reader, we provide the details at the end of this section. In particular, since f is bounded in $D \cap B(x_0, 2r)$, it is bounded on D.

A sweeping argument, which is a simplified version of Lemma 10 in [11], proves then that f is a regular harmonic function in D: Let σ_n be the sequence of consecutive exit times from alternately $D \cap B(x_0, 4r)$ and $D \setminus B(x_0, r)$. That is, $\sigma_0 = 0$ and $\sigma_{n+1} = \sigma_n + \tau_V \circ \vartheta_{\sigma_n}$, where $V = D \cap B(x_0, 4r)$ when n is even and $V = D \setminus B(x_0, r)$ when n is odd (and ϑ_τ is the shift operator).

Clearly, $\sigma_n \le \tau_D < \infty$. Since σ_n is increasing, by quasi-left continuity, $X(\sigma_n)$ has a limit as $n \to \infty$. Therefore, it is impossible that $\sigma_n < \tau_D$ for infinitely many n. It follows that with probability one, eventually $\sigma_n = \tau_D$.

Since $f(x) = \mathbf{E}_x f(X(\sigma_n))$ and f is bounded, by dominated convergence we have $f(x) = \mathbf{E}_x f(X(\tau_D)) = 0$, as desired. □

We have thus proved the representation (29) for harmonic functions f which are zero in the complement of D. The general case is handled as in Lemma 13 in [11].

Proof of Theorem 3(d) Let D_n be an ascending sequence of open sets such that $\overline{D_n} \subseteq D$ and $\bigcup_{n=1}^{\infty} D_n = D$. Then, by Lemma 2,

$$f(x) = \mathbf{E}_x f(X(\tau_{D_n})) \ge \int_{\mathfrak{X}\setminus D} \left(\int_{D_n} G_{D_n}(x, y)\nu(y, z)m(dy) \right) f(z)m(dz).$$

The integrand in the right-hand side increases as $n \to \infty$, and therefore by monotone convergence,

$$f(x) \geq \int_{\mathfrak{X}\setminus D} \left(\int_D G_D(x,y)\nu(y,z)m(dy) \right) f(z)m(dz). \tag{30}$$

Let $g(x)$ be equal to the right-hand side of Eq. 30 for $x \in D$, and to $f(x)$ for $x \in \mathfrak{X}\setminus D$. From Lemma 2 and the property (25) of the Green function it follows easily that g is harmonic: if U is open and $\overline{U} \subseteq D$, then

$$\begin{aligned} \mathbf{E}_x g(X(\tau_U)) &= \mathbf{E}_x(f\mathbf{1}_{\mathfrak{X}\setminus D})(X(\tau_U)) \\ &\quad + \mathbf{E}_x \int_{\mathfrak{X}\setminus D} \left(\int_D G_D(X(\tau_U),y)\nu(y,z)m(dy) \right) f(z)m(dz) \\ &= \int_{\mathfrak{X}\setminus D} \left(\int_D \big(G_U(x,y) + \mathbf{E}_x G_D(X(\tau_U),y)\big)\nu(y,z)m(dy) \right) f(z)m(dz) \\ &= g(x). \end{aligned}$$

Therefore, $f - g$ is a non-negative harmonic function which is equal to zero in $\mathfrak{X} \setminus D$, and so it has a unique representation (29).

Finally, the outer integral in Eq. 30 is finite, and so points at which the inner integral is infinite cannot contribute to the integral. It follows that we can change the outer integral to an integral over $\mathfrak{X} \setminus (D \cup \partial_m D)$. The proof of Eq. 12 is complete. □

Proof of Theorem 3(e) By the boundary Harnack inequality, if the right-hand side of Eq. 12 is finite at some $x \in D$, it is finite everywhere in D. Indeed, let f be given by Eq. 12. If $f(x) = \infty$ for some $x \in D$, by Theorem 1 f is infinite at every point of a ball $B(x,r)$ contained in D. If $y \in D$, then again using Theorem 1 (for a ball centred at y), f is infinite at y.

Finally, harmonicity of the right-hand side of Eq. 12, whenever it is finite, follows from property (25) of the Green function, harmonicity of the Martin kernel and Fubini. □

At the end of this section, we present the proof of Lemma 9 in [11], adapted to our setting. This result was used in the proof of Theorem 3(c).

Lemma 5 (Lemma 9 in [11]) *Let U and D be open subsets of $\mathfrak{X}$ such that $U \subseteq D$. If $0 \leq f(x) \leq g(x)$ for all $x \in \mathfrak{X}$, f and g are harmonic in D, g is a regular harmonic function in U and $g(x) = 0$ for $x \in \mathfrak{X} \setminus D$, then f is a regular harmonic function in U.*

Proof Let D_n be an ascending sequence of open sets such that $\overline{D_n} \subseteq D$ and $\bigcup_{n=1}^{\infty} D_n = D$, and let $U_n = U \cap D_n$. Then τ_{U_n} increases to τ_U, and, by quasi-left continuity, $X(\tau_{U_n})$ converges to $X(\tau_U)$ with probability one. It follows that if $X(\tau_U) \in D \setminus U$, then eventually $\tau_{U_n} = \tau_U$ for n large enough up to an event of probability zero. Hence,

$$\lim_{n\to\infty} \mathbf{E}_x(g\mathbf{1}_{D\setminus U})(X(\tau_{U_n})) = \mathbf{E}_x(g\mathbf{1}_{D\setminus U})(X(\tau_U)) = g(x).$$

Therefore,

$$\mathbf{E}_x(f\mathbf{1}_{U\setminus U_n})(X(\tau_{U_n})) \leq \mathbf{E}_x(g\mathbf{1}_{U\setminus U_n})(X(\tau_{U_n})) = g(x) - \mathbf{E}_x(g\mathbf{1}_{D\setminus U})X(\tau_{U_n}))$$

converges to zero as $n \to \infty$. It follows that

$$f(x) = \lim_{n\to\infty} \mathbf{E}_x f(X(\tau_{U_n})) = \lim_{n\to\infty} \mathbf{E}_x (f \mathbf{1}_{D\setminus U})(X(\tau_{U_n}))$$
$$= \mathbf{E}_x (f \mathbf{1}_{D\setminus U})(X(\tau_U)) = \mathbf{E}_x f(X(\tau_U)),$$

as desired. □

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Decay rate of harmonic functions for non-symmetric strictly α-stable Lévy processes

by

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Abstract. We investigate functions that are harmonic with respect to the non-symmetric strictly α-stable Lévy processes on an open set $D \subset \mathbb{R}^d$. We obtain an explicit formula for their boundary decay rate at parts of the boundary of D outside of which they vanish.

1. Introduction. With rare exceptions, explicit boundary decay rates of harmonic functions for jump Markov type processes, or non-local operators, have been studied under the symmetry assumption. The only results for non-symmetric processes or operators known to the author are [13, 8, 11]. Here we provide the boundary decay rate for functions harmonic with respect to general stable Lévy processes, an important class of Markov processes with numerous applications. Our result requires relatively mild assumptions on the jump kernel, and works for sufficiently smooth sets.

An important tool in investigating the behaviour of harmonic functions near the boundary or existence of their limits is the boundary Harnack inequality. It is a statement about positive harmonic functions in an open set D, which are equal to zero on a part of the boundary. It states that if D is regular enough (for example, a Lipschitz domain), z is a boundary point of D, f and g are positive and harmonic in D, and both f and g converge to 0 on $\partial D \cap B(z,R)$, then for every $r \in (0,R)$,

$$\sup_{x\in D\cap B(z,r)} \frac{f(x)}{g(x)} \le c_{BHI} \inf_{x\in D\cap B(z,r)} \frac{f(x)}{g(x)}, \tag{1.1}$$

where the constant c_{BHI} does not depend on f and g.

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BHI for harmonic functions of the Laplacian Δ in Lipschitz domains was proved in 1977–78 by B. Dahlberg [9], A. Ancona [1] and J.-M. Wu [23]. In 1989, R. Bass and K. Burdzy [2] proposed an alternative probabilistic proof based on elementary properties of the Brownian motion.

It is possible to define harmonicity in more probabilistic terms. Let X be the Brownian motion and let P_t be its transition semigroup defined by

$$P_t f(x) = \mathbb{E}^x f(X_t).$$

Then the generator of P_t is the Laplacian Δ. Moreover, every function f is harmonic in an open set D if and only if for any $x \in B$ where $\overline{B} \subset D$ we have

$$f(x) = \mathbb{E}^x f(X_{\tau_B} \mathbf{1}_{\{\tau_B < \infty\}}), \qquad x \in D,$$

where τ_B is first exit time of X from B.

It is possible to extend the definition of Laplacian and corresponding harmonic functions to non-local operators by changing the underlying stochastic process.

In 1997, K. Bogdan [4] proved BHI for the fractional Laplacian $\Delta^{\alpha/2}$ (and isotropic α-stable Lévy processes) for $0 < \alpha < 2$ and Lipschitz sets. In 1999, R. Song and J.-M. Wu [22] extended the result to all open sets with c_{BHI} depending on d, D, z, r, and in 2007, K. Bogdan, T. Kulczycki and M. Kwaśnicki [5] extended that result by showing that c_{BHI} in fact only depends on α and d. In 2008, P. Kim, R. Song and Z. Vondraček [15] proved BHI for subordinate Brownian motions in "fat" sets and in 2011 they extended it to a more general class of isotropic Lévy processes and arbitrary domains [16]. In 2014, K. Bogdan, T. Kumagai and M. Kwaśnicki [6] proved BHI for a wide class of non-symmetric processes in duality. In 2016, a similar result was obtained by Z.-Q. Chen, Y.-X. Ren and T. Yang [7] for κ-fat sets and some processes without dual process. Finally, in 2016, X. Ros-Oton and J. Serra [21] proved BHI for arbitrary open sets and operators with kernels which are comparable with stable kernels.

In most of the cases mentioned above the constant c_{BHI} in (1.1) converges to 1 as $r \to 0$ giving the existence of boundary limits of ratios of harmonic functions. Methods used in those proofs involve the so-called *reduction of oscillation*. For jump-type processes this requires additional assumptions (scale invariance of BHI or uniformity of BHI). One of the results, which we will refer to in this paper, was found independently by M. Kwaśnicki and the author [20], and by P. Kim, R. Song and Z. Vondraček [18], where the existence of the limits is proven for a wide class of non-symmetric processes and arbitrary open sets.

