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Report of the doctoral dissertation

Title of the dissertation: Nonautonomous Linear Differential Equations with Delays and Skew-product Dynamical Systems

Author of the dissertation: mgr Marek Piotr Kryspin

Supervisor of the dissertation: prof. dr hab. Janusz Mierczyński

The report was prepared at the request of the Scientific Discipline Council Mathematics at Wrocław University of Science and Technology on July 9, 2024, which appointed me to serve as a reviewer in the proceedings for the academic degree of Doctor of Mathematical Sciences in the discipline of Mathematics to Mr. Marek Piotr Kryspin, MSc. The purpose of the report is to assess whether the doctoral dissertation of Mr. Marek Piotr Kryspin meets the conditions specified in Article 13, Section 1, of the Act on Academic Degrees and Titles.

1. The scope of the dissertation

In the dissertation, a general theory of contentious dependence of mild solution of solutions of a linear system of partial differential equations with constant delay (Chapter 3), linear partial differential equation with varying delay (Chapter 4), and linear ordinary differential equations with general delay (Chapter 5) on initial data, parameters, and delays (Chapters 4 and 5) is given. Chapters 6 and 7 present results regarding the existence of exponential separation of linear random differential equations with discrete delay.

2. Formal and editorial aspects of the dissertation

The dissertation is 166 pages, not counting the table of contents, bibliography, and index of symbols, which add 35 pages. It consists of eight chapters, two appendixes, the bibliography, and the index of symbols.

In Chapter 1 entitled *Introduction*, a summary of the main results is presented along with some words about the methodology used in the dissertation.

In Chapter 2 entitled *Preliminaries and Notation* the notation used in the dissertation is introduced as well as basic definitions and and some general facts used later are presented.

In Chapter 3 entitled *PDEs with constant delay* a general system of linear parabolic partial differential equations with constant delay is considered. The Author defines weak and mild solutions of such a system, but in fact, he is interested in the second one. A general theorem assuring existence and uniqueness (under suitable conditions and in the proper space) is proved. This is a generalization of the known results presented for example in the book of J. Wu (position [102]). The main novelty is a general theorem about the continuous dependence of the solutions on all coefficients and initial data.

In Chapter 4 entitled *PDEs with varying delay* similar results to those proved in Chapter 3 are presented. However here, delay can depend on time, but instead of a system of equations, only a single equation is considered. Again weak and mild solutions are considered and the existence theorem as well as continuous dependence on all parameters (including delays) are proved.

Chapter 5 entitled *ODEs with general delay* is devoted to a linear system of ordinary differential equations with general delays. Again mild type of solutions is considered. The general delay kernels are represented by measures (that fulfill some assumptions) and the integration is taken with respect to those measures. The usual approach with a delay kernel is a special case of the problem considered in this chapter when the delay measure has a density. A continuous dependence of solutions on initial data as well on those measures is proved.

Chapter 6 entitled *Oseledets decomposition* is a short one in which some facts about oseledets decomposition are proved.

Chapter 7 entitled *Random ODEs with constant delay* is devoted to a (system) of random linear ordinary differential equation with one constant time delay. The existence of exponential separation for such an equation is proved.

In Chapter 8 entitled *Possible continuation* possible continuation of the results presented in the dissertation is discussed.

Appendices contain results about semiprocess equivalence and some easier (more natural) conditions that guarantee that technical conditions from previous chapters hold.

I have no other critical remarks regarding the formal and editorial aspects of the dissertation.

3. Critical remarks

I find the relation of the results presented in Chapters 3, 4, and 5 of the dissertation with classical and known results to be weak. In the book of J. Wu (position [102]) there are theorems regarding the existence of mild solutions of partial differential equations with delay, thus they are connected with the results presented in Chapters 3 and 4. After careful examination, the difference (and novelty of the results from the dissertation) can be seen, however, I think these novelties are not stated clearly enough. In Chapter 5, Section 4 a case where measures are Dirac delta are considered. In this case, the problem considered in the chapter should be equivalent to the classical linear ordinary differential equation with a discrete delay that depends on time. Such problems were considered in the classical theory of DDEs (books by Y. Kuang [58], J.K Hale, and S.M. Verduyn Lunel [47]). It is not clear to me if the results from Section 5.4 are just different formulations of known results (as a corollary of more general theory) or if there is some novelty in it. There are some words in Chapter 1 about the novelty of the research, but generally, it is not clear enough. I am satisfied only with the detrition of new results presented in Chapters 6 and 7.

Moreover, in Chapters 3, 4, and 5 some results are given without proof (or as proof a reference to the literature is given). I assume that these Lemmas/Propositions/Theorems are stated here to have a complete theory. However, this is not stated clearly, and I find it difficult to recognize which results I should evaluate. Only in the introductory part of Chapter 7, there is a clear statement that divides the chapter content between important results and important facts given for completeness. Sometimes it seems that the fact formulated in the dissertation is a slight modification of already known results (as Proposition 3.3.5 seems to be a simple generalization of Proposition 2.1.7 form [73]), but it is not written clearly.

It is clearly stated that in the dissertation the results obtained with collaborations with other scientists are presented. In particular, the Authors refer to position [55], which is co-authored by Sylvia Novo and Rafael Obaya. It would be very helpful if he had written more precisely what is his contribution to this work.

It is not clear what is the difference between the results presented in section 3.3.2 and the classical theory of PDEs.

On page 50, the Author uses The Contraction Mapping Principle. I wonder why he is not using the Banach Fixed Point Theorem instead.

At the beginning of Chapter 4, two special cases of theorem proved later are presented. It is said that Theorem 4.0.1 is a special case of Theorem 4.4.7, but what about Theorem 4.0.2?

Remark 4.1.1 is not clear to me. I do not understand why problem (4.1.1)-(4.1.2) would not be meaningful if $t \mapsto t - R(t)$ is a constant function. Moreover, is it impossible if $T > 1$ as it is assumed $R(t) \leq 1$.

Why is the Author not using the name of Sobolev spaces in section 4.3?

4. Evaluation of the dissertation and final conclusions

In my opinion, some of the results presented in the dissertation are new and important although not surprising. I have never seen theorems about the continuous dependence of solutions of PDEs with delays on initial data and parameters in such general formulation as in the dissertation. Similarly, results devoted to continuous dependence of solution of linear ODE with delay om measures (that is on delays) is a very nice generalization of the known results. Although these results are not surprising, they are important. The results from Chapter 7 are somehow different from the rest of the dissertation (the results from Chapter 6 are abstract results needed later in Chapter 7) but I found it probably the most interesting. The limiting factor of the results presented in all chapters is the linearity of equations. The proof depends strongly on the existence of a semigroup operator. I wonder if these results could be generalized (under some assumptions) to non-linear systems. For me, it would be particularly interesting for the results from Chapter 5 – in applications, equations are rather non-linear.

I have not found any errors in the proofs presented in the dissertations. They are usually written with sufficient details. Although standard estimation procedures are used, the proofs are not easy, and they indicate the Author's knowledge of functional analysis as well as the studied topic. I think Mr. Marek Kryspin would be an excellent scientist.

Final conclusion In summary, despite some critical remarks I have, I conclude that this dissertation meets the conditions specified in Article 13, paragraph 1 of the Act of March 14, 2003, on academic degrees and scientific titles, and on degrees and titles in the field of art (Journal of Laws No. 65, item 595, as amended), as well as usual and may be the subject of public defense in the field of Natural Sciences in the discipline of Mathematics. I request the admission of the doctoral dissertation of Mr. Marek Piotr Kryspin, MSc to the public defense.

Bodnar

Marek Bodnar