Summary of scientific achievements

1. Name and surname:

Kamil Kaleta

2. Scientific degrees:

2008 M.Sc. in Mathematics
Institute of Mathematics and Computer Sciences
Faculty of Fundamental Problems of Technology
Wroclaw University of Technology
Master's dissertation: *Potential theory of α-stable motion on fractals*
Supervisors: Prof. Dr. Sc. Tomasz Byczkowski and Dr. Sc. Mateusz Kwaśnicki

2011 Ph.D. in Mathematics
Institute of Mathematics and Computer Sciences
Faculty of Fundamental Problems of Technology
Wroclaw University of Technology
Doctoral dissertation: *Potential theory for fractional powers of Laplace operator and related Schrödinger operators*
Supervisor: Prof. Dr. Sc. Tadeusz Kulczycki

3. Information on previous employment in scientific institutions:

2011–2014	Assistant in the Institute of Mathematics and Computer Science Wroclaw University of Technology (2012–2014 on leave)
2014–2015 [1]	Assistant in Department of Mathematics, Wroclaw University of Technology (on leave)
2012–2015	Research Assistant in the Institute of Mathematics University of Warsaw (Post-doc position)
2016–2017	Post-doc position at Technische Universität Dresden, Germany
2015–present	Assistant Professor in Faculty of Pure and Applied Mathematics, Wroclaw University of Science and Technology

[1]this change was caused by the organizational changes at the university, i.e. the transformation of the Institute of Mathematics and Computer Science into Department of Mathematics on Faculty of Fundamental Problem of Technology

4. The indication of the scientific achievement:

(a) The title of the scientific achievement:

Feynman–Kac semigroups of Lévy processes with direct jump property

(b) The list of papers constituting the scientific achievement:

[H1] K. Kaleta, J. Lőrinczi, *Pointwise eigenfunction estimates and intrinsic ultracontractivity-type properties of Feynman–Kac semigroups for a class of Lévy processes*, Annals of Probability 43 (3), 1350-1398 (2015).

[H2] K. Kaleta, P. Sztonyk, *Small time sharp bounds for kernels of convolution semigroups*, Journal d'Analyse Mathématique 132 (1), 355-394 (2017).

[H3] K. Kaleta, J. Lőrinczi, *Fall-off of eigenfunctions for non-local Schrödinger operators with decaying potentials*, Potential Analysis 46 (4), 647-688 (2017).

[H4] K. Kaleta, M. Kwaśnicki, J. Lőrinczi, *Contractivity and ground state domination properties for non-local Schrödinger operators*, Journal of Spectral Theory 8 (1), 165-189 (2018).

(c) Discussion of the above-mentioned papers and the obtained results, together with a discussion of their possible use

I. Introduction

The classical Schrödinger equation, formulated in 1926 by the Austrian physicist Erwin Schrödinger, is one of the key equations of non-relativistic quantum mechanics. It describes the evolution of the quantum state of the physical system. The Schrödinger operator, which acts as the energy operator (the so-called Hamiltonian) of the system, is the main object of the theory. For a single particle moving in a potential V, this operator takes the following form in the position representation

$$H = H_0 + V,$$

where $H_0 = -\Delta$, and V is the multiplication operator. The stationary states of the system are described by the time-independent Schrödinger equation, which takes the form of the eigenequation $H\varphi = \lambda\varphi$. Since the Laplace operator Δ is the infinitesimal generator of the Brownian motion, the properties of H and the solutions of this eigenproblem can be effectively investigated by probabilistic methods [66].

It follows from the special theory of relativity that in the case of particles moving at high velocities (the case of high energies) the kinetic term H_0 should be chosen differently (see e.g. [50]). Certain approximations of relativistic theory are often realized through models based on the so-called *non-local Schrödinger operators*. In this case, $H_0 = -L$, where L is the generator of some Lévy process with jumps. The key examples are the operators H based on

$$L = -(-\Delta + m^{2/\alpha})^{\alpha/2} + m \quad \text{and} \quad L = -(-\Delta)^{\alpha/2}, \quad \alpha \in (0, 2), \ m > 0,$$

called respectively *quasi-relativistic* and *ultra-relativistic* (or fractional) Schrödinger operators.

The evolution semigroups of non-local Schrödinger operators can be represented via the *Feynman–Kac formula* based on Lévy processes with jumps. This allows one to study their properties using probabilistic methods. In recent years, Markov processes with jumps and non-local pseudo-differential operators have received much attention in both pure and applied mathematics. Jump processes provide new methods in scientific modelling, in particular they allow us to model discontinuous phenomena, providing realistic correctives and refinements to established theories.

The theory of non-local Schrödinger operators underwent rapid development over the last 30 years. Spectral and analytic properties of such operators together with their evolution semigroups have been investigated by many excellent mathematicians and physicists by both analytic and probabilistic methods. Among researchers who contributed to the subject one can mention Bañuelos, Bogdan, Byczkowski, Carmona, Chen, Fefferman, Frank, Garbaczewski, Hansen, Herbst, Hiroshima, Jakubowski, Kim, Kulczycki, Kwaśnicki, Lieb, Lőrinczi, Seiringer, Simon, Song, Takeda, Weder, Wang [1, 8, 9, 11, 13, 20, 21, 23, 24, 30, 31, 33, 34, 35, 37, 38, 39, 46, 47, 48, 69, 70]. Let us note that many of these research works, especially in mathematical physics, have been strongly influenced by the investigations of E. Lieb and his collaborators on the stability of relativistic matter [27, 50, 51].

The present scientific achievement is concerned with non-local Schrödinger operators $H = -L + V$, where L is the generator of a *Lévy process with the direct jump property*, and Feynman-Kac semigroups associated with H. Their properties have been studied by probabilistic methods, in particular by the techniques of modern probabilistic potential theory. In the paper [H1] we have considered operators H with confining Kato-decomposable potentials (i.e. $V(x) \to \infty$ as $|x| \to \infty$), obtaining sharp two-sided pointwise estimates of the ground state eigenfunctions and the upper bound for other eigenfunctions. These results have been then used in proving the necessary and sufficient conditions for the intrinsic contractivity properties of the corresponding semigroups ([H1] and [H4]). In the paper [H3] we have investigated the fall-off rates of eigenfunctions corresponding to negative eigenvalues for operators H with decaying potentials (i.e. $V(x) \to 0$ as $|x| \to \infty$). In [H2] we gave sharp two-sided estimates for the heat kernels of operators L in finite time horizon. An important feature of this scientific achievement is that it provides a general framework and methods which allow one to obtain the sharp estimates for Lévy processes with various types of intensities of large jumps. This covers important families of processes with both heavy and light tails (e.g. isotropic *stable* and *relativistic* or *tempered stable processes*).

The following discussion of the series of papers [H1]–[H4] is divided into five chapters. In Chapter I, we give definitions, set the notation, and discuss the basic properties of the processes and operators under consideration. In Chapter II, we present estimates of eigenfunctions for confining potentials. Chapter III is devoted to the presentation of our results on intrinsic contractivity properties. In Chapter IV, we discuss the estimates of eigenfunctions for decaying potentials. Chapter V gives our estimates for heat kernels of the operators L.

Notation

Positive constants appearing in the statements of our assumptions and results in this summary will be denoted by the same symbol C without numbering. However, locally, we use the notation $\widetilde{C}$ or numbering $C_1, C_2, \ldots$ to distinguish between constants in a given assumption, lemma or theorem. Our results are quite general and, therefore, the constants typically depend on the process, the potential and the dimension of the space. This will not be indicated below. If the dependence (or the independence) of a particular constant on a given parameter is essential, this will be clearly pointed out. If f and g are non-negative functions, then the notation $f(x) \asymp g(x)$, $x \in A$, means that there exists a constant $C \geq 1$ such that

$$C^{-1}g(x) \leq f(x) \leq Cg(x), \quad x \in A.$$

We also write $f(x) \stackrel{C}{\asymp} g(x)$ to allow for further reference to the constant C. A two-sided estimate

$$C_1 g(x) \leq f(x) \leq C_2 h(x), \quad x \in A,$$

will be called *sharp* if $g(x) \asymp h(x)$, $x \in A$.

Lévy processes and the direct jump property

Let $\{X_t\}_{t\geq 0}$ be a Lévy process in $\mathbb{R}^d$, $d \geq 1$. Recall that $\{X_t\}_{t\geq 0}$ is a time- and space-homogeneous Markov process with respect to its natural filtration, with strong Markov property and càdlàg paths. Denote by $\mathbb{P}^x$ and $\mathbb{E}^x$ the probability measure and expectation for the process starting from $x \in \mathbb{R}^d$. Lévy processes are uniquely determined by the Lévy-Khintchine formula

$$\mathbb{E}^0 e^{i\xi\cdot X_t} = e^{-t\psi(\xi)}, \quad \xi \in \mathbb{R}^d,\ t > 0,$$

where

$$\psi(\xi) = -i\xi\cdot b + \xi\cdot A\xi + \int \left(1 - e^{i\xi\cdot y} + i\xi\cdot y \mathbb{1}_{B(0,1)}(y)\right)\nu(dy), \quad \xi \in \mathbb{R}^d, \tag{1}$$

$A = [a_{ij}]_{i,j=1,\ldots,d}$ is a symmetric, non-negative definite matrix, $b \in \mathbb{R}^d$, and ν is a Lévy measure on $\mathbb{R}^d\setminus\{0\}$, i.e., $\int_{\mathbb{R}^d\setminus\{0\}} (1 \wedge |z|^2)\,\nu(dz) < \infty$. Denote by $\{P_t : t \geq 0\}$ the transition semigroup of the process $\{X_t\}_{t\geq 0}$. Below we use the same symbol P_t to denote the distribution of the random variable X_t (tj. $P_t(E - x) = \mathbb{P}^x(X_t \in E)$) and to denote the operator defined by this measure (i.e. $P_t f(x) = \mathbb{E}^x f(X_t) = \int_{\mathbb{R}^d} f(y+x)P_t(dy)$). The infinitesimal generator L (of the transition semigroup) of the process $\{X_t\}_{t\geq 0}$ is a homogeneous, non-local, pseudo-differential operator defined by

$$\widehat{Lf}(\xi) = -\psi(\xi)\widehat{f}(\xi), \quad \xi \in \mathbb{R}^d, \quad f \in D(L) := \{h \in L^2(\mathbb{R}^d) : \psi\widehat{h} \in L^2(\mathbb{R}^d)\} \tag{2}$$

($\widehat{f}(\xi)$ denotes the Fourier transform $\int_{\mathbb{R}^d} f(x)e^{i\xi\cdot x}dx$ of the function f; we also write $\mathcal{F}f(\xi)$). The function ψ is called the characteristic (Lévy-Khintchine) exponent of the process or symbol. For smooth functions f with compact support, the action of L takes the form

$$L\,f(x) = b\cdot\nabla f(x) + \sum_{i,j=1}^{d} a_{ij}\partial_{x_i}\partial_{x_j} f(x) + \int \left(f(x+z) - f(x) - \mathbb{1}_{B(0,1)}(z)z\cdot\nabla f(x)\right)\nu(dz). \tag{3}$$

In this chapter and in the next three Chapters II-IV, which contain the description of the results obtained in [H1], [H3] and [H4], respectively, **we always assume** that

$$b = 0, \quad \nu(-E) = \nu(E) \tag{4}$$

and

$$\nu(\mathbb{R}^d \setminus \{0\}) = \infty, \quad \nu(dx) = \nu(x)dx, \tag{5}$$

i.e., ν is an infinite measure which is absolutely continuous with respect to the Lebesgue measure (in the remaining of this chapter and in the next Chapters II-IV we only use the density of the Lévy measure and we denote it by the same symbol ν). The condition (4) implies that the process $\{X_t\}_{t\geq 0}$ is *symmetric*, i.e., $-X_t$ and X_t are equally distributed, and (5) guarantees that the measures P_t are absolutely continuous with respect to the Lebesgue measure [61, Theorem 27.7] (the corresponding densities are denoted by $p_t(x)$). Equivalently, the process $\{X_t\}_{t\geq 0}$ has the strong Feller property: the function $x \mapsto P_t f(x) = \mathbb{E}^x f(X_t)$ is bounded and continuous on $\mathbb{R}^d$, for every $f \in L^\infty(\mathbb{R}^d)$ and $t > 0$. Thanks to (4)–(5) the formula (1) reduces to

$$\psi(\xi) = \xi \cdot A\xi + \int (1 - \cos(\xi \cdot y))\, \nu(y) dy, \quad \xi \in \mathbb{R}^d, \tag{6}$$

ψ is an unbounded (even if $A \equiv 0$) function with values in $[0, \infty)$, and $-L$ is a self-adjoint, non-negative definite unbounded operator on $L^2(\mathbb{R}^d)$.

Below we will also need information on the Lévy process killed on exiting an open nonempty set $D \subset \mathbb{R}^d$. Denote by τ_D the first exit time of the process from D:

$$\tau_D := \inf\{t > 0 : X_t \notin D\}.$$

Transition probability densities of such a process are given by the Dynkin–Hunt formula

$$p_D(t, x, y) = p_t(y - x) - \mathbb{E}^x \left[\tau_D < t; p_{t-\tau_D}(y - X_{\tau_D})\right], \quad x, y \in D,\ t > 0. \tag{7}$$

We always assume that $p_D(t, x, y) = 0$, whenever $x \notin D$ or $y \notin D$. We have, $\mathbb{P}^x(t < \tau_D) = \int_D p_D(t, x, y)\, dy$, $x \in D$, $t > 0$. The Green function of the killed process is then given by $G_D(x, y) = \int_0^\infty p_D(t, x, y)\, dt$. For further details on Lévy processes and their evolution semigroups we refer the reader to [6, 17, 41, 61].

Let us recall that in the papers [H1], [H3] and [H4] we consider the symmetric Lévy processes with infinite and absolutely continuous Lévy measures on $\mathbb{R}^d \setminus \{0\}$ (i.e., (4)–(5) hold). We now formulate the assumptions (A1)-(A3) under which we obtain our main results for this class of processes. The first two assumptions are concerned with $\nu(x)$ and $p_t(x)$ and they are of structural importance. The third one, originating from [15], is a technical assumption concerning the Green function of a ball, needed for a potential-theory argument we use.