A natural consequence of the existence of limits of ratios of harmonic functions is the question about explicit decay rate of such functions near the boundary of D. The answer is known for a wide class of symmetric processes. For example, P. Kim, R. Song and Z. Vondraček [17] proved in 2014 a result for subordinate Brownian motions where the Laplace exponent ϕ of the subordinator satisfies mild scaling conditions. It states that if X is a subordinate Brownian motion then for every $C^{1,1}$ set D, every $r > 0$, $z \in \partial D$, and every non-negative function u in $\mathbb{R}^d$ which is harmonic in $D \cap B(z,r)$ with respect to X and vanishing continuously on D^c the limit

$$\lim_{x \to z} \frac{u(x)}{\sqrt{\phi(\delta_D(x)^{-2})}}$$

exists. Another result is the work of T. Grzywny, K.-Y. Kim and P. Kim [14] from 2015, who obtained the decay rates for a large class of symmetric pure jump Markov processes dominated by isotropic unimodal Lévy processes with weak scaling conditions for sets of class $C^{1,\varrho}$ for $\varrho \in (\alpha/2, 1]$.

To our knowledge not much is known about decay rates in the non-symmetric case. Here we would like to mention the work of X. Fernández-Real and X. Ros-Oton [13] for symmetric α-stable processes with drift, an ongoing work of Z.-Q. Chen and L. Wang [8], and another work in progress by S. Dipierro, X. Ros-Oton, J. Serra and E. Valdinoci [11].

The goal of this article is to obtain the explicit decay rate of harmonic functions in sufficiently regular sets for non-symmetric, strictly α-stable processes. The following is our main result.

THEOREM 1.1. *Let X be a (possibly non-symmetric) $\mathbb{R}^d$-valued strictly α-stable process with $\alpha \in (0,2)$ and with the Lévy measure given by*

$$\nu(A) = \int_A \frac{1}{|x|^{d+\alpha}} \vartheta\left(\frac{x}{|x|}\right) dx,$$

where ϑ is strictly positive and C^{ϵ} on the unit sphere for some $\epsilon > 0$. Let D be a bounded, open $C^{1,1}$ set if $\alpha < 1$, and $C^{2,\alpha-1+\epsilon}$ if $\alpha \geq 1$. Let $z \in \partial D$. Then for every non-negative function f, harmonic in $D \cap B(z, R_0)$ with respect to the process X and vanishing continuously on $D^c \cap B(z, R_0)$, the limit

$$\lim_{\substack{x \to z \\ x \in D}} \frac{f(x)}{|x - x_D|^{\beta(x)}}$$

exists, where $x_D \in \partial D$ is the boundary point nearest to x and the exponent β is given by

$$\beta(x) = \alpha \mathbb{P}^0(\langle X_t, x - x_D \rangle > 0).$$

2. Preliminaries

2.1. Notation, definitions and technical lemmas. Throughout this paper, $d \geq 2$. We denote by $\langle \cdot, \cdot \rangle$ the usual dot product in $\mathbb{R}^d$. We denote the Euclidean distance between x and y by $|x - y|$ and the Euclidean distance between x and D^c by $\delta_D(x)$. Each constant, unless otherwise stated, is positive. By c and c_i, $i \in \mathbb{N}$, we denote constants that are less important, and may vary even within one lemma or theorem. By $c(a)$ we denote that the constant c depends on a.

We denote by $B(x,r)$ the (open) ball of radius r with centre at x and by S^d the unit sphere in $\mathbb{R}^d$. We also define $D_r = D \cap B(0,r)$ and $D_r^* = \{x \in D : \delta_D(x) < r\}$. For $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ we write $x = (\tilde{x}, x_d)$, where $\tilde{x} = (x_1, \ldots, x_{d-1})$.

Let C_0 denote the class of continuous functions on $\mathbb{R}^d$ converging to 0 as $x \to \infty$, and let C_c be the class of compactly supported continuous functions.

By *changing the coordinate system* in $\mathbb{R}^d$ we mean applying an isometrical transformation of $\mathbb{R}^d$. Similarly, *scaling* means an application of a dilation of $\mathbb{R}^d$. For example, by an appropriate change of the coordinate system, every open half-space with a distinguished boundary point z can be transformed into $\mathbb{H} = \{x \in \mathbb{R}^d : x_d > 0\}$ in such a way that z is mapped to the origin 0. Similarly, by an appropriate change of the coordinate system and scaling, every open half-space with a distinguished interior point x can be transformed into $\mathbb{H}$ in such a way that the image of x is $(0, 0, \ldots, 0, 1)$.

Definition 2.1. For a compact set K, we write $f \in C^{n,\gamma}(K)$ if the nth order partial derivatives of f are Hölder continuous on K with exponent γ $(0 < \gamma \le 1)$. Such functions form a Banach space with norm

$$\|f\|_{C^{n,\gamma}(K)} = \begin{cases} \|f\|_{L^\infty(K)} + \sup\left\{\frac{|f(y)-f(x)|}{|y-x|^\gamma} : x, y \in K\right\} & \text{if } n = 0, \\ \sum_{|\nu|<n} \|D^\nu f\|_{C^{0,\gamma}(K)} & \text{if } n \neq 0. \end{cases}$$

For simplicity we write $C^\kappa := C^{n,\gamma}$, where $n = \lfloor\kappa\rfloor, \gamma = \kappa - \lfloor\kappa\rfloor$, when $\kappa > 0$ is not an integer.

Definition 2.2. For an open set D, we write $f \in C^{n,\gamma}(D)$ if $f \in C^{n,\gamma}(K)$ for every compact subset K of D.

Definition 2.3. An open set D in $\mathbb{R}^d$ is of *class* $C^{n,\gamma}$ if there exists a radius $r > 0$ and a constant C such that for every $z \in \partial D$ there exist an isometry $\phi : \mathbb{R}^d \to \mathbb{R}^d$ and a function $f \in C^{n,\gamma}(\mathbb{R}^{d-1})$ such that $\phi(z) = 0$, $\|f\|_{C^{n,\gamma}(\mathbb{R}^{d-1})} \le C$ and $\phi(D) \cap B(0,r) = \{x \in \mathbb{R}^d : x_d > f(\tilde{x})\} \cap B(0,r)$.

Recall that a random variable X has a *strictly stable distribution* if for every $a, b > 0$ there exist $c > 0$ such that $aX_1 + bX_2$ and cX have the same distribution whenever X_1, X_2 are independent copies of X. In this case there exists $\alpha \in (0,2]$ such that $a^\alpha + b^\alpha = c^\alpha$. We say that α is the *index of stability* of X.

Recall also that $X = \{X_t\}_{t\in[0,\infty)}$ is a *Lévy process* if it is an $\mathbb{R}^d$-valued stochastic process with $X_0 = 0$, stationary and independent increments and càdlàg paths.

A Lévy process is described by the characteristic exponent Ψ, which is given by the Lévy–Khinchin formula:

$$\Psi(\xi) = \log(\mathbb{E}e^{i\langle\xi,X_1\rangle}) = -\langle A\xi,\xi\rangle + i\langle\gamma,\xi\rangle - \int_{\mathbb{R}^d\setminus\{0\}} (1 - e^{i\langle\xi,z\rangle} + i\langle\xi,z\rangle\mathbf{1}_{B(0,1)}(z))\,\nu(dz) \tag{2.1}$$

for $\xi \in \mathbb{R}^d$, where A is the (non-negative definite) covariance matrix of the Gaussian part, $\gamma \in \mathbb{R}^d$ is the linear term related to the drift of X_t and ν is a non-negative measure such that $\int_{\mathbb{R}^d\setminus\{0\}} \min(1,|z|^2)\,\nu(dz) < \infty$, called the *Lévy measure*. We denote by $\mathbb{E}^x$ the expectation corresponding to the process X_t with the condition $X_0 = 0$ a.s. replaced by $X_0 = x$ a.s. We denote by τ_D the first time the process X exits an open set D, that is,

$$\tau_D = \inf\{t > 0 : X_t \notin D\}.$$

We say that $X = \{X_t\}_{t\in(0,\infty)}$ is a *strictly α-stable Lévy process* when it is a Lévy process such that X_t has a strictly α-stable distribution for every $t > 0$.

Definition 2.4. We define the *transition operator* p_t of the process X by the formula

$$p_t f(x) = \mathbb{E}^x f(X_t)$$

and the *generator* $\mathcal{L}$ of the process X applied to a function f by the formula

$$\mathcal{L}f(x) = \lim_{t\to 0^+} \frac{p_t f(x) - f(x)}{t} \tag{2.2}$$

for every $f \in C_0$ such that the above limit exists uniformly on $\mathbb{R}^d$.

Definition 2.5. We define the *Dynkin generator* $\mathcal{L}_{\mathcal{D}}$ of the process X applied to a function f at a point x by the formula

$$\mathcal{L}_{\mathcal{D}}f(x) = \lim_{r\to 0^+} \frac{\mathbb{E}^x f(X_{\tau_{B(x,r)}}) - f(x)}{\mathbb{E}^x \tau_{B(x,r)}} \tag{2.3}$$

for every $f \in C_0$ and $x \in \mathbb{R}^d$ such that the above limit exists.

It is known that if f is in the domain of the generator $\mathcal{L}$, then $\mathcal{L}_{\mathcal{D}}f(x)$ is well-defined for every x and $\mathcal{L}_{\mathcal{D}}f(x) = \mathcal{L}f(x)$. Conversely, if $f \in C_0$ and $\mathcal{L}_{\mathcal{D}}f(x)$ is well-defined for every x, and $\mathcal{L}_{\mathcal{D}}f \in C_0$, then f is in the domain of $\mathcal{L}$. We refer to [12, Chapter V] for a proof and further discussion.

For every open set D there exists a *Green function* $G_D(x,y)$ such that $G_D(x,y) \ge 0$ for $x,y \in D$ and $G_D(x,y) = 0$ for $x \in D^c$ or $y \in D^c$ such that $G_D(x,y)$ is a continuous map from $D \times D$ into $[0,\infty]$, and

$$\int_D G_D(x,y)f(y)\,dy = \mathbb{E}^x \int_0^{\tau_D} f(X_t)\,dt$$

for every non-negative function f. In particular,

$$\int_D G_D(x,y)\,dy = \mathbb{E}^x \tau_D$$

for every $x, y \in D$.

