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Report on the Ph.D. thesis "Nonautonomous Linear Differential Equations with Delays and Skew-product Dynamical Systems" by Marek Piotr Kryspin

The thesis deals with nonautonomous and random linearly homogeneous differential equations of different type. It consists of six chapters supplemented by two short appendices. Chapter 1 serves as an introduction, which surveys the central results and the methodology to obtain them. Chapter 2 rather compactly introduces the required notation, terminology and mathematical concepts to understand the thesis. The central contributions of the thesis are finally contained in the Chapters 3-7.

In detail, Chapter 3 addresses systems of partial differential equations

$$u_t(t,x) = A(t,x)u(t,x) + C(t,x)u(t-1,x),$$

where $A(t,x)$ is a vector-valued second order differential operator (elliptic in the 2nd order terms, coefficients depending on t and x from a bounded domain), while $C(t,x)$ denotes a $n \times n$-matrix, equipped with Dirichlet, Neumann or Robin boundary conditions. Under appropriate assumptions, well-posedness results (existence, uniqueness, continuous dependence on the data) in terms of weak solvability in Lebesgue spaces are partly quoted from the literature, partly derived, while a compactifying property of the generated semi-process is actually shown. Similar results are obtained for mild solutions on Lebesgue spaces, as well as growth estimates and smoothing properties.

Chapter 4 restricts to scalar equations, but allows the delay to be time-variant, i.e.

$$u_t(t,x) = A(t,x)u(t,x) + c(t,x)u(t-R(t),x)$$

having a second order differential operator $A(t,x)$ with essentially bounded coefficients and a delay $R(t) \in [0,1]$ a.e.. Again, the well-posedness of weak solutions is partly taken from the literature, partly shown. Concerning global mild solutions, a classical approach using the Contraction Mapping Principle allows to establish their existence and uniqueness. In addition, a compactifying property of the solution operator and continuous dependence on the delay function R are deduced.

In Chapter 5 the author tackles finite-dimensional linear systems of delay equations

$$u_t(t) = A(t)u(t) + \int_{-1}^{0} u(t+s)\mu(ds;t)$$

involving a matrix $A(t) \in \mathbb{R}^{n\times n}$ and $\mu(\,\cdot\,, t)$ being a signed Borel measure on the delay interval $[-1,0]$. The basics on the the delay-free Carathéodory equation $u_t(t) = A(t)u(t)$ could have been presented more compactly[1]. For the problem involving delays, well-posedness (in particular, the sequentially continuous dependence on the measure μ) is derived using the Uniform Contraction Principle. A particular focus is on the situation where μ can be represented using Dirac measures.

In conclusion, the Chapters 3—5 provide the fundamentals guaranteeing that the above evolutionary differential equations give rise to semi-flows or -processes and therefore to tackle them using methods from nonautonomous or random dynamical systems.

The Chapter 6 addresses – quoting Ludwig Arnold – the “Linear Algebra” required to study nonautonomous differential equations (more precisely, measurable linear skew-product semiflows). Most interesting here is the fact that the tool of an Oseledets decomposition in the MET extends to so-called sub-semiflows being defined on continuously embedded subspaces.

A closely related field, but addressing state space decompositions based on principal Floquet subspaces and type-II-exponential separations, is covered in Chapter 7. These results are illustrated using measurable linear skew-product semiflows generated by linear systems of random delay equations

$$\dot{u}_t(t) = A(\theta_t\omega)u(t) + B(\theta_t\omega)u(t-1)$$

with matrices $A(\omega), B(\Omega) \in \mathbb{R}^{n\times n}$. Conditions (cooperativity, irreducibility) are given, which ensure that the semiflow is eventually (strongly) positive and possesses a property denoted as “focussing” (cf. (R3) on p. 136). The abstract results from Section 7.2 allow to deduce the existence of generalized principal Floquet bundles or of a generalized type-II-exponential separation, plus information on the principal Lyapunov exponents. Subsequently the mathematical part of the thesis closes with results addressing state spaces lacking a separable dual.

On merely two pages, the final Chapter 8 indicates potential extensions and possible generalizations of the obtained results. The two appendices essentially consist of supporting results, which would also fit as lemmas in the regular text.

My overall impression is that the thesis is primarily a contribution to the field of linear delayed differential equations (of PDE and Random type) in a sense that they generate (semi-) dynamical systems, and only secondly to the area of Dynamical Systems. Due to the fact that the considered equations contain a delay, the dissertation is on a technically rather advanced level. This demonstrates the advanced knowledge of the author in mastering subjects such as Integration in Banach spaces, Measure Theory resp. applying them to differential equations. In addition, the candidate is highly familiar with methods from

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[1] relying e.g. on B. Aulbach, T. Wanner: *Integral manifolds for Carathéodory type differential equations in Banach space,* in B. Aulbach, F. Colonius (eds.), Six Lectures on Dynamical Systems, 45—119, World Scientific, Singapore etc., 1996

nonautonomous and random Dynamical Systems. This clearly indicates that Marek Kryspin has a general theoretical knowledge required to obtain a Ph.D. degree.

The obtained results are interesting and prepare future applications to nonlinear problems, which promise a wider applicability. This underlines the ability of the candidate for doing independent mathematical research.

Without question, the thesis of Marek Kryspin provides an original solution to a highly nontrivial scientific problem contributing to pure mathematics and promising applications to real world problems.

However, my points of criticism are as follows:

- There is potential for improvement in the organization of the thesis. Indeed, several aspects would make the text more readable. It would be nice to connect different results emphasizing their importance and providing some context. Moreover, introducing notation and terminology when it is needed, should be preferred to collecting most of it in a single chapter put in front (here Chapter 2).
- While the thesis refers to several (e.g. biological) applications, it unfortunately lacks concrete examples which would underline the importance and applicability of the rather theoretical contributions. That said, it is clear that linear equations have limited applicability, but are fundamental for concepts as linearization.

On a formal level, the dissertation is well-written. The results appear to be correct, although not all details were checked. At points, however, the English is a bit awkward, which is understandably for a non-native speaker.

In conclusion, I can **clearly recommend** the dissertation under review to be accepted. The candidate Marek Kryspin definitely deserves a doctoral degree.

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