(A1) ***Lévy density.*** There exists a non-increasing function $g : (0, \infty) \to (0, \infty)$ such that

- **a)** $\nu(x) \asymp g(|x|)$, $x \in \mathbb{R}^d \setminus \{0\}$,
- **b)** there exists $C_1 \geq 1$ such that $g(r) \leq C_1 g(r + 1)$, $\quad r \geq 1$,
- **c)** *direct jump property* (DJP in short): there exists $C_2 > 0$ satisfying

$$\int_{\substack{|x-y|>1\\|y|>1}} g(|x - y|) g(|y|)\, dy \leq C_2 g(|x|), \quad |x| \geq 1.$$

The function g appearing in (A1.a) will be called a profile of the Lévy density.

(A2) ***Transition density.*** There exists $t_b > 0$ such that $\sup_{x\in\mathbb{R}^d} p_{t_b}(x) < \infty$.

(A3) ***Green function.*** For every $0 < p < q < R \leq 1$ it holds that

$$\sup_{x \in B(0,p)} \sup_{y \in B(0,q)^c} G_{B(0,R)}(x, y) < \infty.$$

In our papers [H3]-[H4] the assumptions have been formulated exactly as above (in [H3] we additionally assume that (A3) holds for all $R > 0$), while in the paper [H1], which opens the mono-thematic series of papers, it is done in a somewhat different way. The conditions (A1)-(A3) correspond directly to [H1, Assumptions 2.1-2.3]. The first one of these assumptions has been stated in a more general form, but it holds automatically under (A1) (see the comments given below this assumption in [H1]). Additionally, [H1, Assumption 2.2] claims the existence of $p_t(x)$. However, it is not needed – this follows directly from (5). It suffices to assume (A2).

The convolution condition (A1.c), which determines the class of processes studied in this mono-thematic series of papers, is of structural importance. In some sense, it explains the structure of our results and it has a very natural and suggestive interpretation. Since $\nu \asymp g$ and the profile g is monotone, the condition (A1.c) is in fact equivalent to the existence of a constant $C > 0$, such that

$$\int_{\substack{1<|x-y|<|x| \\ 1<|y|<|x|}} \nu(x-y)\nu(y)\, dy \leq C\nu(x), \quad |x| > 1.$$

The function $\nu(x)$ restricted to $\{x : |x| > 1\}$ can be seen as the intensity of large jumps of the initial Lévy process (after normalizing it is the distribution density of a single jump of its Poissonian component). Therefore, the inequality above means that *the intensity of two consecutive large jumps of the process is dominated by the intensity of a single direct large jump.* It can be understood as follows: the probability of getting by a particle (starting from 0) to a given distant position x by a single direct jump is asymptotically not smaller than the probability of getting there by a combination of several shorter jumps. This interpretation of (A1.c) justifies the name *direct jump property.* This phenomenon can also be understood as the *ability* of the process to *reduce* or *pare* mutliple large jumps (in [H3]-[H4] the condition (A1.c) was referred to as the *jump-paring property*).

Our assumptions (A1)-(A3) cover a large class of Lévy processes with jumps [H1, Section 4], [H4, Section 4], [H3, Sections 4.3-4.4]. The convolution condition (A1.c) was characterized in [H2, Proposition 2] (see also [H2, Example 2 (1)]) for a specific family of profiles g, which covers the most important and interesting examples of processes. For instance, if g has the doubling property, (A1.c) holds automatically. However, let us point out that our assumptions and methods can be applied to processes with various types of intensities of large jumps: polynomial (e.g. jump stable processes), exponential (e.g. relativistic and tempered stable processes), and intermediate (e.g. Weibull-type processes, which are now more and more common in modelling). They can be both isotropic or non-isotropic processes. In the latter case, we only assume that the jump intensities are controlled by monotone and isotropic profiles. Some references to applications of the processes with light tails are given in [P6].

Feynman–Kac semigroups and non-local Schrödinger operators

We first introduce a class of Schrödinger potentials that are studied in our papers (cf. [H1, Definition 2.1], [H3, Definition 2.2] and [H4, Definition 3.2]).

DEFINITION 1 (**Kato class**). We say that the Borel function $V : \mathbb{R}^d \to \mathbb{R}$ belongs to the *Kato-class* $\mathcal{K}$ associated with the Lévy process $\{X_t\}_{t\geq 0}$ if it satisfies

$$\lim_{t\downarrow 0} \sup_{x\in\mathbb{R}^d} \mathbb{E}^x \left[\int_0^t |V(X_s)| ds \right] = 0. \tag{8}$$

Also, we say that V is a *Kato-decomposable potential* (or *X-Kato class* in short), denoted $V \in \mathcal{K}_{\pm}$, whenever

$$V = V_+ - V_-, \quad \text{with} \quad V_- \in \mathcal{K}^X \quad \text{and} \quad V_+ \in \mathcal{K}^X_{\text{loc}},$$

where V_+, V_- denote the positive and negative parts of V, respectively, and $V_+ \in \mathcal{K}_{\text{loc}}$ means that $V_+ 1_B \in \mathcal{K}$ for all compact sets $B \subset \mathbb{R}^d$.

If L is a generator of the Lévy process described by (2) and (4)-(5) and $V \in \mathcal{K}_{\pm}$, then

$$H := -L + V$$

can be defined in a quadratic form sense as a self-adjoint and bounded below operator with the dense domain in $L^2(\mathbb{R}^d)$. Such H is called a *non-local Schrödinger operator based on generator* L. Here $-L$ is often referred to as the *kinetic part* of H and V is the operator of multiplication by a function, which is called a *potential*. As mentioned above, most of the motivations to study the non-local Schrödinger operators come from mathematical physics. The operator H can be interpreted as a *Hamiltonian* of some physical system. The negative part V_- and the positive part V_+ are often called *attractive* and *repulsive* potentials, respectively.

The *Schrödinger semigroup* associated with H has a stochastic representation with respect to the process $\{X_t\}_{t\geq 0}$ [29, Chapter 2.A]. More precisely, it holds that

$$e^{-tH} f(x) = T_t f(x), \quad f \in L^2(\mathbb{R}^d), \quad t > 0, \tag{9}$$

where

$$T_t f(x) := \mathbb{E}^x \left[e^{-\int_0^t V(X_s) ds} f(X_t) \right], \quad f \in L^2(\mathbb{R}^d). \tag{10}$$

The equality (9) is often called the *Feynman–Kac formula.* It should be seen as a generalization of a known fact from the theory of classical Schrödinger operators, for the Brownian motion instead of the process $\{X_t\}_{t\geq 0}$. The Feynman–Kac formula is a very useful tool, which allows for studying various properties of the operators H and e^{-Ht} by probabilistic methods, especially by effective techniques of modern probabilistic potential theory. Very often, various problems related to the non-local Schrödinger operators, including those in mathematical physics, are even formulated in terms of stochastic processes. Here a starting point is the process $\{X_t\}_{t\geq 0}$ and the family of operators $\{T_t : t \geq 0\}$, that can be considered independently of (9). Our approach in the papers [H1], [H3], [H4] is also mainly probabilistic, which fits this trend very well. In the sequel, we will be working only with a stochastic representation of the Schrödinger semigroup associated with the operator H.

The family $\{T_t : t \geq 0\}$ defined in (10) is a strongly continuous semigroup of symmetric operators in $L^2(\mathbb{R}^d)$. It is called the *Feynman-Kac semigroup* of the process $\{X_t\}_{t\geq 0}$ with potential V. When $V \geq 0$ (i.e. $V_- \equiv 0$), then $\{T_t : t \geq 0\}$ is a transition semigroup of a Markov process, whose paths are being *killed* with a random intensity described by V. If $V_- \neq 0$, then due to *mass creation effect* the semigroup $\{T_t : t \geq 0\}$ has not a direct probabilistic interpretation

We now recall some basic facts from the theory of Feynman–Kac semigroups.

Lemma 2. *Let $\{X_t\}_{t\geq 0}$ be a symmetric Lévy process with infinite Lévy measure $\nu(dx) = \nu(x)dx$ such that (A2) holds, and let $V \in \mathcal{K}_{\pm}$.*

(a) The operators $T_t : L^p(\mathbb{R}^d, dx) \to L^p(\mathbb{R}^d, dx)$ are bounded for $t \geq 0$ and $1 \leq p \leq \infty$. Moreover, $T_t : L^p(\mathbb{R}^d, dx) \to L^\infty(\mathbb{R}^d, dx)$ are bounded for $t \geq t_b$ and $1 \leq p < \infty$.

(b) The function $x \mapsto T_t f(x)$ is bounded and continuous on $\mathbb{R}^d$ for every $f \in L^\infty(\mathbb{R}^d, dx)$ and $t > 0$ (i.e. the semigroup $\{T_t : t \geq 0\}$ is strong Feller).

(c) All operators T_t, $t > 0$, have integral kernels: for every $t > 0$ there exists a continuous and symmetric function $u_t(\cdot,\cdot)$ on $\mathbb{R}^d \times \mathbb{R}^d$ such that

$$T_t f(x) = \int_{\mathbb{R}^d} u_t(x,y) f(y) dy, \quad f \in L^p(\mathbb{R}^d, dx), \quad 1 \leq p \leq \infty.$$

Moreover, $u_t(x,y) > 0$ for $t > 0$ and $x, y \in \mathbb{R}^d$, and $\sup_{x,y\in\mathbb{R}^d} u_t(x,y) < \infty$, for $t \geq t_b$.

A more detailed introduction to the theory of Feynman–Kac semigroups can be found in monographs [29] and [25, Chapter 3.2].

Since H is a self-adjoint operator, $\operatorname{spec} H \subset \mathbb{R}$. We say that λ is an eigenvalue of H if there exists $\varphi \in L^2(\mathbb{R}^d, dx)$ such that $H\varphi = \lambda\varphi$ (or, equivalently, $T_t\varphi = e^{-\lambda t}\varphi$, $t > 0$). Denote:

$$\lambda_0 := \inf \operatorname{spec} H.$$

When λ_0 is an eigenvalue, then, by following the terminology from mathematical physics, we say that H has a *ground state*. Since $u_t(x,y) > 0$, λ_0 is unique (i.e. H has a *non-degenerate ground state*), and the corresponding eigenfunction φ_0 is strictly positive (see e.g. [58, Theorem XIII.43]). Below we call λ_0 and φ_0 the *ground state eigenvalue* and *eigenfunction* of H, respectively. We always assume that $\|\varphi_0\|_2 = 1$. Observe that if φ is an eigenfunction, then the properties (a) and (b) in Lemma 2 imply that $\varphi = e^{\lambda t_b} T_{t_b}\varphi \in L^\infty(\mathbb{R}^d, dx)$ and $\varphi \in C_b(\mathbb{R}^d)$. This means that φ has a bounded and continuous version (and, additionally, φ_0 is strictly positive) on $\mathbb{R}^d$. In particular, the eigenequations $T_t\varphi(x) = e^{-\lambda t}\varphi(x)$ hold pointwise, for every $x \in \mathbb{R}^d$.

We now define two disjoint subclasses of Kato-decomposable Schrödinger potentials that are studied in our papers.

DEFINITION **3.** Let $V \in \mathcal{K}_{\pm}$. Then

- $V \in \mathcal{K}_{\pm}^{\infty}$ if $\lim_{|x|\to\infty} V(x) = \infty$,
- $V \in \mathcal{K}_{\pm}^{0}$ if $\lim_{|x|\to\infty} V(x) = 0$.

The class $\mathcal{K}_{\pm}^{\infty}$, called the class of *confining potentials*, is studied in the first and the fourth paper from the series (cf. [H1, Assumption 2.4] and [H4, Assumption (A4)]). The second class $\mathcal{K}_{\pm}^{0}$, called the class of *decaying potentials*, is investigated in the third paper (cf. [H3, Assumption (A4)]). In this summary, this material is presented in Chapters II-III and IV, respectively.

When $V \in \mathcal{K}_{\pm}^{\infty}$, the operators T_t, $t > 0$, are compact and the spectra of T_t and H consist of isolated eigenvalues of finite multiplicty. More precisely, there exists a countable orthonormal basis $\{\varphi_n\}_{n\geq 0}$ in $L^2(\mathbb{R}^d, dx)$ such that $H\varphi_n = \lambda_n\varphi_n$ and $T_t\varphi_n = e^{-\lambda_n t}\varphi_n$ for $t > 0$, where $-\infty < \lambda_0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots \to \infty$ (for convenience, in this representation, we count all eigenvalues without multiplicity, but every λ_n appears only finitely many times; λ_0 is unique). In particular, H has a non-degenerate ground state.

When $V \in \mathcal{K}_{\pm}^{0}$, then the operators T_t are not compact and the spectra of T_t and H look completely different. The essential spectrum of H coincides with the essential spectrum of $-L$ and is equal to $[0, \infty)$. Therefore, if there exists an isolated eigenvalue λ (of finite multiplicity) of H, then $\lambda < 0$. The existence and properties of negative eigenvalues have been widely studied for both local and non-local Schrödinger operators. Some references, together with a short discussion, can be found at the beginning of [H3, Section 4.1].

Ground state transformed semigroup and related Markov process

We now introduce the definition of a ground state transformed semigroup, which is one of the basic objects studied in the papers [H1] and [H4] (this material will be presented in Chapter III of this summary). Let $V \in \mathcal{K}_{\pm}^{\infty}$. Recall that in this case there exists a unique $\varphi_0 \in L^2(\mathbb{R}^d, dx)$, $\varphi_0 > 0$, $\|\varphi_0\|_2 = 1$, such that $H\varphi_0 = \lambda_0 \varphi_0$ and $T_t \varphi_0 = e^{-\lambda_0 t}\varphi_0$, $t > 0$, where $\lambda_0 := \inf \operatorname{spec} H$. Let $\mu(dx) = \varphi_0^2(x)dx$.

We define

$$\widetilde{T}_t f(x) = \frac{1}{\varphi_0(x)} e^{\lambda_0 t} T_t(f\varphi_0)(x), \quad f \in L^2(\mathbb{R}^d, \mu), \quad t \geq 0, \tag{11}$$

where $\{T_t : t \geq 0\}$ is the Feynman–Kac semigroup of the process $\{X_t\}_{t\geq 0}$ with potential V. One can easily check that the operators defined in this manner are symmetric and they form a strongly continuous semigroup in $L^2(\mathbb{R}^d, \mu)$, which is called a *ground state transformed* (or *intrinsic) semigroup*. It is generated by the operator $-\widetilde{H}$, where $\widetilde{H} f = \varphi_0^{-1} H(f\varphi_0) - \lambda_0 f$ is defined for those $f \in L^2(\mathbb{R}^d, \mu)$, for which $f\varphi_0$ belongs to the domain of H. The operators $\widetilde{H}$ i $H - \lambda_0$ are then unitarily equivalent. Moreover, $\widetilde{T}_t \mathbf{1} = \mathbf{1}$, $t \geq 0$.