We will use the *Ikeda–Watanabe formula*, which states that for every open set D, $x \in D$ and a Lévy process X with Lévy measure ν we have

$$\mathbb{E}^x(f(X_{\tau_D})) = \int_D G_D(x,y) \int_{D^c} \nu(z-y) f(z)\,dz\,dy$$

for every non-negative function f such that $f = 0$ in $\overline{D}$.

Definition 2.6. We say that a function f is *harmonic* for X in an open set D if for every bounded open set B such that $\overline{B} \subset D$ and $x \in B$ we have

$$\mathbb{E}^x f(X_{\tau_B}) = f(x).$$

We say that a function is *regular harmonic* when the above equality holds also for $B = D$. If a function is regular harmonic in an open set D, then it is regular harmonic in any open subset of D.

Remark 2.7. If a function f is harmonic in an open set D, then for every x in D we have $\mathcal{L}_{\mathcal{D}} f(x) = 0$.

We need two elementary, technical results.

Lemma 2.8. *For $p, q \in (0,1)$, $x, y > 0$ and $\eta \in (0,1]$ there exists $c = c(q,\eta)$ such that*

$$|x^q - y^q| \le c \max(x,y)^{q-\eta} |x-y|^\eta, \tag{2.4}$$

$$|x^p - x^q| \le |\ln(x)| \max(x^p, x^q) |p-q|. \tag{2.5}$$

Proof. Without loss of generality we assume that $x > y$. We have

$$\begin{aligned} |x^q - y^q| &= \frac{|x^q - y^q|}{|x-y|^\eta} |x-y|^\eta \\ &= \frac{|1-(y/x)^q|}{|1-y/x|^\eta} x^{q-\eta} |x-y|^\eta = \frac{|1-s^q|}{|1-s|^\eta} x^{q-\eta} |x-y|^\eta \end{aligned}$$

for $s = y/x \in [0,1)$. Since $0 < \eta \le 1$, by l'Hospital's rule the limit

$$\lim_{s\to 1} \frac{1-s^q}{(1-s)^\eta} = \lim_{s\to 1} \frac{q s^{q-1}}{\eta(1-s)^{\eta-1}}$$

is equal to 0 for $\eta < 1$ and q for $\eta = 1$. Since the function $\frac{1-s^q}{(1-s)^\eta}$ is continuous on $[0,1)$ and has a limit as $s \to 1$, it is bounded on $[0,1]$ by some constant c. It follows that

$$|x^q - y^q| \le c \max(x,y)^{q-\eta} |x-y|^\eta.$$

For the second inequality we write

$$|x^p - x^q| = \left|\int_p^q \ln(x)x^t\,dt\right| \le |\ln(x)|\max(x^p,x^q)|p-q|. \ \blacksquare$$

LEMMA 2.9. *For any closed, convex set K and any $f \in C^\gamma(K)$ with $1 < \gamma < 2$, we have*

$$|f(x) - f(y) - \langle x-y, \nabla f(y)\rangle| \le \|f\|_{C^\gamma(K)}|x-y|^\gamma$$

for every $x, y \in K$.

Proof. By the mean value theorem,

$$\begin{aligned} f(x) - f(y) - \langle x-y, \nabla f(y)\rangle &= \langle x-y, \nabla f(x_1)\rangle - \langle x-y, \nabla f(y)\rangle \\ &= \langle x-y, \nabla f(x_1) - \nabla f(y)\rangle \end{aligned}$$

for some $x_1 = (1-s)x + sy$, $s \in [0,1]$. Thus we get

$$\begin{aligned} |f(x) - f(y) - \langle x-y, \nabla f(y)\rangle| &\le \|f\|_{C^\gamma(K)}|x-y||x_1-y|^{\gamma-1} \\ &\le \|f\|_{C^\gamma(K)}|x-y|^\gamma. \ \blacksquare \end{aligned}$$

2.2. Assumptions and properties of the process X

ASSUMPTION A. X is a strictly α-stable d-dimensional Lévy process with $d \ge 2$ and $\alpha \in (0,2)$. The Lévy measure of X is absolutely continuous with respect to the Lebesgue measure and it is given by

$$\nu(A) = \int_A \frac{1}{|x|^{d+\alpha}}\vartheta\left(\frac{z}{|z|}\right)dz,$$

where $\vartheta \in C^\epsilon(S)$ for some $\epsilon > 0$ and $\vartheta(z) > 0$ for all $z \in S$.

Assumption A implies that if $\alpha \ne 1$ the Lévy–Khinchin exponent of the process X has coefficients a and γ equal to 0. If $\alpha = 1$, the coefficient a is 0 and the function ϑ is symmetric. Moreover, strictly α-stable processes are scaling invariant.

DEFINITION 2.10. We define the *pointwise generator* $\mathcal{A}$ of a process X at a point x by

$$\mathcal{A}f(x) = \begin{cases} \int_{\mathbb{R}^d} (f(y) - f(x))\nu(y-x)\,dy & \text{if } \alpha < 1, \\ \langle \gamma, \nabla f(x)\rangle + \int_{\mathbb{R}^d} (f(y) - f(x) \\ \quad -\langle \nabla f(x), y-x\rangle \mathbf{1}_{B(x,r)}(y))\nu(y-x)\,dy & \text{if } \alpha = 1, \\ \int_{\mathbb{R}^d} (f(y) - f(x) - \langle \nabla f(x), y-x\rangle)\nu(y-x)\,dy & \text{if } \alpha > 1, \end{cases} \tag{2.6}$$

for every function f for which the integral is finite at x. In particular, this is the case for any bounded function f which is $C^{\alpha+\epsilon}$ in some neighbourhood of x for some $\epsilon > 0$; see [19]. Note that in case $\alpha = 1$, since the Lévy measure of the process is symmetric, the definition of $\mathcal{A}$ does not depend on $r > 0$.

DEFINITION 2.11. For any unit vector $u \in S^d$, we define the one-dimensional Lévy process $X^u = \{\langle X_t, u\rangle\}_{t\in\mathbb{R}^+}$ which is the orthogonal projection of X onto the line $\{xu : x \in \mathbb{R}\}$. We denote by ν_u its Lévy measure. Also, for $z \in \mathbb{R}^d$ we denote by $\mathbb{H}_{u,z}$ the half-space $\{x : \langle x - z, u\rangle > 0\}$.

LEMMA 2.12. *The process X^u is a one-dimensional strictly α-stable Lévy process. Its Lévy measure ν_u is absolutely continuous with respect to the Lebesgue measure and its density $\nu_u(z)$ is given by*

$$\nu_u(z) = \begin{cases} \frac{1}{z^{\alpha+1}} \int_{S_u} \vartheta(w)\langle u, w\rangle^\alpha \, dw & \text{if } z > 0, \\ \nu_{-u}(-z) & \text{if } z < 0, \end{cases} \tag{2.7}$$

where $S_u = S^d \cap \mathbb{H}_{u,0}$ and dw is the surface measure on the unit sphere.

Proof. We begin by calculating the tail of the measure ν_u. Let $x \in \mathbb{R}^d$ and $z_0 > 0$. We have

$$\int_{z_0}^{\infty} \nu_u(dz) = \int_{\mathbb{H}_{u,z_0 u}} |x|^{-d-\alpha} \vartheta\left(\frac{x}{|x|}\right) dx.$$

We use spherical coordinates:

$$\int_{z_0}^{\infty} \nu_u(dz) = \int_{S_u} \int_{z_0/\langle u,w\rangle}^{\infty} r^{-1-\alpha} \vartheta(w) \, dr \, dw = \frac{1}{\alpha} \frac{1}{z_0^\alpha} \int_{S_u} \vartheta(z)\langle u, w\rangle^\alpha \, dw.$$

By differentiation, we get (2.7). The case of $z_0 < 0$ is very similar. ∎

Since X^u is a one-dimensional α-stable Lévy process, below we recall some facts about harmonic functions for those processes.

THEOREM 2.13 (see [3, Lemma VII.11]). *Let Y be a one-dimensional α-stable Lévy process. Let $\beta = \alpha\mathbb{P}(Y_1 > 0)$. Then the function*

$$h(x) = x^\beta \mathbf{1}_{(0,\infty)}(x)$$

is regular harmonic for Y in $(0, a)$ for every $a > 0$.

Recall that for a one-dimensional strictly α-stable Lévy process, the Lévy measure μ is absolutely continuous with respect to the Lebesgue measure and its density is given by

$$\mu(z) = C^- \frac{1}{|z|^{\alpha+1}} \mathbf{1}_{(-\infty,0)}(z) + C^+ \frac{1}{|z|^{\alpha+1}}(z) \mathbf{1}_{(0,\infty)}, \tag{2.8}$$

where $C^-, C^+ \geq 0$, $C^- + C^+ > 0$ and if $\alpha = 1$ then necessarily $C^- = C^+$. In that case the parameter β can be given explicitly by the formula (see [24])

$$\beta = \frac{\alpha}{2} + \frac{1}{\pi}\arctan\left(\frac{C^+ - C^-}{C^+ + C^-}\tan\frac{\alpha\pi}{2}\right) \tag{2.9}$$

if $\alpha \neq 1$, while for $\alpha = 1$ we have $C^+ = C^- > 0$ and

$$\beta = \mathbb{P}(X_1 > 0) = \int_0^\infty \frac{1}{\pi}\frac{C^+}{(C^+)^2 + (x-b)^2}\,dx = \frac{1}{2} + \frac{1}{\pi}\arctan\frac{b}{C^+}, \tag{2.10}$$

where b is the the drift of the process.

In the remaining part of this article we use the objects defined in Theorem 2.13 for the projections X^u. In this case we denote the dependence on u by writing $C^+(u)$, $C^-(u)$ and $\beta(u)$.

