REPORT ON THE

PHD THESIS

"HARDY INEQUALITIES AND NONLOCAL GEOMETRY"

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This is a report on the PhD Thesis *"Hardy inequalities and nonlocal geometry"* presented by Julia Anna Lenczewska at Wroctaw University of Science and Technology (Advisor: dr hab. inz. Tomasz Grzywny).

Description of the contents and main results

This thesis considers different mathematical objects related to nonlocal operators: Hardy inequalities, capacities, heat contents, and curvatures. The point that these diverse issues have in common is that all of them involve applications of energy forms and semigroup theory (something that shows the versatility of these techniques).

First, the author deals with Hardy inequalities related to the fractional Laplacian $\Delta^{\alpha/2}$ for $\alpha \in (0,2)$. Associated with this operator there is a quadratic from defined as

$$\mathcal{E}[u] := A(d,\alpha) \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}}\, dy\, dx.$$

Here $A(d,\alpha)$ is an explicit contant that behaves as $A(d,\alpha) \sim 2-\alpha$ as $\alpha \nearrow 2$.

In this thesis the following form in L^p is considered

$$\mathcal{E}_p[u] := A(d,\alpha) \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(u(x)-u(y))(u(x)^{<p-1>}-u(y)^{<p-1>})}{|x-y|^{d+\alpha}} dydx$$

(here $a^{<p-1>} = |a|^{p-1} sgn(a)$) and the following inequality is proved

$$\mathcal{E}_p[u] \geq \kappa \int_{\mathbb{R}^d} \frac{|u(x)|^p}{|x|^\alpha} dx, \qquad \forall u \in L^p(\mathbb{R}^d).$$

Here κ is the optimal constant. This inequality is then used to show the contractivity of the corresponding semigroups on $L^p(\mathbb{R}^d)$.

After this is done the goal is to obtain similar inequalities for the half-space and more general convex domains. To this end, the author replace $\mathbb{R}^d$

by an open set D and restrict the set of functions to continuous functions with compact support inside D. Here the relevant inequality looks like

$$\mathcal{E}_{p,D}[u] \geq \kappa \int_D \frac{|u(x)|^p}{dist(x, \partial D)^\alpha} dx,$$

for D convex.

The second part of the thesis is concerned with the study of capacities related to general measures in fractional Sobolev spaces. Let $p \in [1, \infty)$ and consider ν be a Borel measure satisfying

$$\int_{\mathbb{R}^d} (1 \wedge |y|^p) \nu(dy) < \infty.$$

The main goal now is to look for the the capacity generated by the nonlocal Sobolev space

$$W_\nu = \{f : \|f\|_\nu := \|f\|_p + [f]_{W_p^\nu} < \infty\},$$

with

$$[f]_{W_p^\nu} = \left(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (u(x+h) - u(x))^p \, \nu(dh) \, dx \right)^{1/p}.$$

The final part of the thesis contains the study of geometric quantities related to nonlocal operators which are generalizations of the fractional Laplacian. First, the author introduces the concept of a nonlocal perimeter, which generalizes the concept of the classical perimeter, as follows:

$$Per_\nu(E) = \int_E \int_{E^c - x} \nu(dy) \, dx.$$

In this part of the thesis some notions closely connected to the nonlocal perimeter of a set are also studied. In particular, the heat content of a set at time t (and its bahaviour as $t \approx 0$) and the curvature of the boundary. The asymptotic expansion of the heat content involve quite delicate estimates.

Concerning the curvature of a set, in this thesis more general nonlocal curvatures are defined using a kernel which is symmetric and bounded from above by a radial and radially non-increasing profile. The general version of a nonlocal mean curvature introduced in this dissertation includes various variants of nonlocal curvatures that have already appeared in the literature, including fractional mean curvature and anisotropic fractional mean curvature. Here the author studies the limit behaviour of the introduced nonlocal curvatures under an appropriate limiting procedure. In this way she recovers known asymptotic results for fractional curvature and obtain a new result for anisotropic fractional curvature, where the limit object is identified as a curvature being the first variation of a related anisotropic perimeter. Finally, also in the framework of general nonlocal curvatures a

proof of existence, uniqueness and stability of viscosity solutions to the corresponding level-set parabolic Cauchy problem is also included. This past part of the thesis contains very nice results with a clear geometric flavor.