All operators $\widetilde{T}_t$, $t > 0$, are of integral type. More precisely, if $u_t(x, y)$ is the integral kernel of T_t, then

$$\widetilde{T}_t f(x) = \int_{\mathbb{R}^d} \widetilde{u}_t(x, y) f(y) \mu(dy), \quad f \in L^2(\mathbb{R}^d, \mu), \quad t > 0,$$

where

$$\widetilde{u}_t(x, y) := \frac{e^{\lambda_0 t} u_t(x, y)}{\varphi_0(x)\varphi_0(y)}.$$

The semigroup $\{\widetilde{T}_t : t \geq 0\}$ defines a stationary Markov process with càdlàg paths, the so-called *ground state transformed Markov process* associated with the Schödinger operator H. Observe that φ_0 is a positivie harmonic funtion of the shifted Schrödinger operator $H - \lambda_0 = -L + V - \lambda_0$. This means that from the probabilistic point of view the construction (11) is in fact a special case of conditioning in the sense of Doob. In stochastic analysis, such a change of measure is also often referred to as a jump-type Girsanov transformation [32, Chapter 6.3], [62] (see also [57]).

For classical Schrödinger operators based on the Laplace operator, the above contruction leads to a diffusion process in which the drift term is determined by $\nabla \log \varphi_0$. For instance, if $H = -\Delta + |x|^2$ (harmonic oscillator), then the transformed process is the Ornstein-Uhlenbeck diffusion. More generally, when the potential is a polynomial of an even degree, then the transformed diffusion is called a $P(\phi)_1$*-process* associated with H (see e.g. [59]). It is worth mentioning that such processes play a central role in the Nelson stochastic quantum mechanics [55].

Transformed processes have also been partly investigated in the non-local case [33, 34, 35, D1, P6, P11].

Tools of probabilistic potential theory

The most of our main results in the papers [H1], [H3] and [H4] are concerned with a fall-off of eigenfunctions of operators H. Our methods are based on the pointwise estimates of functions that are harmonic with respect to the process $\{X_t\}_{t\geq 0}$ whose paths are being killed with random intensity given by the potential V.

Let $0 \leq V \in \mathcal{K}_{\pm}$. A non-negative Borel funtion f on $\mathbb{R}^d$ is called *(X,V)-harmonic* in an open set $D \subset \mathbb{R}^d$, if it has the mean value property

$$f(x) = \mathbb{E}^x \left[\tau_U < \infty; e^{-\int_0^{\tau_U} V(X_s)ds} f(X_{\tau_U})\right], \quad x \in U, \tag{12}$$

for every open set U with its closure contained in D, and is called *regular (X,V)-harmonic* in D if (12) holds for $U = D$ (recall that τ_U is the first exit time of the process from U). By the strong Markov property of $\{X_t\}_{t\geq 0}$ every regular (X,V)-harmonic function in D is (X,V)-harmonic in D. Below we mainly consider the case when D is an open ball or a complement of a closed ball in $\mathbb{R}^d$. The latter case is often referred to as *harmonicity at infinity.*

We often consider the case when $V \equiv \eta$, for a positive number η. In this case, if the support of f is separated from the set D, then the expression on the right hand side of (12) can be effectively estimated by means of the *Ikeda-Watanabe formula* (cf. [40, Theorem 1]): if $D \subset \mathbb{R}^d$ is an open and bounded set, $\eta \geq 0$, and f is bounded or non-negative Borel function on $\mathbb{R}^d$ such that $\operatorname{dist}(\operatorname{supp} f, D) > 0$, then the formula

$$\mathbb{E}^x \left[e^{-\eta\tau_D} f(X_{\tau_D})\right] = \int_D \int_0^\infty e^{-\eta t} p_D(t,x,y) dt \int_{D^c} f(z)\nu(z-y)dzdy, \quad x \in D, \tag{13}$$

holds.

The Green (or potential) operator for the semigroup $\{T_t : t \geq 0\}$ is defined by

$$G^V f(x) = \int_0^\infty T_t f(x) dt = \mathbb{E}^x \left[\int_0^\infty e^{-\int_0^t V(X_s)ds} f(X_t) dt\right],$$

for all boundedd or non-negative Borel functions f on $\mathbb{R}^d$, and the Green operator for an open set D is given by

$$G_D^V f(x) = \int_0^\infty \mathbb{E}^x \left[t < \tau_D; e^{-\int_0^t V(X_s)ds} f(X_t)\right] dt = \mathbb{E}^x \left[\int_0^{\tau_D} e^{-\int_0^t V(X_s)ds} f(X_t) dt\right], \quad x \in D,$$

for all bounded or non-negative Borel functions f on D. Note that when $V \geq 0$ on D, then the expression $G_D^V \mathbf{1}(x) = \mathbb{E}^x \left[\int_0^{\tau_D} e^{-\int_0^t V(X_s)ds} dt\right]$, $x \in D$, can be understood as the mean exit time from D of the process starting from $x \in D$, whose paths are being killed at the rate given by V. It follows from the strong Markov property of $\{X_t\}_{t\geq 0}$ that

$$G^V f(x) = G_D^V f(x) + \mathbb{E}^x \left[\tau_D < \infty; e^{-\int_0^{\tau_D} V(X_s)ds} G^V f(X_{\tau_D})\right], \quad x \in D. \tag{14}$$

The following two-sided estimate for functions that are (X,V)-harmonic in balls is a key technical tool in our papers [H1] and [H3].

Lemma 4 ([H1, Lemma 3.1, Corollary 3.1]). *Let $\{X_t\}_{t\geq 0}$ be a symmetric Lévy process with infinite Lévy measure $\nu(dx) = \nu(x)dx$ such that $\nu(x) \asymp g(|x|)$, $x \in \mathbb{R}^d \setminus \{0\}$, for some non-increasing profile funtion $g : (0,\infty) \to (0,\infty)$. Moreoever, assume that (A3) holds. Then* *there exists a constant $C > 0$ such that for every potential $V \in \mathcal{K}_{\pm}$, $V \geq 0$ on $B(x_0, 1)$ and for every non-negative function on $\mathbb{R}^d$ which is regular (X, V)-harmonic in $B(x_0, 1)$ we have*

$$f(y) \stackrel{C}{\lesssim} G^V_{B(x_0,1)} 1(y) \int_{B(x_0,3/4)^c} f(z)\nu(z - x_0)dz, \quad y \in B(x_0, 1/2).$$

This fact follows directly from the strong result which was obtained by Bogdan, Kumagai and Kwaśnicki in a much larger generality [15, Lemma 3.2 (b), Theorem 3.5]. Let us point out that estimates of this type play a fundamental role in the boundary potential theory of Markov processes with jumps. For our applications it is crucial that the constant in this estimate is uniform with respect to f, V and x_0.

Investigation of the potential theory of non-local Schrödinger operators and their Feynman–Kac semigroups has been initiated for the fractional case by Bogdan and Byczkowski in the pioneering papers [8, 9]. Many further results in this direction can also be found in the papers [23, 24, D3] (for more general approach see the monographs [25, 29]). Our basic references on potential theory for general Markov and Lévy processes are monographs [6, 7, 10] and papers [15, 60, 63].

Decomposition of paths of the process

Our methods in [H1] are based on an appropriate decomposition of paths of the process, which is determined by the first exit and hitting times of some annuli in $\mathbb{R}^d$ centered at the origin. We apply the iterative scheme from the papers [18, 47]. For sufficiently large $n_0 \in \mathbb{N}$ and $n, k \in \mathbb{N}$ such that $n, k \geq n_0$ we let (see [H1, p. 1371-1372])

$$R_k := \begin{cases} \left\{x \in \mathbb{R}^d : k - 1 < |x| \leq k\right\} & \text{if } k \geq n_0 + 2, \\ \left\{x \in \mathbb{R}^d : |x| \leq n_0 + 1\right\} & \text{if } k = n_0 + 1, \\ \left\{x \in \mathbb{R}^d : |x| \leq n_0\right\} & \text{if } k = n_0, \end{cases}$$

$$D_n := \begin{cases} \left\{x \in \mathbb{R}^d : |x| > n - 2\right\} & \text{if } n \geq n_0 + 2, \\ \mathbb{R}^d & \text{if } n \in \left\{n_0 n_0 + 1\right\}, \end{cases}$$

and

$$\sigma_{R_k} := \inf\left\{t \geq 0 : X_t \in R_k\right\} \quad \text{(the first hitting time of } R_k),$$
$$\tau_{D_n} := \inf\left\{t \geq 0 : X_t \notin D_n\right\} \quad \text{(the first exit time from } D_n).$$

This decomposition is based on iteration of the scenario in which the process exits the complement of some ball (i.e. the set D_n) moving to one of the shells R_k, $k \leq n - 2$ (inside this ball). More precisely, for $n - 2 \geq k \geq n_0$ and $t > 0$ we define

$$S(n, k, 1, t) = \left\{X_{\tau_{D_n}} \in R_k, \sigma_{R_k} < t\right\},$$
$$S(n, k, l, t) = \bigcup_{p=k+2}^{n-2} S(n, p, l - 1, t) \cap S(p, k, 1, t), \quad l > 1.$$

The event $S(n, k, 1, t)$ describes the scenario in which the process hits R_k on exiting D_n, before time t. Similarly, $S(n, k, l, t)$ is defined inductively, by the l-fold iteration of this scenario.

For instance, if $k+2 \leq p \leq n-2$, then the event $S(n,p,1,t) \cap S(p,k,1,t)$ corresponds to those paths which get to R_p on exiting D_n and then, hit R_k on exiting D_p. All these happen before time t.

The iterative structure of this scheme allowed us to find sufficiently sharp estimates for probabilities of such scenarios for the process with DJP with the potential.

Lemma 5 ([H1, Lemma 3.6]). *Let $\{X_t\}_{t \geq 0}$ be a symmetric Lévy process with infinite Lévy measure $\nu(dx) = \nu(x)dx$ such that (A1)-(A3) hold, and let $V \in \mathcal{K}_{\pm}^{\infty}$. Then there exists $n_0 \in \mathbb{N}$ and a constant $C > 0$ such that for $n-1 < |x| \leq n$, $n_0 \leq k \leq n-2$, $n,k,l \in \mathbb{N}$ and $t > 0$ we have*

$$\mathbb{E}^x \left[S(n,k,l,t); e^{-(1/2)\int_0^{\sigma_{R_k}} V(X_s)ds} \right] \leq \frac{C}{2^l \inf_{|y| \geq n-2} V(y)} \int_{R_k} \nu(x-z)dz.$$

The above estimate was proven by induction with respect to $l \in \mathbb{N}$ and its proof is mainly based on the convolution condition (A1.c). The key step is to establish the claimed bound for $l=1$. This follows from the next lemma.

Lemma 6 ([H1, Lemma 3.5]). *Let $\{X_t\}_{t \geq 0}$ be a symmetric Lévy process with infinite Lévy measure $\nu(dx) = \nu(x)dx$ such that (A1)-(A3) hold. Then there exist $n_0 \in \mathbb{N}$, a constant $C > 0$ and $\theta_0 > 0$ such that for all $\theta > \theta_0$, $n-1 < |x| \leq n$, $n_0 \leq k \leq n-2$, $n,k \in \mathbb{N}$, and $t > 0$ we have*

$$\mathbb{E}^x \left[\tau_{D_n} < t, X_{\tau_{D_n}} \in R_k; e^{-\theta \tau_{D_n}} \right] \leq \frac{C}{\theta} \int_{R_k} \nu(x-z)dz.$$

Lemmas 5–6 extend the analogous results from [47], where the above scheme was proposed. Our proof of Lemma 6 is based on a new idea and substantially differs from that in the cited paper. The argument is based on Lemma 4, some self-improving estimate and (A1.c). It is essential for our applications below that the constants do not depend on t.

II. Pointwise estimates of eigenfunctions for $V \in \mathcal{K}_{\pm}^{\infty}$

Let us recall that if $V \in \mathcal{K}_{\pm}^{\infty}$, then the operators T_t, $t > 0$, of the Feynman–Kac semigroup are compact, the spectra of T_t and H consist of countably many isolated eigenvalues of finite multiplicity, and the corresponding eigenfunctions $\{\varphi_n\}_{n \geq 0}$ form an orthonormal basis in $L^2(\mathbb{R}^d, dx)$. In particular, there exists a nondegenerate ground state of H. This chapter is devoted to the presentation of the two-sided sharp estimates at infinity of φ_0 and the upper bounds for other φ_n's. These results were obtained in [H1].

Motivations and background. Investigations of the localization and geometric properties of eigenfunctions are one of the most important challenges of spectral analysis. Let us recall that in the non-relativistic (local) and quasi-relativistic (non-local) models of quantum mechanics the eigenfunctions φ_n are solutions of the time-independent Schrödinger equation with the energy operator H (or, simply, the eigenproblem for Hamiltonian H). They encode full information about the so-called stationary states of the system, in particular their squares are the probability densities of the particle positions in the configuration space. Knowing the rate of decay at infinity of the ground state φ_0 turns out to be crucial for understanding the evolution properties of the semigroup $\{T_t : t \geq 0\}$ for large times. If the semigroup $\{T_t : t \geq 0\}$ is *intrinsically ultracotractive* (see Definition 10 (iii) below), then the factorization

$$u_t(x, y) \asymp e^{-\lambda_0 t} \varphi_0(x) \varphi_0(y), \quad t \geq t_0, \quad x, y \in \mathbb{R}^d, \tag{15}$$

holds. This means that φ_0 describes the spatial behavior of the kernels $u_t(x, y)$ for large times (this will be discussed in the next chapter).

In several classical models the eigenfunctions of the operator $-\Delta + V$ can be computed explicitly. The most well known example is the quantum harmonic oscillator for which the eigenfunctions are Hermite functions. The literature concerning the estimates of asymptotic behaviour of eigenfunctions for classical Schrödinger operators is abundant (e.g. [2, 19, 65] and the survey [67]). From the point of view of this scientific achievement, the most interesting contribution is due to Carmona [19], who proved the pointwise estimates by using the Feynman–Kac formula. It is worth noting that the results obtained by probabilistic methods were typically more general and they required much less assumptions than those obtained by analytical methods [66]. The probabilistic approach shed new light on these issues ([67, point (V), p. 3]). A common feature of eigenfunctions of classical Schrödinger operators with Kato-decomposable potentials is that their decay rates at infinity are not slower than exponential ([19, Proposition 3.3]) and [65, Theorems C.3.3-C.3.4]).