LEMMA 2.14. *There are constants* $\beta_{\min} = \beta_{\min}(X) > \max\{0, \alpha - 1\}$ *and* $\beta_{\max} = \beta_{\max}(X) < \min(\alpha, 1)$ *such that* $\beta_{\min} + \beta_{\max} = \alpha$ *and*

$$\beta_{\min} \leq \beta(u) \leq \beta_{\max} \quad \text{for every } u \in S^d.$$

Proof. There exists a constant $c = c(X) \in (0, 1)$ such that

$$-1 + c \leq \frac{C^+(u) - C^-(u)}{C^+(u) + C^-(u)} \leq 1 - c$$

for any $u \in S$. By (2.9) we have

$$\frac{\alpha}{2} - \frac{1}{\pi}\left|\arctan\left(\tan\frac{\alpha\pi}{2}\right)\right| < \beta(u) < \frac{\alpha}{2} + \frac{1}{\pi}\left|\arctan\left(\tan\frac{\alpha\pi}{2}\right)\right|$$

for $\alpha \in (0, 2) \setminus \{1\}$. Since

$$\arctan\left(\tan\frac{\alpha\pi}{2}\right) = \begin{cases} \alpha\pi/2 & \text{if } \alpha < 1, \\ (\alpha - 2)\pi/2 & \text{if } \alpha > 1, \end{cases}$$

and since $\beta(u)$ is a continuous function on a compact set, we have

$$\max(0, \alpha - 1) < \beta(u) < \min(\alpha, 1)$$

for $\alpha \in (0, 2) \setminus \{1\}$. When $\alpha = 1$, the desired result follows from (2.10). ∎

Note that the constants $\beta_{\min}$ and $\beta_{\max}$, even though depending on X, are invariant under the orthogonal changes of the coordinate system and scaling. We keep the notation $\beta_{\min}$ and $\beta_{\max}$ till the end of this article.

REMARK 2.15. By (2.8) and (2.7), the function $C^+(u)$ is a spherical convolution of a $C^\alpha(S^d)$ "zonal" function $w \mapsto \max(\langle u, w\rangle, 0)^\alpha$ and a $C^\epsilon(S^d)$ function θ. Thus, $C^+(u)$ and $C^-(u) = C^+(-u)$ belong to $C^{\alpha+\epsilon}(S^d)$. By (2.9) for $\alpha \neq 1$, and by (2.10) for $\alpha = 1$, the function $\beta(u)$ is in $C^{\alpha+\epsilon}(S^d)$.

LEMMA 2.16. *Let X be a d-dimensional strictly α-stable Lévy process. Let $u \in S^d$. Then the function*

$$h_{u,z}(x) = (\delta_{\mathbb{H}_{u,z}}(x))^{\beta(u)}$$

is a regular harmonic function for X in $D \cap \mathbb{H}_{u,z}$ for every bounded open set D.

Proof. By an appropriate change of the coordinate system and scaling, we may assume that $z = 0$ and $u = (0, \dots, 0, 1)$. Let D be a bounded open set. Let $h(x_d)$ be defined as in Theorem 2.13 for $Y = X^u$. We define $U_t = \{z \in \mathbb{R}^d : 0 < x_d < t\}$ and we choose t such that $D \cap \mathbb{H}_{u,z} \subset U_t$. We have

$$\mathbb{E}^x h_{u,z}(X_{\tau_{U_t}}) = \mathbb{E}^{x_d} h(X^u_{\tau_{(0,t)}}) = h(x_d) = h_{u,z}(x), \tag{2.11}$$

where $\mathbb{E}^x$ and $\mathbb{E}^{x_d}$ are the expectations for the d-dimensional process X and its orthogonal projection X^u, respectively. By (2.11), the function $h_{u,z}$ is regular harmonic in U_t. Since $D \cap \mathbb{H}_{u,z} \subset U_t$, $h_{u,z}$ is also regular harmonic in $D \cap \mathbb{H}_{u,z}$. ■

COROLLARY 2.17. *The function $\mathcal{A}h_{u,z}(x)$ is well-defined for every $x \in \mathbb{H}_{u,z}$ and*

$$\mathcal{A}h_{u,z}(x) = 0 \quad \text{for } x \in \mathbb{H}_{u,z}.$$

Proof. Let $x \in \mathbb{H}_{u,z}$ and let r be a radius such that $B(x,r) \subset \mathbb{H}_{u,z}$. Then $h_{u,z} \in C^\infty(B(x,r))$. Since it is harmonic in $\mathbb{H}_{u,z}$, by Remark 2.7 it belongs to the domain of the Dynkin generator $\mathcal{L_D}$ at the point x and $\mathcal{L_D} h_{u,z}(x) = 0$. We define

$$h^*_{u,z}(y) = h_{u,z}(y) \quad \text{for } y \in B(x,r)$$

and extend it to a smooth and compactly supported function. The function $h^*_{u,z}$ also belongs to the domain of the Dynkin generator $\mathcal{L_D}$, as well as to the domain of the pointwise generator $\mathcal{A}$, and

$$\mathcal{L_D} h^*_{u,z}(x) = \mathcal{A} h^*_{u,z}(x). \tag{2.12}$$

The difference $h_{u,z}(x) - h^*_{u,z}(x)$ is equal to 0 on $B(x,r)$, thus, by the Ikeda–Watanabe formula, we have

$$\mathcal{L_D}(h_{u,z} - h^*_{u,z})(x) = \lim_{s\to 0^+} \frac{\int_{B(x,s)} G_{B(x,s)}(x,y) \int_{B(x,r)^c} \nu(v-y)(h_{u,z} - h^*_{u,z})(v)\,dv\,dy}{\int_{B(x,s)} G_{B(x,s)}(x,y)\,dy}.$$

where $G_{B(x,s)}(x,z)$ is the Green function of $B(x,s)$. Observe that if $y \in B(x,r/2)$ and $v \in B(x,r)^c$, then $|v-y| > r/2$, and so $\nu(v-y)(h_{u,z} - h^*_{u,z})$ is a continuous function of $y \in B(x,r/2)$ and $v \in B(x,r)^c$, bounded by an integrable function of $v \in B(x,r)^c$ uniformly in $y \in B(x,r/2)$. It follows that

$\int_{B(x,r)^c} \nu(z-y)(h_{u,z}-h^*_{u,z})(z)\,dz$ is a continuous function of $y \in B(x,r/2)$. As $s \to 0$, the measures $\frac{G_{B(x,s)}(x,y)\,dy}{\int_{B(x,s)} G_{B(x,s)}(x,y)\,dy}$ converge vaguely to the Dirac measure at x, and thus

$$\mathcal{L}_{\mathcal{D}}(h_{u,z}-h^*_{u,z})(x) = \int_{B(x,r)^c} \nu(z-x)(h_{u,z}-h^*_{u,z})(z)\,dz = \mathcal{A}(h_{u,z}-h^*_{u,z})(x). \tag{2.13}$$

By combining (2.12) with (2.13) we get the desired result. ∎

2.3. Regularity of the domain D

Assumption B. If $\alpha \in (0,1)$, then D is a bounded $C^{1,1}$ open set. If $\alpha \in [1,2)$, then D is a bounded $C^{2,\alpha+\epsilon-1}$ set for some $\epsilon > 0$.

Remark 2.18. If D is a $C^{1,1}$ open set, it satisfies the uniform exterior and the uniform interior ball conditions: for some $r(D) > 0$, for every $z \in \partial D$ there are points x_1, x_2 such that $B(x_1,r) \subset D$, $B(x_2,r) \subset D^c$ and $z \in \overline{B}(x_1,r) \cap \overline{B}(x_2,r)$.

Recall that $D^*_r = \{x \in D : \delta_D(x) < r\}$.

Definition 2.19. Let r be as in Remark 2.18. For $x \in D^*_r$, we let $z(x)$ be the unique point on ∂D such that $\delta_D(x) = |x - z(x)|$. If $x \in \partial D$ we define $z(x) = x$. We define $n(x)$ to be the inward-pointing normal vector to the boundary of D at $z(x)$.

Lemma 2.20. *Let D satisfy Assumption B. There exists $R(D) > 0$ such that the functions $z(x)$ and $n(x)$ are Lipschitz continuous on $\overline{D^*_R}$ for $\alpha < 1$, and are $C^{\alpha+\epsilon}(\overline{D^*_R})$ functions for some $\epsilon > 0$ if $\alpha \geq 1$.*

Proof. By [10, Theorem 3.1] the distance function $\delta_D(x)$ is in $C^{1,1}(\overline{D^*_R})$ for some R. Note that $\nabla\delta_D(x) = n(x)$, thus $n(x)$ is a Lipschitz function on $\overline{D^*_R}$. Since $z(x) = x - \delta_D(x)\nabla\delta_D(x)$, it is also a Lipschitz function on $\overline{D^*_R}$. If $\alpha \geq 1$ and D is a $C^{2,\alpha-1+\epsilon}$ set for some $\epsilon > 0$ then, again by [10, Theorem 3.1], $\delta_D(x)$ is in $C^{2,\alpha-1+\epsilon}(\overline{D^*_R})$, thus $\nabla\delta_D$ and z are in $C^{\alpha+\epsilon}(\overline{D^*_R})$. ∎

Remark 2.21. Since harmonic functions for the process X are scale-invariant, the constants $\beta_{\max}, \beta_{\min}$ and the function $h_{u,z}$ that will be used later in this article do not change if we scale the process X or (equivalently) scale the coordinate system. To simplify the notation, till the end of the article we choose a coordinate system, together with its scale, in such a way that $0 \in \partial D$,

$$e_d := n(0) = (0,\dots,0,1),$$

the radius r defined in Remark 2.18 is not less than 2, and the radius R defined in Lemma 2.20 is greater than or equal to 1.

Corollary 2.22. *Let D satisfy Assumption B. The function $\beta(n(x))$ is in $C^{\alpha+\epsilon}(\overline{D_1^*})$ for some $\epsilon > 0$ and $\|\beta(n(\cdot))\|_{C^{\alpha+\epsilon}(\overline{D_1^*})} \le C(X,D)$ for some $C(X,D) > 0$.*

Proof. If $\alpha < 1$, by Remark 2.15 and Lemma 2.20 we have $\beta \in C^{\alpha+\epsilon}(S^d)$ for some $\epsilon > 0$ and $n(x)$ is a Lipschitz continuous function, thus the composition $\beta(n(\cdot))$ belongs to $C^{\alpha+\epsilon}(\overline{D_1^*})$ for some $\epsilon > 0$. If $\alpha \ge 1$, by Remark 2.15 and Lemma 2.20 we have $\beta \in C^{\alpha+\epsilon}(S^d)$ and $n \in C^{\alpha+\epsilon}(\overline{D_1^*})$, thus the composition belongs to $C^{\alpha+\epsilon}(\overline{D_1^*})$. ∎

To simplify the notation, we write $\beta(x)$ instead of $\beta(n(x))$ if $x \in \overline{D_1^*}$. Recall that $D_r = D \cap B(0,r)$.