This dissertation consists of an introduction, that describes and contextualizes the main results and seven chapters whose contents are briefly described as follows:

(1) Chapter 2 presents some preliminary results and sets the notation used in the thesis.

(2) Chapter 3 contains the analysis of the fractional Hardy inequalities in L^p.

(3) Chapter 4 contains the results concerning Hardy inequalities in open convex sets with weights that are powers of the distance to the boundary.

(4) In Chapter 5 the author deals with estimates for the capacity of sets in fractional Sobolev spaces with respect to general measures.

(5) In Chapter 6 the asymptotics of the heat content are obtained. This chapter contains the most delicate estimates of the thesis.

(6) Chapter 7 contains results of geometric nature concerning very general nonlocal mean curvatures. This chapter contain very nice results.

(7) A complete and up to date bibliography is gathered at the end of the thesis.

Publications.

The results contained in this thesis are contained in the following publications:

(1) K. Bogdan, T. Jakubowski, J. Lenczewska, and K. Pietruska-Paluba. *Optimal Hardy inequality for the fractional Laplacian on L^p.* J. Funct. Anal., 282(8): Paper No. 109395, 2022.

(2) W. Cygan, T. Grzywny, and J. Lenczewska. *Asymptotics and geometric flows for a class of nonlocal curvatures.* arXiv e-prints, 2023.

(3) T. Grzywny and J. Lenczewska. *Asymptotic expansion of the nonlocal heat content.* Studia Math., 270(3):339–359, 2023.

(4) M. Kijaczko and J. Lenczewska. *Sharp Hardy inequalities for Sobolev-Bregman forms.* Math. Nachr., 297(2):549–559, 2024.

Conclusions.

The Thesis under review presents an extensive amount of work, innovative, deep and of high technical quality. It contains a large number and a wide variety of regularity results combining subtle techniques for nonlocal operators and functionals including viscosity solutions of PDEs.

The presented results are relevant as they constitute nontrivial extensions of the well-known theory for the usual Laplacian or the usual fractional Laplacian to more general operators. These extensions involve the introduction of new ideas and techniques that could be useful in other contexts.

The thesis is very well written and quite nice and easy to read.

In this work the author demonstrates knowledge of a wide spectrum in mathematics and a large set of actual references and techniques. It is clear that the candidate has a solid background and it could be expected that she will continue doing research at a very high level.

The thesis shows clearly the author's strong command of the area of mathematics involved in the thesis (a mixture between Analysis, Probability and Partial Differential Equations) and testifies her capacity to perform independent research as well as to produce collaborative results. The results contained in the thesis are very interesting and the totality makes a nice showcase for nonlocal techniques that can be used in different contexts. Most of the papers included in this dissertation are already published in first class journals.

To conclude, I want to congratulate Julia Anna Lenczewska for an impressive thesis, and state that in my view there is no doubt that the thesis largely meets the standard for the PhD degree.

According to the criteria for distinguished doctoral dissertations at Wroctaw University of Science and Technology I assert the following:

- This thesis contains a large amount of results of first quality that largely exceeds the requirements for a PhD Thesis.
- The Thesis give rise to several papers published in first class journals, among them I want to highlight J. Funct. Anal. that is considered a top journal in the discipline.
- The obtained results concerning Hardy inequalities, capacity, heat content asymptotics and general nonlocal mean curvatures are highly nontrivial and require new and innovative techniques.

In conclusion, I recommend the *acceptance* of this thesis and propose the *Distinguish Doctoral Dissertation* grade with strong enthusiasm.

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