The localization problem for eigenfunctions of non-local Schrödinger operators with confining potentials has been studied as well. In this case, one can not expect to get any explicit formulas. Certain asymptotic expansions for eigenfunction of $H = \sqrt{-\Delta} + |x|^2$ were obtained in [52]. The famous work of Carmona, Masters and Simon, based entirely on probabilistic techniques, contains a result linking the exponential decay of eigenfunctions with the existence of an exponential moment of the Lévy measure [20, Proposition IV.4]. More specifically, for a wide class of Lévy processes and confining potentials, the following implication is true: if φ is an eigenfunction of H and $\int_{|y|>1} e^{b|y|} \nu(dy) < \infty$ for some $b > 0$, then there are constants $C_1, C_2 > 0$ for which $|\varphi(x)| \leq C_1 e^{-C_2|x|}$, $x \in \mathbb{R}^d$. Later, only the decay of φ_0 was studied, and that was only in the context of intrinsic ultracontactivity of the semigroup $\{T_t : t \geq 0\}$. Kulczycki and Siudeja dealt with the relativistic α-stable process ($\alpha \in (0, 2)$) and non-negative potentials from certain subclass of $\mathcal{K}_{\pm}^{\infty}$ with sufficiently regular growth at infinity. Under the condition that the semigroup $\{T_t : t \geq 0\}$ is intrinsically ultracontractive, they obtained two-sided sharp estimates of φ_0 at infinity. Another important contribution to this field is the work of Kwaśnicki [48], which is concerned with isotropic stable semigroups on unbounded subsets of $\mathbb{R}^d$. The author proved sharp two-sided estimates of φ_0 and then applied them to characterize the intrinsic ultracontractivity. These results do not apply directly to Schrödinger semigroups, but a natural adaptation of techniques from this article allowed us to prove equally strong results for Feynman–Kac semigroups of isotropic stable processes associated with *fractional Schrödinger operators* [D3, D1]. The key idea of Kwaśnicki was based on a pioneering application of the boundary Harnack inequality (proven in [14]) to obtain some strategic estimates of α-harmonic functions, which appear naturally within this framework. Both works [47, 48] were the inspiration for our research in [H1].

Our results. The main goal of the paper [H1], which initiated the mono-thematic series of articles, was to investigate the decay properties of the ground state φ_0 and the intrinsic contractivity properties of the semigroup $\{T_t : t \geq 0\}$ for a large class of Lévy processes with jumps and confining potentials. We wanted to understand how do these properties depend on the process and the potential. We expected to find a proper framework which would unify old and new results. These goals have been achieved for a wide class of processes with the *direct jump property*, which turned out to be optimal for our results. It includes both the isotropic jump-type stable processes and processes with exponential intensities of large jumps such as relativistic and tempered stable ones. Our results apply to arbitrary confining potentials and they do not require any additional regularity assumptions on the semigroup.

Our first main result is the upper estimate for non-negative Borel functions φ on $\mathbb{R}^d$, which satisfy the following condition: for some $\lambda \geq 0$ the inequality

$$\varphi(x) \leq e^{\lambda t} T_t \varphi(x), \quad t > 0, \quad x \in \mathbb{R}^d, \tag{16}$$

holds.

Theorem 7 ([H1, Theorem 2.1]). *Let $\{X_t\}_{t\geq 0}$ be a symmetric Lévy process with infinite Lévy measure $\nu(dx) = \nu(x)dx$ such that (A1)–(A3) hold, and let $V \in \mathcal{K}_{\pm}^{\infty}$. If φ is a non-negative and bounded Borel function on $\mathbb{R}^d$ satisfying the condition* (16) *for some $\lambda \geq 0$, then there exist $C = C(\lambda)$ and $R = R(\lambda)$ such that*

$$\varphi(x) \leq C \left\| \varphi \right\|_\infty \nu(x), \quad |x| \geq R.$$

Observe that if

$$\varphi(x) = e^{\lambda t} T_t \varphi(x), \quad t > 0, \quad x \in \mathbb{R}^d, \tag{17}$$

holds for some $\lambda > 0$, then $|\varphi|$ satifies the condition (16). This allows us to improve the upper bound in the theorem above for functions satisfying (17), even if they change the sign.

Theorem 8 ([H1, Theorem 2.2]). *Let $\{X_t\}_{t\geq 0}$ be a symmetric Lévy process with infinite Lévy measure $\nu(dx) = \nu(x)dx$ such that (A1)–(A3) hold, and let $V \in \mathcal{K}_{\pm}^{\infty}$. If φ is a bounded Borel function on $\mathbb{R}^d$ satisfying the condition* (17) *for some $\lambda > 0$, then there exist $C = C(\lambda)$ and $R = R(\lambda)$ such that*

$$|\varphi(x)| \leq C \left\| \varphi \right\|_\infty G^V_{B(x,1)} \mathbf{1}(x)\, \nu(x), \quad |x| \geq R.$$

If, in addition, φ is positive, then there are $\widetilde{C} = \widetilde{C}(\lambda, \varphi)$ and $\widetilde{R} = \widetilde{R}(\lambda, \varphi)$ such that

$$\varphi(x) \geq \widetilde{C} G^V_{B(x,1)} \mathbf{1}(x)\, \nu(x), \quad |x| \geq \widetilde{R}.$$

Recall that $G^V_{B(x,1)}\mathbf{1}(x) = \mathbb{E}^x\left[\int_0^{\tau_{B(x,1)}} e^{-\int_0^t V(X_s)ds}dt\right]$.

The main ideas of proofs of Theorems 7 - 8 will be discussed at the end of this chapter. First we focus on applications of these results in proving pointwise estimates of eigenfunctions of the operator H. Since for every $\eta \geq 0$ such that $\eta + \lambda_0 > 0$ (then also, $\eta + \lambda_n > 0$, $n = 1, 2, ...$) we have

$$e^{-(\lambda_n+\eta)t}\varphi_n(x) = e^{-\eta t}T_t\varphi_n(x) = \mathbb{E}^x\left[e^{-\int_0^t(V(X_s)+\eta)ds}\varphi_n(X_t)\right], \quad x \in \mathbb{R}^d, \quad t > 0,$$

every eigenfunction φ_n satisifes (17) with respect to the Feynman–Kac semigroup with shifted potential $V + \eta$ and $\lambda = \lambda_n + \eta$. This immediately gives the following corollary.

Corollary 9 ([H1, Theorems 2.3-2.4]). *Let $\{X_t\}_{t\geq 0}$ be a symmetric Lévy process with infinite Lévy measure $\nu(dx) = \nu(x)dx$ such that (A1)-(A3) hold, and let $V \in \mathcal{K}^\infty_\pm$. Then for every $n \in \{0, 1, 2, 3, ...\}$ and $\eta \geq 0$ such that $\lambda_0 + \eta > 0$, there are $C = C(n, \eta)$ and $R = R(n, \eta)$ such that*

$$|\varphi_n(x)| \leq C G^{V+\eta}_{B(x,1)}\mathbf{1}(x)\,\nu(x), \quad |x| \geq R.$$

Furthermore,

$$\varphi_0(x) \asymp G^{V+\eta}_{B(x,1)}\mathbf{1}(x)\,\nu(x), \quad |x| \geq R.$$

The rate of decay of φ_0 at infinity is determined by the product of functions $\nu(x)$ and $G^{V+\eta}_{B(x,1)}\mathbf{1}(x)$. First of them describes the intensity of jumps of the free process, while the other can be interpreted as thr mean exit time from the ball $B(x, 1)$ for the process starting from x and evolving under the potential $V + \eta$ (recall that $V \geq 0$ outside a bounded set). One can check (cf. [H1, (3.2)]) that there is a constant $C \geq 1$ such that

$$\frac{1}{C}\frac{1}{\sup_{y\in B(x,1)} V(y)} \leq G^{V+\eta}_{B(x,1)}\mathbf{1}(x) \leq C\frac{1}{\inf_{y\in B(x,1)} V(y)},$$

for sufficiently large x. This allows us to make the estimate in Corollary 9 more explicit (see [H1, Corollary 2.2]). However, first of all, this gives a full factorization of the estimates and explains how the fall-off rate of φ_0 at infinity depends on the process and the potential. In particular, if there exists $C \geq 1$ such that $\sup_{y\in B(x,1)} V(y) \leq C \inf_{y\in B(x,1)} V(y)$, for $|x| \geq R$, then

$$\varphi_0(x) \asymp \frac{\nu(x)}{V(x)}, \quad |x| \geq R + 1.$$

The estimates of the ground state eigenfunction of the operator H presented above are crucial in proving our further results in the papers [H1, H4] (this will be discussed in the next chapter).

Let us notice a further, quite surprising, consequence of our estimates from Corollary 9 (see [H1, Corollary 2.1]): for every $n \geq 1$ there is a constant $C = C(n)$ such that

$$|\varphi_n(x)| \leq C\varphi_0(x), \quad x \in \mathbb{R}^d. \tag{18}$$

This means that all eigenfunctions of non-local Schrödinger operators with confining potentials decay at infinity not slower than the ground state eigenfunction. Interestingly, such a domination property is not true in general for classical Schrödinger operators $-\Delta + V$ with $V \in \mathcal{K}^\infty_\pm$. The simplest example is again the harmonic oscillator (i.e. $V(x) = |x|^2$). Clearly, the Hermite functions do not satisfy (18) with any contants.

We now describe the main ideas of the proofs of Theorems 7–8. Recall that these results corresponds to Theorems 2.1–2.2 in [H1]. Their proofs can be found on pages 1377-1383 of this paper.

First of all, let us point out that Theorem 7 is fundamental for our further results in the papers [H1, H4] (it also naturally motivates our main questions in [H3]). This was the most laborious part of the research. The main difficulty was caused by the fact that we expected the result which said that the decay rate of $\varphi(x)$ at infinity was dominated by the intensity $\nu(x)$. Moreoever, we wanted to include the processes with exponential intensities of large jumps. As mentioned above, some rough estimates for such processes have been found by Carmona, Masters and Simon. However, the more difficult question about the exact domination of $\varphi(x)$ by $\nu(x)$ remained unanswered. Known methods, working well in the polynomial case, were not sufficient for this problem. Below we describe our idea which led us to the solution.

We discovered that it is possible to prove the following estimate: for sufficiently large $n_0 > 0$ there exists $C > 0$ such that

$$\varphi(x) \leq C \|\varphi\|_\infty \left(\int_{|y| \leq n_0} \nu(x-y) \right)^{\sum_{i=1}^{p} 2^{-i}}, \quad |x| > n_0 + 3, \quad p \in \mathbb{N}. \tag{19}$$

This is enough for the proof of Theorem 7. Indeed, the constant C is uniform not only in x, but also in p. The expected upper bound is then a consequence of taking the limit $p \to \infty$ and application of (A1.b). The inductive argument leading to (19) is based on certain *self-improving estimate*. The starting point is the inequality (16) and the decomposition of paths that was discussed in the previous chapter. Thanks to them, we can write (cf. [H1, (3.11) and the estimate above it])

$$\begin{aligned} \varphi(x) \leq e^{\lambda t} T_t \varphi(x) \leq \mathbb{E}\left[\tau_{D_n} > t; e^{-\int_0^t (V(X_s)-\lambda)ds} \varphi(X_t)\right] \\ + \sum_{k=n_0+2}^{n-2} \sum_{l=1}^{\infty} \mathbb{E}\left[S(n,k,l,t), \tau_{D_k} > t; e^{-\int_0^t (V(X_s)-\lambda)ds} \varphi(X_t)\right] \\ + \sum_{k=n_0}^{n_0+1} \sum_{l=1}^{\infty} \mathbb{E}\left[S(n,k,l,t); e^{-\int_0^t (V(X_s)-\lambda)ds} \varphi(X_t)\right], \end{aligned} \tag{20}$$

for $t > 0$, $n-1 < |x| \leq n$ and sufficiently large natural n. Recall that for $k = n_0$ and $k = n_0 + 1$ we have set $D_k = \mathbb{R}^d$ (and so $\tau_{D_{n_0}} = \tau_{D_{n_0+1}} = \infty$). This means that after getting to R_{n_0} and R_{n_0+1} the process may freely evolve in the whole space up to time t. In particular, its paths may approach the region in which V_+ is not bounded away from zero, or even the support of V_-. In consequence, the last two terms of the above sum have to be treated separately.

The proof of (19) consists of two steps. First we prove the claimed bound for $p = 1$. The condition $V(x) \to \infty$, $|x| \to \infty$, and Lemma 5 allow us to estimate effectively all expected values on the right hand side of (20). We can then sum up all the appropriate upper bounds with respect to l and k. After some rearrangements this leads to the inequality

$$\varphi(x) \leq \|\varphi\|_\infty \left(e^{-Ct}(1 + C_1) + e^{Ct} C_2 \int_{|y| \leq n_0} \nu(x-y)dy \right), \tag{21}$$

valid for all $t > 0$, $n-1 < |x| \leq n$ and sufficiently large natural n, with uniform constants. Now, taking

$$t := -\frac{1}{2C} \log \left(C_3 \int_{|y| \leq n_0} \nu(x-y)dy \right) > 0, \tag{22}$$

we get the claimed bound (19) for $p = 1$. Observe that the first term in the brackets on the right hand side of (21) contains e^{-Ct}, while the second one, which already includes the integral $\int_{|y|\leq n_0} \nu(x-y)dy$, contains e^{Ct} (the positive constant in the exponent appears here because we can not control the values of the potential in balls $B(0, n_0)$ and $B(0, n_0+1)$; it was explained above). This means that (22) is indeed the optimal choice of t in the first step of the proof.

In the second part of the proof we verify the induction step. Starting from (19) with p and using the estimate (20), we improve (19) by increasing the range of summation up to $p+1$. Here the argument is similar as before, but it is more delicate and technical. All terms on the right hand side of (20) are estimated in a similar way, with use of Lemma 5, and by taking into account the values of φ within the sets D_k. Proper summation of all terms with respect to k requires subtle rearrangements of multiple integrals that are based on the Tonelli theorem, the convolution inequality (A1.c), and the properties of the Lévy density. Similarly as above, the final bound is obtained by a proper choice of t. Our inductive procedure allows us to get the desired bound in Theorem 7 via countably many iterations of (20). The convolution inequality (A1.c), determining the class of processes with direct jump property, is of structural importance for this reasoning.