Lemma 2.23. *Let D satisfy Assumption B. For every $x = (\tilde{x}, x_d) \in \overline{D}_1$, we have*

$$|\delta_D(x) - x_d| \le \tfrac{1}{2}|\tilde{x}|^2. \tag{2.14}$$

Proof. Let f be a function such that $\partial D \cup B(0,2)$ is contained in the graph of f (see Definition 2.3). By the uniform exterior ball condition with radius 2, we have

$$f(\tilde{x}) \ge -2 + \sqrt{4 - |\tilde{x}|^2} \ge -\tfrac{1}{2}|\tilde{x}|^2$$

for $|\tilde{x}| \le 2$. Thus, for $x \in \overline{D_2}$,

$$\delta_D(x) \le d(x, (\tilde{x}, f(\tilde{x}))) = |x_d - f(\tilde{x})| = x_d - f(\tilde{x}) \le x_d + \tfrac{1}{2}|\tilde{x}|^2. \tag{2.15}$$

On the other hand, since for $|x| \le 1$ we have $2 - x_d + \frac{1}{2}|\tilde{x}|^2 \ge \sqrt{|\tilde{x}|^2 + (2-x_d)^2}$ (which follows by squaring both sides), and by the uniform interior ball condition, it follows that

$$\delta_D(x) \ge \delta_{B(2e_d,2)}(x) = 2 - \sqrt{|\tilde{x}|^2 + (2-x_d)^2} \ge x_d - \tfrac{1}{2}|\tilde{x}|^2. \tag{2.16}$$

By combining (2.15) and (2.16), we get (2.14). ∎

3. Proof of the main theorem. The main goal of this section is to provide explicit decay rates of harmonic functions at a boundary point z of the set D. In the remainder of the article we will always assume that the process X satisfies Assumption A and the set D satisfies Assumption B. We choose the coordinate system (and scaling) as in Remark 2.21 and we fix $x_0 = (0, 0, \ldots, x_{0d}) = x_{0d}e_d$ such that $\delta_D(x_0) \le 1/2$. Finally, we define $\mathbb{H}(x) = \mathbb{H}_{e_d,0}(x)$ and $h(x) = h_{e_d,0}(x)$.

Definition 3.1. We define

$$g(x) = (\delta_D(x))^{\beta(x)} \mathbf{1}_{\overline{D_1^*}}(x),$$

where the function β is given in Corollary 2.22; see Figure 1.

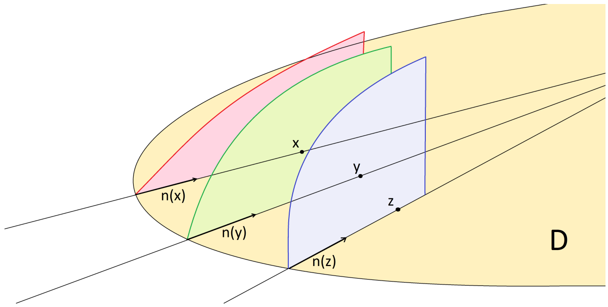

Fig. 1. The function g is of power type with different exponents for different directions.

REMARK 3.2. By Lemma 2.20 and Corollary 2.22, $g \in C^{\alpha+\epsilon}(\overline{D_1^*})$ for some $\epsilon > 0$. Moreover, g is bounded by 1.

REMARK 3.3. By Lemma 2.14, the interval $(\beta_{\max}, \min(\alpha, 1))$ is non-empty. In the remainder of this article we fix

$$\eta \in (\beta_{\max}, \min(\alpha, 1)), \tag{3.1}$$

so that, since $\beta_{\min} + \beta_{\max} = \alpha$, we have

$$2\eta-\alpha-2<-1, \quad \beta_{\min}+\eta-\alpha>0, \quad -1<\beta_{\min}-\eta<0, \quad 2\eta-\alpha-1>-1, \tag{3.2}$$

which we will use later.

DEFINITION 3.4. We define

$$\begin{aligned} f_1(x) &= ((\delta_{\mathbb{H}}(x))^{\beta(x)} - (\delta_{\mathbb{H}}(x))^{\beta(x_0)})\mathbf{1}_{D_1\cap\mathbb{H}}(x), \\ f_2(x) &= ((\delta_D(x))^{\beta(x)} - (\delta_{\mathbb{H}}(x))^{\beta(x)})\mathbf{1}_{D_1\cap\mathbb{H}}(x), \\ f_3(x) &= (g(x) - h(x))\mathbf{1}_{(\mathbb{R}^d\setminus(D_1\cap\mathbb{H}))}(x). \end{aligned} \tag{3.3}$$

Since for $x \in D_1 \cap \mathbb{H}$ we have

$$\begin{aligned} g(x) - h(x) &= (\delta_D(x))^{\beta(x)} - (\delta_{\mathbb{H}}(x))^{\beta(x_0)} \\ &= (\delta_D(x))^{\beta(x)} - (\delta_{\mathbb{H}}(x))^{\beta(x)} + (\delta_{\mathbb{H}}(x))^{\beta(x)} - (\delta_{\mathbb{H}}(x))^{\beta(x_0)}, \end{aligned}$$

by (3.3),

$$g(x) - h(x) = f_1(x) + f_2(x) + f_3(x). \tag{3.4}$$

LEMMA 3.5. *There exist $\epsilon = \epsilon(X, D) > 0$ and a constant $c = c(X, D)$ such that*

$$\begin{aligned} |f_1(x)| &\le c|x - x_0|^{\alpha+\epsilon} && \textit{if } \alpha < 1, \\ |f_1(x) - \langle \nabla f_1(x_0), x - x_0\rangle| &\le c|x - x_0|^{\alpha+\epsilon} && \textit{if } \alpha \ge 1, \end{aligned}$$

for $x \in D_1 \cap \mathbb{H}$.

Proof. Let $x \in D_1 \cap \mathbb{H}$. When $\alpha < 1$, by (3.3) and (2.5) we have

$$|f_1(x)| \le |\ln(\delta_{\mathbb{H}}(x))| \max(\delta_{\mathbb{H}}(x)^{\beta(x)}, \delta_{\mathbb{H}}(x)^{\beta(x_0)})|\beta(x) - \beta(x_0)|.$$

Since $\beta(x), \beta(x_0) \in [\beta_{\min}, \beta_{\max}]$, it follows that there exists $c(X, D)$ such that

$$|f_1(x)| \le c|\beta(x) - \beta(x_0)|\mathbf{1}_{D_1\cap\mathbb{H}}(x).$$

By Corollary 2.22, β is in $C^{\alpha+\epsilon}(\overline{D_1^*})$ for some $\epsilon > 0$. Thus,

$$|f_1(x)| \le c|x - x_0|^{\alpha+\epsilon}\mathbf{1}_{D_1\cap\mathbb{H}}(x)$$

for some $\epsilon > 0$, as desired.

We now consider $\alpha \ge 1$. For the notational convenience, till the end of this proof we write $\beta_0 := \beta(x_0)$, $\beta_1 := \beta(x)$, $\delta_0 := \delta_{\mathbb{H}}(x_0)$, $\delta_1 := \delta_{\mathbb{H}}(x)$ and $v := \nabla\beta(x_0)$. By Corollary 2.22, β is in $C^{\alpha+\epsilon}(\overline{D_1^*})$ for some $\varepsilon > 0$. In particular, $\nabla\beta$ exists and is bounded by a constant $c(X, D)$. By a simple calculation,

$$\nabla f_1(x_0) = (\ln \delta_0)\delta_0^{\beta_0} v.$$

For later use, we record that as a consequence, for every $\epsilon > 0$ there exists a constant $c = c(X, D, \epsilon)$ such that

$$|\nabla f_1(x_0)| = |\ln(\delta_0)\delta_0^{\beta_0} v| \le c\delta_0^{\beta_0-\epsilon} = c\delta_{\mathbb{H}}(x_0)^{\beta(x_0)-\epsilon}. \tag{3.5}$$

We come back to the proof of the lemma. Observe that by the triangle and Cauchy–Schwarz inequalities,

(3.6)

$$\begin{aligned}
|f_1(x) - \langle\nabla f_1(x_0), x - x_0\rangle| &= |\delta_1^{\beta_1} - \delta_1^{\beta_0} - \langle\delta_0^{\beta_0}\ln(\delta_0)v, x - x_0\rangle| \\
&\le |\delta_1^{\beta_1} - \delta_1^{\beta_0} - \delta_1^{\beta_0}\ln(\delta_1)(\beta_1 - \beta_0)| \\
&\quad + |\delta_1^{\beta_0}\ln(\delta_1)(\beta_1 - \beta_0) - \langle\delta_1^{\beta_0}\ln(\delta_1)v, x - x_0\rangle| \\
&\quad + |\langle\delta_1^{\beta_0}\ln(\delta_1)v, x - x_0\rangle - \langle\delta_0^{\beta_0}\ln(\delta_0)v, x - x_0\rangle| \\
&\le |\delta_1^{\beta_1} - \delta_1^{\beta_0} - \delta_1^{\beta_0}\ln(\delta_1)(\beta_1 - \beta_0)| \\
&\quad + |\beta_1 - \beta_0 - \langle v, x - x_0\rangle|\delta_1^{\beta_0}|\ln(\delta_1)| \\
&\quad + |\delta_1^{\beta_0}\ln(\delta_1) - \delta_0^{\beta_0}\ln(\delta_0)|\,|v|\,|x - x_0|.
\end{aligned}$$

Recall that $\beta_1, \beta_0 \in [\beta_{\min}, \beta_{\max}]$. By Taylor expansion, there exist $\beta_2 \in (\beta_{\min}, \beta_{\max})$ lying between β_0 and β_1, and $c(\beta_{\min})$, such that

$$|\delta_1^{\beta_1} - \delta_1^{\beta_0} - \delta_1^{\beta_0}\ln(\delta_1)(\beta_1 - \beta_0)| = \tfrac{1}{2}|\delta_1^{\beta_2}\ln^2(\delta_1)(\beta_1 - \beta_0)^2| \le c|\beta_1 - \beta_0|^2.$$

Since β is a Lipschitz continuous function on $\overline{D_1^*}$, there exists a constant $c = c(X, D)$ such that

$$|\delta_1^{\beta_1} - \delta_1^{\beta_0} - \delta_1^{\beta_0}\ln(\delta_1)(\beta_1 - \beta_0)| \le c|x - x_0|^2. \tag{3.7}$$

By Corollary 2.22, β is in $C^{\alpha+\epsilon}(\overline{D_1^*})$ for some $\epsilon > 0$. Thus, by Lemma 2.9, there exists a constant $c = c(X, D)$ such that for some $\epsilon = \epsilon(X, D) > 0$ we have

$$|\beta_1 - \beta_0 - \langle v, x - x_0\rangle|\delta_1^{\beta_0}|\ln(\delta_1)| \le c|x - x_0|^{\alpha+\epsilon}. \tag{3.8}$$

As $\beta_0 > \alpha - 1$, for some $\epsilon = \epsilon(X, D) > 0$ we have $\delta_{\mathbb{H}}(\cdot)^{\beta_0}\ln(\delta_{\mathbb{H}}(\cdot)) \in C^{\alpha-1+\epsilon}(\overline{D_1^*})$. Since $|\nabla\beta|$ is bounded, there exists a constant $c = c(X, D)$ such that

$$|\delta_1^{\beta_0}\ln(\delta_1) - \delta_0^{\beta_0}\ln(\delta_0)|\,|v|\,|x - x_0| \le c|x - x_0|^{\alpha+\epsilon} \tag{3.9}$$

for some $\epsilon > 0$. By combining (3.6)–(3.9) we get the desired result for $\alpha \ge 1$. ∎

Recall that η was chosen in Remark 3.3 so that it satisfies (3.2).