We now turn to a brief discussion of the proof of Theorem 8. Since φ satisfies (17) for some $\lambda > 0$, we may integrate the equality $e^{-\lambda t}\varphi(x) = T_t\varphi(x)$, $t > 0$, over $t \in (0, \infty)$, getting $\varphi(x) = \lambda G^V \varphi(x)$, pointwise in $x \in \mathbb{R}^d$. Now, by using the formula (14) with $f = \varphi$ and $D = B(x, 1)$, we have

$$\varphi(x) = \lambda G^V_{B(x,1)}\varphi(x) + \mathbb{E}^x \left[e^{-\int_0^{\tau_{B(x,1)}} V(X_s)ds} \varphi(X_{\tau_{B(x,1)}}) \right], \quad x \in \mathbb{R}^d, \tag{23}$$

which gives

$$|\varphi(x)| \leq \lambda G^V_{B(x,1)}|\varphi|(x) + \mathbb{E}^x \left[e^{-\int_0^{\tau_{B(x,1)}} V(X_s)ds} |\varphi(X_{\tau_{B(x,1)}})| \right] =: I + II, \quad x \in \mathbb{R}^d.$$

The term I can be first estimated from above by $\lambda G^V_{B(x,1)}\mathbf{1}(x) \sup_{y\in B(x,1)} |\varphi(y)|$, and then by the bound from Theorem 7 and the property (A1.b). The proof of the upper bound for II consists of two steps. Here we use the non-negativity of V on $B(x, 1)$, the bound in Theorem 7, the assumption (A1), the Ikeda-Watanabe formula (13), and the estimate of harmonic functions from Lemma 4.

The lower bound follows directly from the fact that φ is positive. We just omit the first term in (23) and apply the Ikeda-Watanabe formula to the second one.

Let us point out that if (A1.c) does not hold, then the ground state eigenfunction φ_0 does not satisfy the upper bound as above. This was not discussed in [H1], but it follows by the same argument as in Theorem 19 below.

III. Intrinsic contractivity-type properties and domination by φ_0

In this chapter, we still assume that $V \in \mathcal{K}_{\pm}^{\infty}$, which gives the compactness of T_t for every $t > 0$. We now present our results on intrinsic contractivity properties of Feynman–Kac semigroups associated with non-local Schrödinger operators. They include necessary and sufficient conditions for intrinsic ultra- and hypercontractivity, and some consequences of them. These results come from the second part of [H1] and from [H4].

Definitions, motivations and background. Let us recall that λ_0 and φ_0 denote the ground state eigenvalue and eigenfunction of the Schrödinger operator H, respectively, and $\{\widetilde{T}_t : t \geq 0\}$ is the corresponding ground state transformed semigroup defined in (11). Also, recall the notation $\mu(dx) = \varphi_0^2(x)dx$.

We study of the following properties.

DEFINITION **10** **(Contractivity properties).**

(i) The semigroup $\{\widetilde{T}_t : t \geq 0\}$ *is* supercontractive, *if for every* $p \in (2, \infty)$ *and* $t > 0$ *the operators* $\widetilde{T}_t : L^2(\mathbb{R}^d, \mu) \to L^p(\mathbb{R}^d, \mu)$ *are bounded. We then say that the initial semigroup* $\{T_t : t \geq 0\}$ *is* intrinsically supercontractive *(ISC in short).*

(ii) The semigroup $\{\widetilde{T}_t : t \geq 0\}$ *is* hypercontractive, *if for every* $p \in (2, \infty)$ *there exists* $t_p > 0$ *such that for every* $t \geq t_p$ *the operators* $\widetilde{T}_t : L^2(\mathbb{R}^d, \mu) \to L^p(\mathbb{R}^d, \mu)$ *are bounded. We then say that the initial semigroup* $\{T_t : t \geq 0\}$ *is* intrinsically hypercontractive *(IHC in short).*

(iii) The semigroup $\{\widetilde{T}_t : t \geq 0\}$ *is* ultracontractive, *if for every* $t > 0$ *the operators* $\widetilde{T}_t : L^2(\mathbb{R}^d, \mu) \to L^\infty(\mathbb{R}^d, \mu)$ *are bounded. We then say that the initial semigroup* $\{T_t : t \geq 0\}$ *is* intrinsically ultracontractive *(IUC in short).*

(iv) The semigroup $\{\widetilde{T}_t : t \geq 0\}$ *is* asymptotically ultracontractive, *if there exists* $t_\infty > 0$ *suh that for every* $t \geq t_\infty$ *the operators* $\widetilde{T}_t : L^2(\mathbb{R}^d, \mu) \to L^\infty(\mathbb{R}^d, \mu)$ *are bounded. We then say that the initial semigroup* $\{T_t : t \geq 0\}$ *is* asymptotically intrinsically ultracontractive *(AIUC in short).*

Observe that it is enough to assume the boundedness for some $t_p, t_\infty > 0$ in *(ii)* and *(iv)*, respectively. Indeed, thanks to the semigroup property this extends to $t \geq t_p$ and $t \geq t_\infty$. Properties *(i)-(iii)* have appeared in the literature in the context of classical differential operators. The notion of IHC was introduced by Nelson [54], and IUC has been first studied by Davies and Simon [28], Davis [26] and Bañuelos [4] (see also [5]). We have investigated AIUC for fractional Schrödinger operators in [D1]. The above properties have many important applications. It, however, should be pointed out that AIUC is equivalent with the two-sided sharp large time estimate (15) and with the uniform estimate $|\widetilde{u}(t,x,y) - 1| \leq Ce^{-(\lambda_1 - \lambda_0)t}$, $t > t_0$, which implies very strong uniform ergodicity property (see e.g. [D1, Lemma 4.1]). In light of our results, which will be presented below, it is also worth emphasising that for classical Schrödinger operators there exists a natural hierarchy between properties *(i)-(iii)*: IUC is stronger than ISC, and ISC is stronger than IHC. Moreover, AIUC is also stronger than IHC. This can be illustrated by the following example: for classical Schrödinger semigroup of $H = -\Delta + |x|^\alpha \log(1 + |x|)^\beta$, with $\alpha > 0$ and $\beta \geq 0$, it is known that [28]:

- if $\alpha > 2$ or $\alpha = 2$, $\beta > 2$, then IUC holds;
- if $\alpha = 2$, $0 < \beta \leq 2$, then ISC holds, but IUC does not hold;

- if $\alpha = 2$, $\beta = 0$ (harmonic oscillator), then IHC holds, but ISC does not hold;
- if $\alpha < 2$, then even IHC does not hold.

In addition, we verified that if $\alpha = 2$, $\beta = 0$ (harmonic oscillator), then also AIUC does not hold [H4, Example 4.4]. Let us mention that for classical Schrödinger semigroups of $H = -\Delta + V$ with potentials that have non-decreasing profiles (cf. (26)), the IUC was characterized in 2009 [3] (cf. Corollary 16 below). Properties *(i)-(iii)* have also been partly studied for non-local Schrödinger operators: IUC in [47, D3], and AIUC in [D1]. For a class of Feynman–Kac semigroups of non-homogeneous Markov processes, IUC, ISC and IHC have been investigated in a recent paper [21] via Dirichlet forms.

Our results on intrinsic contractivity properties. The chronology of this research was the following: first in [H1] we studied the case $p = \infty$ (intrinsic ultracontractivity, including its asymptotic version) for Lévy processes with DJP; then, in [H4], we observed that our results also extends to $p \in (2, \infty)$ (hyper- and supercontractivity) for the same class of processes. Note, however, that our first main theorems in [H4] apply to a much larger class of semigroups. Below we try to present the material from the papers [H1] and [H4] as a coherent entity. Additionally, for more clarity, we restrict our presentation to the case of Lévy processes with jumps.

It is crucial for our investigations that the boundedness properties of operators $\widetilde{T}_t : L^2(\mathbb{R}^d, \mu) \to L^p(\mathbb{R}^d, \mu)$ can be refined in terms of certain integrability properties of ratios $T_t\mathbf{1}/\varphi_0$.

Definition **11** ([H4, Definition 2.1] **Ground state domination properties**). *Let $p \in (2, \infty]$. We say that*

*(i) the operator T_t is L^p-*ground state dominated *(abbreviated as L^p-GSD) if*

$$\frac{T_t\mathbf{1}}{\varphi_0} \in L^p(\mathbb{R}^d, \mu), \tag{24}$$

*(ii) the semigroup $\{T_t : t \geq 0\}$ is L^p-*ground state dominated *(abbreviated as L^p-GSD) if for every $t > 0$ the operators T_t are L^p-ground state dominated,*

(iii) the semigroup $\{T_t : t \geq 0\}$ is asymptotically L^p-ground state dominated *(abbreviated as L^p-AGSD) if there exists $t_p > 0$ such that for every $t > t_p$ the operators T_t are L^p-ground state dominated. If the specific value of t_p is essential, we write (t_p, L^p)-AGSD to emphasize this.*

In our first paper [H1], we only considered the case $p = \infty$ and, therefore, we wrote there AGSD and GSD instead of L^∞-AGSD and L^∞-GSD (cf. [H1, Definition 2.3]).

The following observation, already mentioned above, was crucial in proving the characterization of intrinsic contractivity properties in our papers [H1] and [H4].

Lemma 12 ([H4, Lemma 2.1]). *Let $\{X_t\}_{t\geq 0}$ be a symmetric Lévy process with infinite Lévy measure $\nu(dx) = \nu(x)dx$ such that (A2) holds with $t_b > 0$, and let $V \in \mathcal{K}_\pm$. Suppose $p \in (2, \infty]$ and consider the following two conditions.*

(1) For some $t > 0$ the operator T_t is L^p-GSD.

(2) For some $t > 0$ the operator $\widetilde{T}_t$ is bounded from $L^2(\mathbb{R}^d, \mu)$ to $L^p(\mathbb{R}^d, \mu)$.

Then we have the following:

(i) If (1) holds for some $t = s > 0$, then (2) follows for $t = s + t_b$.

(ii) If (2) holds for some $t = s > 0$ and

$$\varphi_0^{1-\frac{1}{p-1}} \in L^1(\mathbb{R}^d, dx), \tag{25}$$

then (1) follows for $t = 2s$.

Note that [H4, Lemma 2.1] was formally formulated for $p \in (2,\infty)$ only, but the argument for $p = \infty$ is the same (in (25) we use the convention $1/\infty = 0$). The case $p = \infty$ was studied in the first paper in the series. Such an equivalence was not formulated there as a separate lemma, but it was proven and applied within the proof of [H1, Theorem 2.5]. The assumption (25) (i.e. $\varphi_0 \in L^1(\mathbb{R}^d, dx)$ was automatically satisfied there, but the estimates are general.

Further discussion of (25) is included in [H4, Remark 2.1]. It holds for a large class of semigroups associated with local and non-local Schrödinger operators.

The following characterization theorem follows directly from Lemma 12.

Theorem 13. *Under the assumptions of Lemma 12 we have the following.*

*(i) **(IHC and L^p-AGSD)** If the semigroup $\{T_t : t \geq 0\}$ is L^p-AGSD for all $p \in (2,\infty)$, then it also IHC. If the semigroup $\{T_t : t \geq 0\}$ is IHC and, in addition, $\varphi_0^\delta \in L^1(\mathbb{R}^d, dx)$, for some $\delta \in (0,1)$, then it is also L^p-AGSD for all $p \in (2,\infty)$.*

*(ii) **(ISC and L^p-GSD)** If the semigroup $\{T_t : t \geq 0\}$ is L^p-GSD for all $p \in (2,\infty)$, then it is also ISC. If the semigroup $\{T_t : t \geq 0\}$ is ISC and, in addition, $\varphi_0^\delta \in L^1(\mathbb{R}^d, dx)$, for some $\delta \in (0,1)$, then it is also L^p-GSD for all $p \in (2,\infty)$.*

*(iii) **(AIUC and L^∞-AGSD)** If the semigroup $\{T_t : t \geq 0\}$ is L^∞-AGSD, then it is also AIUC. If the semigroup $\{T_t : t \geq 0\}$ is AIUC and, in addition, $\varphi_0 \in L^1(\mathbb{R}^d, dx)$, then it is also L^∞-AGSD.*

*(iv) **(IUC and L^∞-GSD)** If the semigroup $\{T_t : t \geq 0\}$ is L^∞-GSD and $\sup_{x\in\mathbb{R}^d} p_t(x) < \infty$ for very $t > 0$, then it is also IUC. If the semigroup $\{T_t : t \geq 0\}$ is IUC and, in addition, $\varphi_0 \in L^1(\mathbb{R}^d, dx)$, then it is also L^∞-GSD.*

Assertions (i)–(ii) are versions of [H4, Theorems 2.1-2.2], formulated for Lévy processes with jumps. The proof of assertions (iii)–(iv) is the same as that of [H1, Theorem 2.5], which was obtained directly for Lévy processes with DJP. Recall that for this class of processes the condition $\varphi_0 \in L^1(\mathbb{R}^d, dx)$ automatically holds.

Note that the L^p-domination properties can be effectively studied by using probabilistic methods. This is the main advantage of the above characterizations.

We now present our necessary and sufficient conditions for L^p-GSD and L^p-AGSD obtained under framework assumptions (A1)-(A3). These are the next main results in [H1] and [H4]. The first theorem presents jointly the results from [H1, Theorems 2.6 (1) and 2.7 (1)] and [H4, Proposition 3.2].

Theorem 14 (Sufficient conditions for L^p-GSD and L^p-AGSD). *Let $\{X_t\}_{t\geq 0}$ be a symmetric Lévy process with infinite Lévy measure $\nu(dx) = \nu(x)dx$ such that (A1)-(A3) hold, and let $V \in \mathcal{K}_\pm$.*

(i) If there exist $C > 0$ and $R > 0$ such that

$$\frac{V(x)}{|\log \nu(x)|} \geq C, \quad |x| \geq R,$$

then for every $p \in (2, \infty]$ the semigroup $\{T_t : t \geq 0\}$ is (t_0, L^p)-AGSD with $t_0 = 4/C$.

(ii) If

$$\lim_{|x|\to\infty} \frac{V(x)}{|\log \nu(x)|} = \infty,$$

then for every $p \in (2, \infty]$ the semigroup $\{T_t : t \geq 0\}$ is L^p-GSD.

The next theorem presents jointly our results obtained in [H1, Theorems 2.6 (2) and 2.7 (2)] and [H4, Theorem 3.1]. Denote $V_r^*(x) = \sup_{y \in B(x,r)} V(y)$, $r > 0$.

Theorem 15 (Necessary conditions for L^p-GSD and L^p-AGSD). *Let $\{X_t\}_{t\geq 0}$ be a symmetric Lévy process with infinite Lévy measure $\nu(dx) = \nu(x)dx$ such that (A1)-(A3) hold, and let $V \in \mathcal{K}_\pm^\infty$.*

(i) If for some $p \in (2, \infty]$ the semigroup $\{T_t : t \geq 0\}$ is L^p-AGSD, then for every $r \in (0, 1)$ there exist $C > 0$ and $R > 0$ such that

$$\frac{V_r^*(x)}{|\log \nu(x)|} \geq C, \quad |x| \geq R.$$

(ii) If for some $p \in (2, \infty]$ the semigroup $\{T_t : t \geq 0\}$ is L^p-GSD, then for every $r \in (0, 1)$

$$\lim_{|x|\to\infty} \frac{V_r^*(x)}{|\log \nu(x)|} = \infty.$$

Note that the usefulness of the above conditions is mainly based on the fact that they are expressed directly in terms of the intensity of jumps $\nu(x)$ and the potential $V(x)$. Moreover, thanks to Theorem 13 our results in Theorems 14-15 in fact give necessary and sufficient conditions for all contractivity properties from Definition 10. For sufficiently regular potentials V, they also lead to a somewhat unexpected characterization results (cf. [H1, Corollary 2.3] and [H4, Theorem 3.2]). Let us define the following two subclasses of functions $V : \mathbb{R}^d \to \mathbb{R}$:

$$V \in \mathcal{V}_1 \iff \text{there exist } r \in (0,1) \text{ and } R > 0 \text{ such that } V_r^*(x) \asymp V(x), \text{ for } |x| > R;$$

$$V \in \mathcal{V}_2 \iff \begin{array}{l} \text{there exists a non-decreasing function } f : (0, \infty) \to (0, \infty) \\ \text{and } R > 0 \text{ such that } V(x) \asymp f(|x|), \text{ for } |x| > R. \end{array} \tag{26}$$

Corollary 16 ([H4, Corollary 3.2] **Equivalence of intrinsic contractivity properties).** *Let $\{X_t\}_{t\geq 0}$ be a symmetric Lévy process with infinite Lévy measure $\nu(dx) = \nu(x)dx$ such that (A1)-(A3) hold, and let $V \in \mathcal{K}_\pm^\infty \cap \mathcal{V}_1$ or $V \in \mathcal{K}_\pm^\infty \cap \mathcal{V}_2$. Suppose, in addition, that $p(t, \cdot)$ is bounded for every $t > 0$. The following conditions are equivalent:*

(i) $\lim_{|x|\to\infty} \frac{V(x)}{|\log \nu(x)|} = \infty$;

(ii) the semigroup $\{T_t : t \geq 0\}$ is L^p-GSD for all $p \in (2, \infty]$;

(iii) the semigroup $\{T_t : t \geq 0\}$ is ISC;

(iv) the semigroup $\{T_t : t \geq 0\}$ is IUC.

Corollary 17 ([H4, Corollary 3.3] **Equivalence of asymptotic intrinsic contractivity properties**). *Let $\{X_t\}_{t\geq 0}$ be a symmetric Lévy process with infinite Lévy measure $\nu(dx) = \nu(x)dx$ such that (A1)-(A3) hold, and let $V \in \mathcal{K}_{\pm}^{\infty} \cap \mathcal{V}_1$ or $V \in \mathcal{K}_{\pm}^{\infty} \cap \mathcal{V}_2$. The following conditions are equivalent:*

(i) there exist $C, R > 0$ such that $\frac{V(x)}{|\log \nu(x)|} \geq C$, for all $|x| \geq R$;

(ii) the semigroup $\{T_t : t \geq 0\}$ is L^p-AGSD for all $p \in (2, \infty]$;

(iii) the semigroup $\{T_t : t \geq 0\}$ is IHC;

(iv) the semigroup $\{T_t : t \geq 0\}$ is AIUC.

The above conclusions show that the analytic properties of semigroups related to non-local Schrödinger operators are essentially different from those of classical Schrödinger semigroups associated with the local operator $-\Delta + V$. As we have already noted at the beginning of this chapter, in the local case (even for very regular potentials) ISC is weaker than IUC, and IHC is weaker than AIUC (see the discussion on the harmonic oscillator in [H4, Example 4.4]).

Let us also mention that it makes sense to consider (at least asymptotically) the so-called *borderline* or *minimal growth of the potential* for the intrinsic contractivity properties. Our results show that this growth is described by $|\log \nu(x)|$. Moreover, our probabilistic approach to this problem allowed for a stochastic and variational interpretation of these properties and the borderline growth. Some results and the heuristic interpretation are included in [H1, Section 2.5].

An extensive list of examples of processes to which the results presented above apply was discussed in [H1, Section 4].

We now briefly discuss the proofs of Theorems 14-15. Thanks to the inclusions $L^p(\mathbb{R}^d, \mu) \subset L^\infty(\mathbb{R}^d, \mu)$, $p > 2$, in the first theorem we only need to show both implications for $p = \infty$ (i.e. L^∞-(A)GSD implies L^p-(A)GSD for all $p > 2$). For $p = \infty$ these are exactly Theorems 2.6 i 2.7 (1) in [H1]. We have to show that the assumption on $V/|\log \nu|$ implies the estimate $T_t\mathbf{1}(x) \leq C\varphi_0(x)$, $x \in \mathbb{R}^d$, for sufficiently large (implication i) or all (implication ii) $t > 0$, with a constant C uniform in x. Since $T_t\mathbf{1}$ is bounded and φ_0 is continuous and strictly positive, it is enough to show such a bound for large x only. The key observation, based on an application of the two-sided estimate for φ_0 in Corollary 9 and [H1, Lemma 3.7], is that in fact we only need to show a weaker bound $T_t\mathbf{1}(x) \leq C\nu(x)$.

Theorem 18 ([H1, Theorem 3.1]). *Let $\{X_t\}_{t\geq 0}$ be a symmetric Lévy process with infinite Lévy measure $\nu(dx) = \nu(x)dx$ such that (A1)-(A3) hold, and let $V \in \mathcal{K}_{\pm}$. If there exist $C > 0$ and $R > 0$ such that*

$$\frac{V(x)}{|\log \nu(x)|} \geq C, \quad |x| \geq R, \tag{27}$$

then for every $t \geq t_0 := 4/C$ there exist $\widetilde{C}, \widetilde{R} > 0$ such that

$$T_t\mathbf{1}(x) \leq \widetilde{C}\nu(x), \quad |x| \geq \widetilde{R}. \tag{28}$$

If, moreover,

$$\lim_{|x|\to\infty} \frac{V(x)}{|\log \nu(x)|} = \infty,$$

then (28) *holds for all* $t > 0$.

The proof of the above theorem (see [H1, p. 1385-1386]) uses a similar argument as in the proof of Theorem 7. The main difference is that now the function under the expectation is equal to one and we need to get more information about the decay from the functional $\exp(-\int_0^t V(X_s)ds)$.

The proof of Theorem 15 is based on a direct lower estimate for the expectation of the Feynman–Kac functional for fixed $t > 0$. Again, the argument uses in an essential way the sharp estimates of φ_0 from Corollary 9. For $p = \infty$ this is the proof of the implication (2) of Theorems 2.6 and 2.7 in [H1] (p. 1387), and for $p \in (2, \infty)$ the proof of Theorem 3.1 in [H4] (p. 179-180).

It should be emphasized that the proofs of Theorems 14-15 could not be carried out without sharp two-sided estimate of the ground state eigenfunction presented in the previous chapter.

IV. Fall-off of eigenfunctions for $V \in \mathcal{K}_{\pm}^0$

In this chapter we will discuss pointwise bounds for eigenfunctions of non-local Schrödinger operators with decaying potentials which correspond to negative eigenvalues. These results were obtained in the paper [H3].

Motivations and background. It is known that in the case of classical Schrödinger operators $-\Delta + V$ with Kato-decomposable potentials such that $V(x) \to 0$ as $|x| \to \infty$, the decay rate at infinity for eigenfunctions corresponding to negative eigenvalues is at least exponential (see e.g. [65, Theorems C.3.4-C.3.5]). The problem of localization of such eigenfunctions for fractional and relativistic Schrödinger operators has been addressed in the aforementioned work of Carmona, Masters and Simon [20]. They showed that for fractional Schrödinger operators $H = (-\Delta)^{\alpha/2} + V$, $\alpha \in (0, 2)$, the decay rate at infinity of an eigenfuntion φ corresponding to a negative eigenvalue λ is dominated by the intensity of jumps of the isotropic stable process, i.e. $|\varphi(x)| \leq C(1 \wedge |x|^{-d-\alpha})$ (moreover, for the ground state one has $\varphi_0(x) \asymp 1 \wedge |x|^{-d-\alpha}$). Their results for relativistic Schrödinger operators $H = \sqrt{-\Delta + m^2} - m + V$, $m > 0$, are less sharp, but lead to the following surprising observation: when $|\lambda| \geq m$, then the rate of decay at infinity of eigenfunction $\varphi(x)$ corresponding to λ is of the order $e^{-m|x|}$, while for $|\lambda| < m$ it is of order $e^{-\sqrt{2m|\lambda| - \lambda^2}\,|x|}$ (recall that the intensity of large jumps for the relativistic process is comparable with the function $e^{-m|x|}|x|^{-(d+2)/2}$). This means that for $|\lambda| < m$ the decay rate of φ at infinity is still exponential, but it depends on $|\lambda|$ and is essentially slower than the decay rate of the intensity of jumps of the process. Note that such a dichotomy does not occur in the fractional case, where the size of $|\lambda|$ does not affect the fall-off rate of φ.

One of the main motivations for our study was to understand this dichotomy.

Our results. In the paper [H3] we wanted to understand what properties of the process generated by L and the potential V guarantee that the decay rates of eigenfunctions of the operator $H = -L + V$ are dominated by the intensity of jumps, ν, at infinity. Working within the class of processes with DJP, we obtained sufficient conditions for this property. However, first of all, we proved that if φ is positive, then it automatically satisfies the analogous lower estimate.

Theorem 19 ([H3, Theorem 4.1]). *Let $\{X_t\}_{t \geq 0}$ be a symmetric Lévy process with infinite Lévy measure $\nu(dx) = \nu(x)dx$ such that (A1.a) holds, and let $V \in \mathcal{K}_{\pm}^0$. Suppose $\varphi \in L^2(\mathbb{R}^d)$ is a positive eigenfunction of the operator H corresponding to eigenvalue $\lambda \in \mathbb{R}$.*

(1) If (A1.b) is satisfied, then there exist $C, R > 0$ (depending on λ) such that

$$\varphi(x) \geq C\,\nu(x), \quad |x| \geq R.$$

(2) Consider the following two disjoint cases:

(i) (A1.b) holds, and (A1.c) does not hold,

(ii) (A1.b) does not hold (and hence also (A1.c) does not hold).

Then in either of cases (i) and (ii) we have

$$\limsup_{|x| \to \infty} \frac{\varphi(x)}{\nu(x)} = \infty.$$

The second part of the theorem states that outside of the class of processes with DJP the domination property in question cannot be expected. It is therefore an optimal assumption for our research. The above result is a direct consequence of the theorem describing analogous properties for the (X,η)-harmonic functions [H3, Theorem 3.1]. We obtained this theorem by using the Ikeda-Watanabe formula and some direct estimates based on the monotonicity of the profile of Lévy density.

The proofs of our further results in [H3], giving the upper bounds for eigenfunctions, are much more technically complicated. In particular, they require subtler methods than those discovered in [H1], where we dealt with the confining potentials. For a potential V decaying to zero the perturbed processes behave far out like free processes. Now, there is no longer a strong killing effect like in the confining case, which helps very much in proving estimates. The contribution of both the process and the potential in the fall-off rates of eigenfunctions is now more subtle. The effect of the potential appears in the relative position of the corresponding eigenvalue from the edge of the continuous spectrum of H.

The following result states that if $|\lambda|$ is large enough with respect to the initial Lévy process, then the decay rate of φ at infinity is dominated by ν.

Theorem 20 ([H3, Theorem 4.2]). *Let $\{X_t\}_{t\geq 0}$ be a symmetric Lévy process with infinite Lévy measure $\nu(dx) = \nu(x)dx$ such that (A1)-(A3) hold, and let $V \in \mathcal{K}^0_{\pm}$. Then there exists η_0 (depending on process and independent of V) such that if $\varphi \in L^2(\mathbb{R}^d)$ is an eigenfunction of H with eigenvalue $\lambda \in (-\infty, -\eta_0)$, then there are $C, R > 0$ (depending on process and λ) satisfying*

$$|\varphi(x)| \leq C \left\|\varphi\right\|_\infty \nu(x), \quad |x| \geq R.$$

The constant η_0 (given by [H3, (4.6)]) can be estimated for some examples of Lévy processes, but it is not optimal for the above result. However, let us emphasize that this result is very general – it applies to arbitrary process with DJP.

We now formulate our next two results. They provide sufficient conditions on the initial Lévy process under which the above domination property holds for any $\lambda < 0$. These results follow directly from the two different theorems describing the behavior at infinity of (X,η)-harmonic functions. In our paper, they are formulated as a single result (see [H3, Theorem 4.3]). However, in this summary, for clarity, we will present them separately. Note that these results require a stronger version of the assumption (A3). In the whole paper [H3] the condition (A3) is formulated for all $R > 0$, but for the proofs of these results we have to additionally assume that there is a constant $C > 0$ such that

$$\sup_{\substack{x,y\in B(0,s)\\ |x-y|\geq s/8}} G_{B(0,s)}(x,y) \leq \frac{C}{\Psi(1/s)s^d}, \quad s \geq 1, \tag{29}$$

where $\Psi(r) := \sup_{|\xi|\leq r} \psi(\xi)$ (see [H3, (2.19)]). The role of this condition is just technical. The potential theory of Markov processes with jumps has undergone rapid development in recent years and it now clear that such a condition is satisfied for a large class of Lévy processes (e.g. for isotropic unimodal ones [36, Theorems 1.2-1.3]). In our settings, it is known that if there are constants $C_1, C_2 > 0$ such that

$$\sup_{|x|\geq r} p_t(x) \leq C_1 t \frac{\Psi(1/r)}{r^d}, \quad t > 0, \quad r \geq 1, \tag{30}$$

and

$$p_t(0) = \int_{\mathbb{R}^d} e^{-t\psi(z)}dz \leq C_2 \left(\Psi_*^{-1}\left(\frac{1}{t}\right)\right)^d, \quad t \geq 1, \tag{31}$$

where $\Psi_*^{-1}(s) = \inf\{r \geq 0 : \Psi(r) = s\}$, $s \geq 0$, then (29) holds [H3, Lemma 2.2].