LEMMA 3.6. *There exists a constant $c = c(X, D) > 0$ such that*

$$|f_2(x)| \le c\max(\delta_D(x), \delta_{\mathbb{H}}(x))^{\beta_{\min}-\eta}|\tilde{x}|^{2\eta}\mathbf{1}_{D_1\cap\mathbb{H}}(x).$$

Proof. Let $x \in D_1 \cap \mathbb{H}$. By Lemma 2.23, (2.4) and (3.3) there exists a constant $c = c(X, D)$ such that

$$\begin{aligned}|f_2(x)| &\le c\max(\delta_D(x), \delta_{\mathbb{H}}(x))^{\beta(x)-\eta}|\delta_D(x) - \delta_{\mathbb{H}}(x)|^{\eta}\\ &\le c\max(\delta_D(x), \delta_{\mathbb{H}}(x))^{\beta(x)-\eta}|\tilde{x}|^{2\eta}.\end{aligned}$$

Since $\max(\delta_D(x), \delta_{\mathbb{H}}(x)) \le 1$ for $x \in D_1 \cap \mathbb{H}$, we have

$$\max(\delta_D(x), \delta_{\mathbb{H}}(x))^{\beta(x)-\eta} \le \max(\delta_D(x), \delta_{\mathbb{H}}(x))^{\beta_{\min}-\eta}.$$

Thus

$$|f_2(x)| \le c\max(\delta_D(x), \delta_{\mathbb{H}}(x))^{\beta_{\min}-\eta}|\tilde{x}|^{2\eta}. \ ∎$$

REMARK 3.7. By Remark 3.3 and the fact that

$$\beta_{\max} \ge \max(\beta(u), \beta(-u)) \ge \tfrac{1}{2}\beta(u) + \tfrac{1}{2}\beta(-u) = \alpha/2,$$

we have $2\eta > 2\beta_{\max} \ge \alpha$. Thus, in the case $\alpha \ge 1$, $\nabla f_2(x_0)$ exists and it is equal to 0.

REMARK 3.8. We have

$$|f_3(x)| \le g(x)\mathbf{1}_{\overline{D_1^*}\setminus(D_1\cap\mathbb{H})} + h(x)\mathbf{1}_{\mathbb{H}\setminus D_1}(x).$$

LEMMA 3.9. *There exists a constant $c_{\text{gen}} = c_{\text{gen}}(X, D) > 0$ such that for $x \in D_{1/2}^*$,*

$$|\mathcal{A}g(x)| \le c_{\text{gen}}.$$

Proof. It is enough to show that $|\mathcal{A}g(x_0)| \le c_{\text{gen}}$ for some constant $c_{\text{gen}} = c_{\text{gen}}(X, D) > 0$. The result in the general case follows then by an appropriate change of coordinates; see Remark 2.21. To estimate $\mathcal{A}g(x_0)$, we will compare the functions g and h. Recall that, by Lemma 2.16, h is harmonic on $\mathbb{H}$.

By (3.4) we have

$$|\mathcal{A}(g-h)(x_0)| \le |\mathcal{A}f_1(x_0)| + |\mathcal{A}f_2(x_0)| + |\mathcal{A}f_3(x_0)|.$$

We claim that each summand on the right-hand side is bounded by some $c(X, D) > 0$. Once this is proved, it follows that $\mathcal{A}(g-h)(x_0)$ is well-defined and $|\mathcal{A}(g-h)(x_0)| \le c(X, D)$. By Lemma 2.16, we have $\mathcal{A}h(x_0) = 0$, and so $\mathcal{A}g(x_0)$ is well defined and $|\mathcal{A}g(x_0)| \le c(X, D)$, as desired.

To estimate $|\mathcal{A}f_1(x_0)|$ we use Lemma 3.5 and the fact that $f_1(x_0) = 0$. If $\alpha < 1$, there exists $\epsilon = \epsilon(X, D) > 0$ and constants $c = c(X, D) > 0$ such that

$$\begin{aligned} |\mathcal{A}f_1(x_0)| &\le \int_{\mathbb{R}^d} |f_1(x) - f_1(x_0)|\nu(x-x_0)\,dx \\ &\le c \int_{B(x_0,2)} |x-x_0|^{\alpha+\epsilon}\nu(x-x_0)\,dx \le c. \end{aligned}$$

If $\alpha > 1$, we have

$$\begin{aligned} |\mathcal{A}f_1(x_0)| &\le \int_{\mathbb{R}^d} |f_1(x) - f_1(x_0) - \langle \nabla f_1(x_0), x-x_0\rangle|\nu(x-x_0)\,dx \\ &\le \int_{B(x_0,1/2)} |x-x_0|^{\alpha+\epsilon}\nu(x-x_0)\,dx \\ &\quad + |\nabla f_1(x_0)| \int_{B(x_0,\delta_D(x_0))^c} |x-x_0|\nu(x-x_0)\,dx. \end{aligned}$$

Using additionally (3.5) with $\epsilon = \beta_{\min} + 1 - \alpha$ (recall that $\beta_{\min} > \alpha - 1$), we find that there exist constants $c = c(X, D) > 0$ such that

$$\begin{aligned} |\mathcal{A}f_1(x_0)| &\le c + c\delta_{\mathbb{H}}(x_0)^{\beta(x_0)-\beta_{\min}-1+\alpha} \int_{B(x_0,\delta_D(x_0))^c} |x-x_0|^{1-d-\alpha}\,dx \\ &\le c + c\delta_D(x_0)^{\beta(x_0)-\beta_{\min}} \le c, \end{aligned}$$

where in the last step we have used the fact that $\beta(x_0) \ge \beta_{\min}$.

Finally, for $\alpha = 1$, we have

$$\begin{aligned} |\mathcal{A}f_1(x_0)| \le \langle \nabla f_1(x_0), b\rangle + \int_{\mathbb{R}^d} \big| f_1(x) - f_1(x_0) \\ - \langle \nabla f_1(x_0), x-x_0\rangle \mathbf{1}_{B(x_0,r)}(x)\big|\nu(x-x_0)\,dx, \end{aligned}$$

where b is the drift of the process X and r is any radius as in (2.6). We set

$r = 2$. As in the case $\alpha > 1$, there exist constants $c = c(X, D) > 0$ such that

$$|\mathcal{A}f_1(x_0)| \le c + \int_{\mathbb{R}^d} \big|f_1(x) - f_1(x_0) - \langle \nabla f_1(x_0), x - x_0\rangle \mathbf{1}_{B(x_0,2)}(x)\big|\nu(x - x_0)\,dx$$

$$\le c + c \int_{D_1\cap\mathbb{H}} |x - x_0|^{\alpha+\epsilon}\nu(x - x_0)\,dx + \int_{(D_1\cap\mathbb{H})^c\cap B(x_0,2)} |\langle\nabla f_1(x_0), x - x_0\rangle|\nu(x - x_0)\,dx$$

$$\le c + |\nabla f_1(x_0)| \int_{B(x_0,\delta_D(x_0))^c\cap B(x_0,2)} |x - x_0|\nu(x - x_0)\,dx.$$

Again using (3.5) with $\epsilon = \beta_{\min}/2$, we obtain

$$|\mathcal{A}f_1(x_0)| \le c + c\delta_{\mathbb{H}}(x_0)^{\beta(x_0)-\beta_{\min}/2}(\ln(2) - \ln(\delta_{\mathbb{H}}(x_0))) \le c,$$

because $\beta(x_0) - \beta_{\min}/2 \ge \beta_{\min}/2$.

By (3.3) we have $f_2(x_0) = f_3(x_0) = 0$ and $\nabla f_3(x_0) = 0$. By Remark 3.7 for $\alpha \ge 1$ we have $\nabla f_2(x_0) = 0$, thus for every $\alpha \in (0, 2)$,

$$\mathcal{A}f_2(x_0) = \int_{\mathbb{R}^d} f_2(x)\nu(x - x_0)\,dx, \qquad \mathcal{A}f_3(x_0) = \int_{\mathbb{R}^d} f_3(x)\nu(x - x_0)\,dx.$$

Recall that $f_2(x) = ((\delta_D(x))^{\beta(x)} - (\delta_{\mathbb{H}}(x))^{\beta(x)})\mathbf{1}_{D_1\cap\mathbb{H}}(x)$. To estimate $|\mathcal{A}f_2(x_0)|$ we use Lemma 3.6:

$$|\mathcal{A}f_2(x_0)| \le \int_{\mathbb{R}^d} |f_2(x)|\nu(x - x_0)\,dx \le c \int_{D_1\cap\mathbb{H}} \max(\delta_D(x), \delta_{\mathbb{H}}(x))^{\beta_{\min}-\eta}|\tilde{x}|^{2\eta}\nu(x - x_0)\,dx;$$

here and below the symbols c denote constants $c = c(X, D) > 0$. By Remark 3.3, $\beta_{\min} - \eta < 0$, and therefore

$$|\mathcal{A}f_2(x_0)| \le c \int_{B(0,1)\cap\mathbb{H}} \delta_{\mathbb{H}}(x)^{\beta_{\min}-\eta}|\tilde{x}|^{2\eta}|x - x_0|^{-d-\alpha}\,dx.$$