Our first result is concerned with Lévy processes whose jump intensities satisfy the following condition:

$$\begin{array}{c} \text{there exists } \; C > 0 \; \text{ such that for every } \; r \geq 1 \text{ we have} \\ \nu(x-y) \leq C\nu(x), \; \text{ whenever } \; |y| \leq r \; \text{ and } \; |x| \geq 2r. \end{array} \tag{32}$$

One can check that under (A1.a) the condition (32) automatically holds if we know that the profile g has a *doubling* property, i.e. there exists $\widetilde{C} > 1$ such that $g(r/2) \leq \widetilde{C} g(r)$, $r \geq 1$. It is then a typical property of the densities polynomially decaying at infinity.

Theorem 21 ([H3, Theorem 4.3 (1)]). *Let $\{X_t\}_{t\geq 0}$ be a symmetric Lévy process with infinite Lévy measure $\nu(dx) = \nu(x)dx$ such that (A1)-(A3) (for $R > 0$) hold, and let $V \in \mathcal{K}_{\pm}^0$. If the conditions* (29) *and* (32) *are satisfied, and $\varphi \in L^2(\mathbb{R}^d)$ is an eigenfunction of H with eigenvalue $\lambda < 0$, then there are $C, R > 0$ (depending on process and λ) such that*

$$|\varphi(x)| \leq C \|\varphi\|_\infty \nu(x), \quad |x| \geq R.$$

When the tail of the process is lighter than polynomial (in particular, (32) does not hold), then the analysis of the fall-off of eigenfunctions φ of H for small $|\lambda|$ is even more complicated. Our next result is about the case

$$\int_{\mathbb{R}^d} |x|^2 \nu(dx) < \infty \tag{33}$$

and it requires more regularity of the transition density:

$$\begin{array}{c} \text{there exist } C \geq 1 \text{ and } R > 0 \text{ such that } p(t,x) \leq Cp(t,y), \\ \text{for all } t > 0 \text{ and } |x| \geq |y| \geq R \text{ with } |x-y| \leq 1. \end{array} \tag{34}$$

Under (33) we have $\Psi(r) \asymp r^2$, for r close to 0. Then the condition (31) holds and (29) reduces to (30).

The sufficient condition in our theorem is expressed in terms of two functions, which describe the system of large jumps of the process: for $1 \leq s_1 < s_2 < s_3 \leq \infty$ we define

$$K_1^X(s_1) := \sup_{|x|\geq s_1} \frac{\int_{|x-y|>s_1, |y|>s_1} \nu(x-y)\nu(y)dy}{\nu(x)} \tag{35}$$

and

$$K_2^X(s_1, s_2, s_3) := \inf\{C \geq 1 : \nu(x-y) \leq C\,\nu(x), \; |y| \leq s_1, \; s_2 \leq |x| < s_3\}. \tag{36}$$

We assume that

$$\text{there exists } \; \kappa_1 \geq 2 \; \text{ such that } \; \lim_{s\to\infty} K_1^X(\kappa_1 s)\, K_2^X(s, \kappa_1 s, \infty) = 0 \tag{37}$$

and

$$\begin{array}{c} \text{there exists } \; \kappa_2 < \infty \; \text{ such that for every } \; s_1 \geq 1 \; \text{ we have} \\ \limsup_{s\to\infty} K_2^X(s_1, s, \infty) \leq \kappa_2. \end{array} \tag{38}$$

For a discussion of the properties of functions K_1^X and K_2^X and the heuristic interpretation of the both conditions (37) and (38) we refer the reader to [H3, Section 2.2]. Let us also mention that the function $K_1^X(s)$ turned out to be the key tool in our recent paper [P9], where the spatial asymptotics at infinity for heat kernels of non-local operators were studied.

Theorem 22 ([H3, Theorem 4.3 (2)]). *Let $\{X_t\}_{t\geq 0}$ be a symmetric Lévy process with infinite Lévy measure $\nu(dx) = \nu(x)dx$ such that (A1)-(A3) (for $R > 0$) and* (30), (33)-(34) *hold, and let $V \in \mathcal{K}_{\pm}^0$. If the conditions* (37)-(38) *are satisfied, and $\varphi \in L^2(\mathbb{R}^d)$ is an eigenfunction of H with eigenvalue $\lambda < 0$, then there are $C, R > 0$ (depending on process and λ) such that*

$$|\varphi(x)| \leq C \left\|\varphi\right\|_{\infty} \nu(x), \quad |x| \geq R.$$

In [H3, Section 4.3] we gave a more detailed discussion of the estimates for various types of jump intensities, and we amply illustrated them with examples. These examples shows that when the intensity of large jumps of the process is sub-exponential, then the eigenfunctions of H corresponding to negative eigenvalues are dominated at infinity by this intensity (see [H3, Corollaries 4.1-4.2 and Remarks 4.2-4.3]). On the other hand, when the order of the decay of ν at infinity is exponential or faster, this domination is broken and a *qualitative transition* in the fall-off rates can be observed. For exponential Lévy densities the following dichotomy occurs. If λ is sufficiently low-lying (with respect to the process), then the fall-off of the corresponding eigenfunction is dominated by ν, while for bottom eigenvalues which are closer to zero (i.e., to the edge of the essential spectrum of H), the fall-off gets much slower, with essential contribution of the eigenvalue into the rate (see also the heuristic description of the mechanism of this dichotomy in [H3, p. 651, second paragraph]). The relevance of the assumptions (37)–(38) in Theorem 21 has been tested in [H3, Proposition 4.2] for certain class of processes.

We now briefly discuss the proofs of the upper bounds presented above. We always use [H3, (4.5)] and reduce the proof to an upper estimate of a certain (X, η)-harmonic function defined in [H3, (3.13)], for some $\eta \in (0, |\lambda|)$. However, the methods worked out in [H1] turned out to be insufficient in the present case. Our argument in [H3] is then based on a new idea, which led us to Lemma 23 formulated below.

Recall that the functions K_1^X, K_2^X were defined in (35)-(36). For $s_1 \geq 1$ and $s_2 \geq 2s_1$ we define

$$h_1(X, s_1, s_2) = K_2^X(s_1, s_2, \infty) \left[C\left(X, \frac{s_1}{16}\right)\left(\widetilde{C}(X, s_1)\left|B(0, s_1)\right| + \mathbb{E}^0[\tau_{B(0,2s_1)}]\right) + 1\right],$$

$$h_2(X, s_1) = C\left(X, \frac{s_1}{16}\right)\left[C\left(X, s_1\right)\widetilde{C}(X, s_1) + \mathbb{E}^0[\tau_{B(0,2s_1)}] \sup_{|y|\geq \frac{s_1}{4}} \nu(y)\right] + \sup_{|y|\geq \frac{s_1}{16}} \nu(y),$$

where

$$C(X, s) = \inf_{f \in \mathcal{B}} \left\|Lf_s\right\|_{\infty}, \qquad f_s(x) = f(x/s), \qquad s > 0,$$

$$\mathcal{B} = \left\{f \in C^2(\mathbb{R}^d) : f(x) = 1 \text{ for } x \in B(0, 1/2), f(x) = 0 \text{ for } x \in B(0,1)^c \text{ and } 0 \leq f \leq 1\right\},$$

and

$$\widetilde{C}(X, s_1) := \sup_{\substack{x,y \in B(0,s_1) \\ |x-y| \geq s_1/8}} G_{B(0,s_1)}(x, y) + \frac{\mathbb{E}^0[\tau_{B(0,2s_1)}]}{\left|B(0, \frac{s_1}{4})\right|}\left(K_2^X\left(\frac{s_1}{4}, \frac{s_1}{2}, s_1\right)\right)^2.$$

Also, denote for a moment by C_1 the constant appearing in (A1.a).

Lemma 23 ([H3, Lemma 3.2]). *Let $\{X_t\}_{t\geq 0}$ be a symmetric Lévy process with infinite Lévy measure $\nu(dx) = \nu(x)dx$ such that (A1)-(A3) (for $R > 0$). Moreover, suppose there are $r_1 \geq 1$, $r_2 \geq 2r_1$ and $r_3 > r_2$ such that*

$$2\, C_1^4\, h_1(X, r_1, r_2)\, K_1^X(r_2) + h_2(X, r_1) \left|B(0, r_2)\right| K_2^X(r_2, r_3, \infty) < \eta. \tag{39}$$

Then for every bounded and non-negative function f which is (X, η)-harmonic in $\overline{B}(0,r)^c$ for some $r > 0$, there exist $C_2, R > 0$ (depending on the process and η) such that

$$f(x) \leq C_2 \|f\|_\infty \nu(x), \quad |x| \geq R.$$

The structure of the condition (39) is determined by the estimate of (X, η)-harmonic functions in [H3, Lemma 3.1], which is a version of the upper bound in [15, Lemma 3.2 (b)]. This version is specialized for use in the case of a process whose path are being killed with the intensity given by the potential V satisfying $V(x) \to 0$ as $|x| \to \infty$. The most technical step of our research was to understand this structure and to prove [H3, Lemmas 3.1-3.2]. Note that the paper [42], dealing with a large class of Feller processes, also contains some estimates for functions harmonic at infinity. However, the assumptions in this paper exclude the processes with light tails and therefore we could not apply these result in our case.

Theorem 20 follows directly from Lemma 23, while the proofs of Theorems 21 and 22 require some preparation. The intermediate steps are [H3, Theorem 3.2] and [H3, Theorem 3.3], respectively. The proof of [H3, Theorem 3.3] is the second most technical part of the paper [H3]. An effective application of Lemma 23 in the proof of this theorem requires some regularization of the density of the Lévy measure.

At the end of this chapter, let us mention that Lemma 23 can also be used to obtain similar results for the class of confining potentials. This is discussed in [H3, Section 4.5].

V. Heat kernel estimates

In this chapter, we present the results obtained in the paper [H2]. These are sharp two-sided estimates of the transition densities for Lévy processes with DJP in finite time horizon. Note that this paper was written as the second one in the series of articles presented in this summary. It is not directly concerned with the properties of the Feynman–Kac semigroups, but the motivations for this study (and also the hope to get such results) came directly from the paper [H1]. The direct jump property proved to be a structural assumption in [H1] and our further studies of Feynman-Kac semigroups required some general systematic knowledge about the properties of such processes. For instance, the results from our paper [H2] were important for further consideration in [H3], especially in the case of Lévy measures with finite second moment. They allowed us to check that the assumption (A1) together with some additional information about the behavior of the profile g near the origin imply the other assumptions (A2)–(A3) and (29), (34), even if the process is non-isotropic [H3, Proposition 3.2].

In [H2] we deal with a slightly different class of Lévy processes than that in [H1], [H3] and [H4]: we assume that $A \equiv 0$ and $b \in \mathbb{R}^d$ is an arbitrary vector. This means that the Lévy-Khintchine exponent of the process $\{X_t\}_{t\geq 0}$ takes the form (cf. (1))

$$\psi(\xi) = -i\xi \cdot b + \int \left(1 - e^{i\xi\cdot y} + i\xi \cdot y \mathbb{1}_{B(0,1)}(y)\right) \nu(dy), \quad \xi \in \mathbb{R}^d. \tag{40}$$

We always assume that $\nu(\mathbb{R}^d) = \infty$. Our most general upper estimate allows for irregular Lévy measures ν, but the strongest characterization result, which will be presented first, requires ν to be symmetric and absolutely continuous with respect to Lebesgue measure.

Denote

$$\Psi(r) = \sup_{|\xi|\leq r} \Re\psi(\xi), \quad r > 0.$$

We note that Ψ is continuous and non-decreasing, and we also have $\sup_{r>0} \Psi(r) = \infty$, since $\nu(\mathbb{R}^d) = \infty$. Moreover, let

$$\Psi^{-1}(s) = \sup\{r > 0 : \Psi(r) = s\} \quad \text{for } s > 0$$

so that $\Psi(\Psi^{-1}(s)) = s$ for $s \in (0, \infty)$ and $\Psi^{-1}(\Psi(s)) \geq s$ for $s > 0$. To shorten the notation below, we set

$$h(t) := \frac{1}{\Psi^{-1}\left(\frac{1}{t}\right)}, \qquad b_r := \begin{cases} b - \int_{r\leq|y|<1} y\,\nu(dy) & \text{if} \quad r < 1, \\ b & \text{if} \quad r = 1, \\ b + \int_{1\leq|y|<r} y\,\nu(dy) & \text{if} \quad r > 1 \end{cases} \tag{41}$$

and $\nu_r(x) = \nu(x)\mathbf{1}_{\{|x|\geq r\}}$, $r > 0$.

The following theorem characterizes certain type of estimates of the kernels $p_t(x)$ for symmetric and absolutely continuous Lévy measures that have non-increasing profiles.

Theorem 24 ([H2, Theorem 1]). *Let $\nu(dx) = \nu(x)dx$ be a Lévy measure such that $\nu(\mathbb{R}^d) = \infty$, $\nu(y) = \nu(-y)$ and $\nu(x) \asymp f(|x|)$, $x \in \mathbb{R}^d \setminus \{0\}$, for some non-increasing profile $f : (0, \infty) \to (0, \infty)$. Assume, moreover, that $b \in \mathbb{R}^d$. Then the following conditions* (1) *and* (2) *are equivalent.*

(1) *There exists $r_0 > 0$ and the constants $C, \widetilde{C} > 0$ such that*

(a) $\nu_{r_0} * \nu_{r_0}(x) \le C\nu(x)$, $|x| \ge 2r_0$,

(b) $\Psi(1/|x|) \le \widetilde{C}|x|^d\nu(x)$, $0 < |x| \le 2r_0$.

(2) *There exist* $t_0, \theta > 0$ *and constants* $C_1 - C_4$ *such that for every* $t \in (0, t_0]$ *we have*

$$C_1\,[h(t)]^{-d} \le p_t(x+tb) \le C_3\,[h(t)]^{-d}, \quad t \in (0,t_0], \quad |x| \le \theta h(t),$$

and

$$C_2\,t\,\nu(x) \le p_t(x+tb) \le C_4\,t\,\nu(x), \quad t \in (0,t_0], \quad |x| \ge \theta h(t).$$

The above result gives sharp two-sided small-time estimates for a wide subclass of semigroups of Lévy processes with DJP, which was studied in the series of articles constituting this scientific achievement. Recall that this class includes processes with exponential tails, like relativistic and tempered stable processes [H2, Example 2]. This is also a characterization result, which shows that outside the class of processes with DJP the behavior of $p_t(x)$ is different [H2, Example 5].