Noticing that $B(0, 1) \cap \mathbb{H}$ is contained in $B^{(d-1)}(0, 1) \times (0, 1)$, and writing $x = (\tilde{x}, t)$, we find that

$$|\mathcal{A}f_2(x_0)| \le c\int_0^1 \int_{B^{(d-1)}(0,1)} t^{\beta_{\min}-\eta}|\tilde{x}|^{2\eta}(|\tilde{x}|^2 + |t - x_{0d}|^2)^{-(d+\alpha)/2}\,d\tilde{x}\,dt$$

$$= c\int_0^1 t^{\beta_{\min}-\eta}\int_0^1 r^{d-2+2\eta}(r^2 + |t - x_{0d}|^2)^{-(d+\alpha)/2}\,dr\,dt.$$

Now we investigate the integral over r. For $b>0$ we have

$$\int_0^1 r^{d-2+2\eta}(r^2+b^2)^{-(d+\alpha)/2}\,dr = b^{2\eta-\alpha-1}\int_0^{1/b} s^{d-2+2\eta}(1+s^2)^{-(d+\alpha)/2}\,ds$$
$$\le b^{2\eta-\alpha-1}\int_0^{1/b}(1+s^2)^{(2\eta-\alpha-2)/2}\,ds \le cb^{2\eta-\alpha-1}$$

(the final integral is bounded because, by Remark 3.3, $2\eta-\alpha-2<-1$). Next we take $b=|t-x_{0d}|$ and we find that

$$\begin{aligned}
&c\int_0^1 t^{\beta_{\min}-\eta}\int_0^1 r^{d-2+2\eta}(r^2+|t-x_{0d}|^2)^{-(d+\alpha)/2}\,dr\,dt\\
&\le c\int_0^1 t^{\beta_{\min}-\eta}|t-x_{0d}|^{2\eta-\alpha-1}\,dt\\
&= cx_{0d}^{\beta_{\min}+\eta-\alpha}\int_0^{1/x_{0d}} u^{\beta_{\min}-\eta}|u-1|^{2\eta-\alpha-1}\,du\\
&= cx_{0d}^{\beta_{\min}+\eta-\alpha}\Big(\int_0^2 u^{\beta_{\min}-\eta}|u-1|^{2\eta-\alpha-1}\,du + c\int_2^{1/x_{0d}} u^{\beta_{\min}+\eta-\alpha-1}\,du\Big).
\end{aligned}$$

The first integral on the right-hand side is bounded, because, by Remark 3.3, $\beta_{\min}-\eta>-1$ and $2\eta-\alpha-1>-1$. The other integral does not exceed $x^{-(\beta_{\min}+\eta-\alpha)}$. Finally, again by Remark 3.3, we have $\beta_{\min}+\eta-\alpha>0$, and we conclude that the right-hand side is bounded by a constant $c(X,D)$.

To estimate $|\mathcal{A}f_3(x_0)|$ we denote $A:=\{x\in\mathbb{R}^d : |\tilde{x}|\le 1,\ |x_d|<\frac{1}{2}|\tilde{x}|^2\}$. By Lemma 2.23, $((D\setminus\mathbb{H})\cup(\mathbb{H}\setminus D))\cap B(x_0,1)\subset A$. We write

$$\begin{aligned}
|\mathcal{A}f_3(x_0)| &\le \int_{\mathbb{R}^d}|f_3(x)|\nu(x-x_0)\,dx\\
&\le c\int_A (g(x)+h(x))\nu(x-x_0)\,dx\\
&\quad+\int_{B(x_0,1)^c}(g(x)+h(x))\nu(x-x_0)\,dx\\
&:= J_1+J_2.
\end{aligned}$$

For $x\in A$ we have $|x_d|/|\tilde{x}|\le\frac{1}{2}|\tilde{x}|<1$, and hence

$$\frac{|x-x_0|}{|x|}=\frac{\sqrt{|\tilde{x}|^2+|x_{0d}-x_d|^2}}{\sqrt{|\tilde{x}|^2+x_d^2}}=\frac{\sqrt{1+|x_{0d}-x_d|^2/|\tilde{x}|^2}}{\sqrt{1+x_d^2/|\tilde{x}|^2}}\ge\frac{1}{\sqrt{2}},$$

and thus by Assumption A we have

$$\nu(x-x_0) \le c\nu(x).$$

Moreover, for $x \in A$ we have $\delta_{\mathbb{H}}(x), \delta_D(x) \le c|\tilde{x}|^2$ and hence

$$g(x)+h(x) \le c|\tilde{x}|^{2\beta_{\min}}.$$

Thus

$$\begin{aligned} J_1 &\le c \int\limits_{B^{d-1}(0,1)} |\tilde{x}^{2\beta_{\min}}| \int\limits_{-|\tilde{x}|^2}^{|\tilde{x}|^2} \frac{1}{(|x_d|^2+|\tilde{x}|^2)^{(d+\alpha)/2}}\,dx_d\,d\tilde{x} \\ &\le c \int\limits_{B^{d-1}(0,1)} |\tilde{x}|^{-d-\alpha+2\beta_{\min}+2}\,d\tilde{x} \le c, \end{aligned}$$

because $\beta_{\min} > \alpha - 1$. By Remark 3.2 we have

$$\begin{aligned} J_2 &\le \int\limits_{B(x_0,1)^c} \nu(x-x_0)\,dx + \int\limits_{B(x_0,1)^c\cap\mathbb{H}} x_d^{\beta(x_0)}\nu(x-x_0)\,dx \\ &\le c + \int\limits_{B(x_0,1)^c\cap\mathbb{H}} x_d^{\beta(x_0)}\nu(x-x_0)\,dx. \end{aligned}$$

Since for x such that $|x-x_0| \ge 1$ we have $x_d \le |x_0|+|x-x_0| \le 1/2+|x-x_0| \le 2|x-x_0|$, it follows that

$$J_2 \le c + c \int\limits_{B(x_0,1)^c} |x-x_0|^{\beta_{\max}}\nu(x-x_0)\,dx = c.$$

We have thus proved that all three summands $|\mathcal{A}f_1(x_0)|$, $|\mathcal{A}f_2(x_0)|$ and $|\mathcal{A}f_3(x_0)|$ are bounded by a constant $c(X,D)$. This completes the proof. ∎

We keep the notation c_{gen} till the end of the article.

We recall the following fundamental result on existence of boundary limits of ratios of harmonic functions.

THEOREM 3.10 ([20, Theorem 2 and Example 1]). *Let D be an open set, and $z \in \partial D$. Suppose that $f_1, f_2 \ge 0$ are regular harmonic functions in $D \cap B(z,r)$ and are zero in $B(z,r) \setminus D$ for $r < R$. Then either one of f_1 and f_2 is zero everywhere in $D \cap B(z,r)$, or the finite, positive boundary limit of $f_1(x)/f_2(x)$ exists as $x \to z$, $x \in D$.*

Following, for example, [5], we introduce the following notation.

DEFINITION 3.11. The *relative oscillation* of a function f on the set D_r is given by the formula

$$RO_r(f) = \frac{\sup_{x\in D_r} f(x)}{\inf_{x\in D_r} f(x)}.$$

Note that if f_1 and f_2 are positive in D_r for some r, then the existence of a finite, positive limit of f_1/f_2 as $x \to 0$ is equivalent to the condition $RO_r(f_1/f_2) \to 1$ as $r \to 0^+$.

DEFINITION 3.12. We define the *harmonic reduction* g_r of the function g by

$$g_r(x) = \mathbb{E}^x(g(X_{\tau_{D_r}})).$$

LEMMA 3.13. *For every $\epsilon > 0$ there exists a radius $r_0 = r_0(X, D, \epsilon) > 0$ such that*

$$RO_r\left(\frac{g_r}{g}\right) \le \frac{1+\epsilon}{1-\epsilon} \tag{3.10}$$

for every $0 < r \le r_0$.

Proof. Let ϕ be a non-negative smooth function such that $\phi(y) = 0$ for $|y| > 1/2$ and $\int_{\mathbb{R}^d} \phi(y)\,dy = 1$. For $k \ge 1$ we define $\phi_k(y) = k^d \phi(ky)$ and

$$g_k(x) := (\phi_k * g)(x) := \int_{\mathbb{R}^d} \phi_k(y) g(x-y)\,dy.$$

For $r \le 1/4$ let $D_r^k := \{y \in D_r : \delta_D(y) \ge 1/k\}$. Since g_k is a smooth function, $\mathcal{A}g_k$ is well-defined everywhere.

Let $x \in D_r^k$ and $z \in B(0, 1/(2k))$. By Lemma 3.9 we have $-c_{\mathrm{gen}} \le \mathcal{A}g(x-z) \le c_{\mathrm{gen}}$. We claim that $\mathcal{A}g_k(x) = \phi_k * \mathcal{A}g(x)$ and consequently, by Lemma 3.9,

$$-c_{\mathrm{gen}} \le \mathcal{A}g_k(x) \le c_{\mathrm{gen}}. \tag{3.11}$$

By Remark 3.2, $g \in C^{\alpha+\epsilon}(D_1^*)$ for some $\epsilon > 0$. Hence, by (2.6), for $\alpha < 1$,

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \phi_k(z)|(g(y-z) - g(x-z))|\nu(y-x)\,dy\,dz \le \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \phi_k(z) \\ \times \big(c|y-x|^{\alpha+\epsilon}\mathbf{1}_{B(x,1/(4k))}(y) + 2 \times \mathbf{1}_{B(x,1/(4k))^c}(y)\big)\nu(y-x)\,dy\,dz < \infty.$$

Now, by the Fubini theorem, we have

$$\begin{aligned}
\mathcal{A}g_k(x) &= \int_{\mathbb{R}^d} (g_k(y) - g_k(x))\nu(y-x)\,dy \\
&= \int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d} \phi_k(z)(g(y-z) - g(x-z))\,dz\Big)\nu(y-x)\,dy \\
&= \int_{|z|<1/(2k)} \phi_k(z)\Big(\int_{\mathbb{R}^d}(g(y-z) - g(x-z))\nu(y-x)\,dy\Big)\,dz \\
&= \int_{|z|<1/(2k)} \phi_k(z)\mathcal{A}g(x-z)\,dz = \phi_k * \mathcal{A}g(x).
\end{aligned}$$

In case $\alpha > 1$, by Remark 3.2 and Lemma 2.9 we write

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \phi_k(z)\big|\big(g(y-z)-g(x-z)-\langle \nabla g(x-z), y-x\rangle\big)\big|\nu(y-x)\,dy\,dz$$
$$\le \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \phi_k(z)\big(c|y-x|^{\alpha+\epsilon}\mathbf{1}_{B(x,1/(4k))}(y) + c(2+|y|)\mathbf{1}_{B(x,1/(4k))^c}(y)\big)\nu(y-x)\,dy\,dz < \infty.$$

Furthermore, $\nabla g_k = (\nabla g) * \phi_k$ in D_r^k. Now, similarly to the case $\alpha < 1$, we write

$$\begin{aligned}\mathcal{A}g_k(x) &= \int_{\mathbb{R}^d} (g_k(y)-g_k(x)-\langle \nabla g_k(x), y-x\rangle))\nu(y-x)\,dy\\ &= \int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d} \phi_k(z)(g(y-z)-g(x-z) - \langle \nabla g(x-z), y-x\rangle)\,dz\Big)\nu(y-x)\,dy\\ &= \int_{|z|<1/(2k)} \phi_k(z)\Big(\int_{\mathbb{R}^d}(g(y-z)-g(x-z) - \langle \nabla g(x), y-x\rangle)\nu(y-x)\,dy\Big)\,dz\\ &= \int_{|z|<1/(2k)} \phi_k(z)\mathcal{A}g(x-z)\,dz = \phi_k * \mathcal{A}g(x).\end{aligned}$$

Finally, in case $\alpha = 1$ we write similarly

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \phi_k(z)\big|(g(y-z)-g(x-z) - \langle \nabla g(x-z), y-x\rangle \mathbf{1}_{B(x,1/(4k))}(y-z))\big|\nu(y-x)\,dy\,dz$$
$$\le \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \phi_k(z)\Big(c|y-x|^{\alpha+\epsilon}\mathbf{1}_{B(x,1/(4k))}(y) + 2\times \mathbf{1}_{B(x,1/(4k))^c}(y)\Big) \times \nu(y-x)\,dy\,dz < \infty.$$

Since $g \in C^{\alpha+\epsilon}(D_1^*)$, ∇g is a continuous function in D_1^*, and $\nabla g_k = (\nabla g) * \varphi_k$ is in D_r^k. Thus,

$$\begin{aligned}\mathcal{A}g_k(x) &= \langle \gamma, \nabla g_k(x)\rangle\\ &\quad + \int_{\mathbb{R}^d} (g_k(y)-g_k(x)-\langle \nabla g_k(x), y-x\rangle \mathbf{1}_{B(x,1/(4k))}(y)))\nu(y-x)\,dy\\ &= \int_{\mathbb{R}^d} \langle \gamma, \nabla g(x-z)\rangle \phi_k(z)\,dz + \int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d} \phi_k(z)(g(y-z)-g(x-z)\\ &\qquad - \langle \nabla g(x-z), y-x\rangle \mathbf{1}_{B(x,1/(4k))}(y-z))\,dz\Big)\nu(y-x)\,dy\end{aligned}$$

$$= \int\limits_{|z|<1/(2k)} \phi_k(z)\Big(\langle\gamma, \nabla g(x-z)\rangle + \int\limits_{\mathbb{R}^d} \big(g(y-z) - g(x-z) - \langle \nabla g(x-z), y-x\rangle \mathbf{1}_{B(x,1/(4k))}(y)\big)\nu(y-x)\,dy\Big)\,dz$$

$$= \int\limits_{|z|<1/(2k)} \phi_k(z)\mathcal{A}g(x-z)\,dz = \phi_k * \mathcal{A}g(x).$$

This completes the proof of our claim (3.11).

Recall that g_k is in $C_c^\infty(\mathbb{R}^d)$ and $\mathcal{A}$ restricted to C_c^∞ coincides with the infinitesimal generator $\mathcal{L}$ of the process X. Denote $\sigma(r,k) = \tau_{D_r^k}$. For $k \geq l$, by Dynkin's formula, for $x \in D_r^l$ we have

$$\mathbb{E}^x \int\limits_0^{\sigma(r,l)} \mathcal{A}g_k(X_t)\,dt = \mathbb{E}^x(g_k(X_{\sigma(r,l)})) - g_k(x).$$

Using (3.11), we get

$$-c_{\mathrm{gen}}\mathbb{E}^x\sigma(r,l) \leq \mathbb{E}^x(g_k(X_{\sigma(r,l)})) - g_k(x) \leq c_{\mathrm{gen}}\mathbb{E}^x\sigma(r,l).$$

As $k \to \infty$, g_k remains bounded by 1 and it converges pointwise to g. Thus,

$$-c_{\mathrm{gen}}\mathbb{E}^x\sigma(r,l) \leq \mathbb{E}^x(g(X_{\sigma(r,l)})) - g(x) \leq c_{\mathrm{gen}}\mathbb{E}^x\sigma(r,l).$$

Now we pass to the limit as $l \to \infty$. Since $\sigma(r,l)$ is an increasing function of l, and since g is bounded and $g(X_{\sigma(r,l)})$ converges almost surely to $g(X_{\tau_{D_r}})$, we infer that for all $x \in D_r$,

$$-c_{\mathrm{gen}}\mathbb{E}^x(\tau_{D_r}) \leq \mathbb{E}^x(g(X_{\tau_{D_r}})) - g(x) \leq c_{\mathrm{gen}}\mathbb{E}^x(\tau_{D_r}). \tag{3.12}$$

In order to proceed we need a technical result comparing $g_r(x)$ with $\mathbb{E}^x(\tau_{D_r})$. Since $\nu(y-z) \geq c|y|^{-d-\alpha}$ for every $z \in D_r$ and $y \in B(0,r)^c$ (here and below $c = c(X,D) > 0$), we have

$$\begin{aligned} g_r(x) &= \int\limits_{D_1^*\setminus D_r} \Big(\int\limits_{D_r} G_{D_r}(x,z)\nu(z-y)\,dz\Big)g(y)\,dy \\ &\geq c\mathbb{E}^x(\tau_{D_r}) \int\limits_{D_1^*\setminus D_r} c|y|^{-d-\alpha}g(y)\,dy, \end{aligned} \tag{3.13}$$

where $G_{D_r}(x,z)$ is the Green function of the set D_r.

For $y \in B(0,1)$ with $2|\tilde{y}| < y_d$ we have

$$|y| = \sqrt{y_d^2 + |\tilde{y}|} \leq \sqrt{y_d^2 + \tfrac{1}{4}y_d^2} \leq cy_d.$$

By (2.16) we have

$$\delta_D(y) \geq y_d - |\tilde{y}|^2 \geq y_d - |\tilde{y}| \geq y_d/2 \geq c|y|,$$

so that

$$g(y) \geq c|y|^{\beta_{\max}}.$$

Thus, by using polar coordinates, we have

$$\int_{D_1^*\setminus D_r} |y|^{-d-\alpha} g(y)\,dy \ge \int_{D_1\setminus D_r} |y|^{-d-\alpha} g(y)\,dy$$
$$\ge c \int_{\{(\tilde{y},y_d):\, 2|\tilde{y}|<y_d,\, r<|y|<1\}} |y|^{-d-\alpha}|y|^{\beta_{\max}}\,dy$$
$$\ge c\int_r^1 u^{-d-\alpha}u^{\beta_{\max}}u^{d-1}\,du = c(r^{\beta_{\max}-\alpha}-1).$$

By combining this and (3.13) we get

$$g_r(x) \ge c(r^{\beta_{\max}-\alpha}-1)\mathbb{E}^x(\tau_{D_r}) \tag{3.14}$$

for $x \in D_r$.

Now suppose that $\epsilon > 0$. By (3.14) there exists $r_0 = r_0(X, D, \epsilon) > 0$ such that

$$c_{\text{gen}}\mathbb{E}^x(\tau_{D_r}) \le \epsilon g_r(x) \tag{3.15}$$

for $x \in B(0,r)$ and $r \le r_0$. By combining (3.15) with (3.12), we find that $-\epsilon g_r(x) \le g_r(x) - g(x) \le \epsilon g_r(x)$, and hence

$$1-\epsilon \le \frac{g(x)}{g_r(x)} \le 1+\epsilon \tag{3.16}$$

for $x \in D_r$ and $r \le r_0$. This implies (3.10). ∎

THEOREM 3.14. *Let f be a non-negative function which is regular harmonic in D_1 and which vanishes on $D^c \cap B(0,1)$. Then either f is zero everywhere in D, or*

$$\lim \frac{f(x)}{\delta_D(x)^{\beta(x)}} > 0 \quad \textit{exists as } x \to 0,\ x \in D.$$

Proof. Let $\epsilon > 0$ and let r_0 be chosen according to Lemma 3.13. By Theorem 3.10 and the fact that g_{r_0} and f are harmonic in D_{r_0}, there exists a radius $r \le r_0$ such that

$$RO_r(f/g_{r_0}) \le 1+\epsilon.$$

For any positive functions f_1, f_2, f_3 we have

$$RO_r\left(\frac{f_1}{f_2}\right) = \frac{\sup_{x\in D_r}\frac{f_1(x)}{f_2(x)}}{\inf_{x\in D_r}\frac{f_1(x)}{f_2(x)}}$$
$$\le \frac{\sup_{x\in D_r}\frac{f_1(x)}{f_3(x)}\sup_{x\in D_r}\frac{f_3(x)}{f_2(x)}}{\inf_{x\in D_r}\frac{f_1(x)}{f_3(x)}\inf_{x\in D_r}\frac{f_3(x)}{f_2(x)}} = RO_r\left(\frac{f_1}{f_3}\right)RO_r\left(\frac{f_3}{f_2}\right).$$

Thus, by Lemma 3.13,

$$RO_r\left(\frac{f}{g}\right) \le RO_r\left(\frac{f}{g_{r_0}}\right)RO_r\left(\frac{g_{r_0}}{g}\right) \le \frac{(1+\epsilon)^2}{1-\epsilon}.$$

Since ϵ was chosen arbitrarily, we have $RO_r(f/g) \to 1$ as $r \to 0$. ■

REMARK 3.15. Note that, in contrast to the 2015 work of T. Grzywny, K.-Y. Kim and P. Kim [14], with our methods we cannot relax the assumption $D \in C^{1,1}$. If $D \in C^{1,\beta}$ for $\beta < 1$, then the function $n(x)$ (and so $\beta(x)$ and $g_z(x)$) is not even a continuous function.

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