It should be emphasized that proving sharp estimates and asymptotic expansions for heat kernels is one of the basic and challenging problems in the modern theory of non-local operators and their evolution semigroups. In recent years, a lot of strong and useful estimates have been obtained [12, 22, 43, 44, 45, 53]. An important feature of our characterization presented above is that it also applies to a large subclass of processes with finite second moments (in this case $|\Re\psi(\xi)| \le C|\xi|^2$ for small ξ). Let us emphasize that this class of processes has been only partly explored so far.

Note that conditions (1.a) and (1.b) refer to disjoint ranges of x. The proof of the above theorem also consists of two parts, for small and for large $|x|$, respectively (see [H2, Theorems 2-3]). Our main contribution was the observation that the uniform comparability $p_t(x+tb) \asymp t\nu(x)$ for small t and large x is in fact equivalent with the condition (1.b), which is exactly the direct jump property (cf. (A1.c)). This was proven in [H2, Theorem 3] under some mild regularity assumption on the behaviour of $\Re\psi$ at infinity (see the condition **(E)** below). Here the key step was to show that DJP implies the upper bound $p_t(x+tb) \le Ct\nu(x)$. In fact, we obtained a much more general version of this result [H2, Theorem 4]. It allows for non-symmetric and strongly singular Lévy measures. We first formulate the assumptions of this theorem.

(E) There exist $C > 0$ and $t_p > 0$ such that

$$\int_{\mathbb{R}^d} e^{-t\Re\psi(\xi)}|\xi|\,d\xi \le C\,[h(t)]^{-d-1}, \quad t \in (0,t_p].$$

(D) There exist a non-increasing profile function $f : (0,\infty) \to (0,\infty)$, $\gamma \in [0,d]$ and a constant $C > 0$ such that

$$\nu(A) \le Cf(\mathrm{dist}(A,0))(\mathrm{diam}(A))^{\gamma},$$

for every Borel set $A \subset \mathbb{R}^d$ with $\mathrm{dist}(A,0) > 0$.

Here $\mathrm{diam}(A)$ denotes the diameter of the set A, and $\mathrm{dist}(A,0)$ its distance to 0. Our last assumption generalizes DJP (see [H2, Lemma 3]).

(C) There exist constants $C, \widetilde{C} > 0$ and $r_0 > 0$ such that for every $|x| \geq 2r_0$ and $r \in (0, r_0]$ we have

$$\int_{|x-y|>r_0,\ |y|>r} f(|y-x|)\,\nu(dy) \leq C\,\Psi(1/r)\,f(|x|) \quad \text{and} \quad f(r) \leq \widetilde{C}\Psi(1/r)r^{-\gamma},$$

with f and γ taken from **(D)**.

Theorem 25 ([H2, Theorem 4]). *Let ν be a Lévy measure such that $\nu(\mathbb{R}^d) = \infty$ and let the assumptions **(E)**, **(D)** and **(C)** be satisfied for some $t_p > 0$, the profile f, the parameter γ and some $r_0 > 0$. Then there is a constant $C > 0$ such that*

$$p_t(x + tb_{h(t)}) \leq C\,t\,[h(t)]^{\gamma-d} f(|x|), \quad |x| > 4r_0,\ t \in (0, t_0],$$

where $t_0 := t_p \wedge \frac{1}{\Psi(1/r_0)}$.

This result applies directly to convolution semigroups built on the so-called product and discrete Lévy measures [H2, Examples 2-3].

The proof of Theorem 25 uses some general method, which is based on estimates for compound Poisson semigroups of the form

$$\bar{P}_t^r = e^{-t|\bar{\nu}_r|} \sum_{n=0}^{\infty} \frac{t^n \bar{\nu}_r^{n*}}{n!}, \quad t, r > 0,$$

where $\bar{\nu}_r(dy) = \mathbf{1}_{B(0,r)^c}(y)\,\nu(dy)$ [H2, Lemma 4]. This approach was proposed in the papers of Bogdan and Sztonyk [16] and Sztonyk [68]. In our paper [H2], we work with a very general class of processes. In particular, we do not assume that the Lévy measures have the doubling properties. It should be emphasized that proving sharp estimates in our settings required an essential modification of this method.

The main step was the following lemma, which gives sharp upper estimates for the n-fold convolutions of restricted Lévy measures.

Lemma 26 ([H2, Lemma 2]). *Let ν be a Lévy measure such that $\nu(\mathbb{R}^d) = \infty$ and let the assumptions **(D)** and **(C)** be satisfied for some profile f, the parameter γ and some $r_0 > 0$. Then the following hold.*

(a) There is a constant $C = C(r_0)$ such that

$$\int_{|x-y|>r_0} f(|y-x|)\bar{\nu}_r^{n*}(dy) \leq (C\Psi(1/r))^n f(|x|), \quad |x| \geq 3r_0, \quad r \in (0, r_0],\ n \in \mathbb{N}. \tag{42}$$

(b) For every bounded Borel set $A \subset \mathbb{R}^d$ such that $\operatorname{dist}(A, 0) \geq 3r_0$ we have

$$\bar{\nu}_r^{n*}(A) \leq C^n [\Psi(1/r)]^{n-1} f(\operatorname{dist}(A,0))\ (\operatorname{diam}(A))^{\gamma}, \quad r \in (0, r_0],\ n \in \mathbb{N}, \tag{43}$$

with a constant $C = C(r_0, \lceil \operatorname{diam}(A)/r_0 \rceil)$.

The *sharpness* of the above upper estimates should be understood as follows: if the conditions **(C)** and **(D)** are satisfied for some profile function f, then the appropriate bounds hold for convolutions of restricted measures with exactly the same profile f (i.e. the dilatations of f are not enough).

In the proof of Theorem 25 we also needed the observations from [H2, Lemma 1].

The proof of Lemma 26 is inductive. The estimate (a) is a natural extension of **(C)**. It follows from the fact that the first inequality in **(C)** can be effectively interated under the assumption **(D)**. On the other hand, the estimate (b) is an extension of **(D)**. The argument here is more subtle: we have to check the induction step for $\mathrm{dist}(A, 0) \geq 3r_0 - r_0/2^n$ instead of $\mathrm{dist}(A, 0) \geq 3r_0$. We then see that our Lemma 26 extends both assumptions **(C)** and **(D)** to arbitrary convolutions of restricted Lévy measures.

5. Description of other scientific achievements

Besides the four papers, which constitute mono-thematic series of publications, after Ph.D., I published ten articles, one is accepted and is waiting for publication, another two were submitted to journals. Total number of my papers is 21, the number of citations, according to the Web of Science database ('Sum of the Times Cited' on 2019-01-03), is 91 (62 without self-citations), and the h-index (Hirsh index) is 6. Total *impact factor* of the journals for four publications included in the *scientific achievement*, according to the Journal Citation Reports, is 4,022; total *impact factor* of the journals for all publications is 18,577, see Table 1.

Table 1: Impact factor of the journals according to Journal Citation Report from the year of publication (or 2018 for publications from 2017)

article	journal	publication year	impact factor
[H1]	Annals Probab.	2015	1,734
[H2]	J. Anal. Math.	2017	0,592
[H3]	Potential Anal.	2017	0,852
[H4]	J. Spectr. Th.	2018	0,844
[P1]	J. Evol. Eq.	2013	0,643
[P2]	Rev. Math. Phys.	2013	1,448
[P3]	CAIM	2014	–
[P4]	Stoch. Proc. Appl.	2015	1,193
[P5]	J. Math. Anal. Appl.	2015	1,014
[P6]	Phys. Rev. E	2017	2,284
[P7]	J. Math. Anal. Appl.	2016	1,064
[P8]	Stoch. Proc. Appl.	2018	1,051
[P9]	Trans. Amer. Math. Soc.	2018	1,496
[P10]	Potential Anal.	2018	0,852
[P11]	Commun. Contemp. Math.	2018	1,155 [2]
[D1]	Stoch. Proc. Appl.	2012	0,953
[D2]	Studia Math.	2012	0,549
[D3]	Potential Anal.	2010	0,853
[M1]	Prob. Math. Stat.	2010	–
		Sum:	18,577

After PhD I published the following papers:

[P1] K. Kaleta, P. Sztonyk, *Upper estimates of transition densities for stable dominated semigroups*, Journal of Evolution Equations 13 (3), 633-650 (2013)

[P2] K. Kaleta, M. Kwaśnicki, J. Małecki, *One-dimensional quasi-relativistic particle in the box*, Reviews in Mathematical Physics 25 (8), 1350014 (2013)

[P3] J. Lőrinczi, K. Kaleta, S.O. Durugo, *Spectral and analytic properties of non-local Schrödinger operators and related jump processes*, Communications in Applied and Industrial Mathematics 6 (2), 534 (2014)

[2] article [P11] is waiting for publication since November 2018

[P4] K. Kaleta, K. Pietruska-Pałuba, *Integrated density of states for Poisson-Schrödinger perturbations of subordinate Brownian motions on the Sierpiński gasket*, Stochastic Processes and their Applications 125 (4), 1244-1281 (2015)

[P5] K. Kaleta, P. Sztonyk, *Estimates of transition densities and their derivatives for jump Lévy processes*, Journal of Mathematical Analysis and Applications 431 (1), 260-282 (2015)

[P6] K. Kaleta, J. Lőrinczi, *Transition in the decay rates of stationary distributions of Lévy motion in an energy landscape*, Physical Review E 93, 022135 (2016)

[P7] K. Kaleta, M. Kwaśnicki, J. Małecki, *Asymptotic estimate of eigenvalues of pseudo-differential operators in an interval*, Journal of Mathematical Analysis and Applications 439 (2), 896-924 (2016)

[P8] K. Kaleta, K. Pietruska-Pałuba, *Lifschitz singularity for subordinate Brownian motions in presence of the Poissonian potential on the Sierpiński gasket*, Stochastic Processes and their Applications 128 (11), 3897-3939 (2018)

[P9] K. Kaleta, P. Sztonyk, *Spatial asymptotics at infinity for heat kernels of pseudo-differential operators*, Transactions of the American Mathematical Society, published online, https://doi.org/10.1090/tran/7538

[P10] K. Kaleta, K. Pietruska-Pałuba, *The quenched asymptotics for non-local Schrödinger operators with Poissonian potentials*, Potential Analysis, published online, https://doi.org/10.1007/s11118-018-9747-x

[P11] K. Kaleta, J. Lőrinczi, *Typical long time behaviour of ground state transformed jump processes*, Communications in Contemporary Mathematics, to appear (2018), text available at arXiv:1806.10657

[Pre1] K. Kaleta, J. Lőrinczi, *Zero-energy bound state decay for non-local Schrödinger operators*, 1-35, submitted (2018), text available at arXiv:1804.04245

[Pre2] K. Kaleta, M. Olszewski, K. Pietruska-Pałuba, *Reflected Brownian motion on simple nested fractals*, 1-37, submitted (2018), text available at arXiv:1804.04228

Before PhD I published the following four papers, which will not be discussed here:

[D1] K. Kaleta, J. Lőrinczi, *Fractional $P(\phi)_1$-processes and Gibbs measures*, Stochastic Processes and their Applications 122 (10), 3580-3617 (2012)

[D2] K. Kaleta, *Spectral gap lower bound for the one-dimensional fractional Schrödinger operator in the interval*, Studia Mathematica 209, 267-287 (2012)

[D3] K. Kaleta, T. Kulczycki, *Intrinsic ultracontractivity for Schrödinger operators based on fractional Laplacians*, Potential Analysis 33 (4), 313-339 (2010)

[M1] K. Kaleta, M. Kwaśnicki, *Boundary Harnack inequality for α-harmonic functions on the Sierpiński triangle*, Probability and Mathematical Statistics 30 (2), 353-368 (2010)

I will now discuss the results obtained in the papers [P1]-[P11] and [Pre1]-[Pre2].

Spatial asymptotics of heat kernels at infinity

In the paper [P9] we have investigated the spatial asymptotic at infinity for heat kernels of homogeneous non-local pseudo-differential operators L of the form (3). More precisely, we gave sufficient conditions under which the limits $\lim_{r\to\infty} \frac{p_t(r\theta-y)}{t\,\nu(r\theta)}$, $t \in T$, $\theta \in E$, $y \in \mathbb{R}^d$, can be computed. Here ν is the corresponding Lévy density, $T \subset (0,\infty)$ is a bounded time-set, and E is a subset of the unit sphere $\mathbb{S}^{d-1}$, $d \geq 1$. Denote by $\Gamma_E := \{y : y/|y| \in E\}$ a generalized cone based on the set $E \subset \mathbb{S}^{d-1}$.

We assume that $A \equiv 0$ or $\inf_{|\xi|=1} \xi \cdot A\xi > 0$, $\nu(dx) = \nu(x)dx$ and that there exists a non-increasing profile function f such that $\nu(x) \leq Cf(|x|)$ (f should fit ν around zero) and

$$K(r) := \sup_{|x|>1} \frac{\int_{\substack{|x-y|>r \\ |y|>r}} f(|x-y|)f(|y|)dy}{f(|x|)} \searrow 0 \qquad \text{as } r \to \infty. \tag{44}$$

Additionally, we assume there is a non-empty and bounded set $T \subset (0,\infty)$ and a constant $\widetilde{C} > 0$ such that

$$\int_{\mathbb{R}^d} e^{-t\Re(\psi(\xi)-\xi\cdot A\xi)}|\xi|d\xi \leq \widetilde{C}\left(\Psi^{-1}\left(\frac{1}{t}\right)\right)^{d+1}, \quad t \in T,$$

where

$$\Psi(r) = \sup_{|\xi|\leq r} \Re(\psi(\xi) - \xi \cdot A\xi) \quad \text{and} \quad \Psi^{-1}(s) = \sup\{r > 0 : \Psi(r) = s\} \quad \text{for } r, s > 0.$$

Then our main result [P9, Theorem 1] states that if for some $E \subset \mathbb{S}^{d-1}$ and $\kappa \geq 0$

$$\lim_{r\to\infty} \frac{\nu(r\theta - y)}{\nu(r\theta)} = e^{\kappa(\theta\cdot y)}, \quad y \in \mathbb{R}^d, \quad \theta \in E, \tag{45}$$

and $\inf_{x\in\Gamma_E} \frac{\nu(x)}{f(|x|)} > 0$, then for every $t \in T$, $\theta \in E$ and $y \in \mathbb{R}^d$ we also have

$$\lim_{r\to\infty} \frac{p_t(r\theta - y)}{t\,\nu(r\theta)} = \begin{cases} 1 & \text{if} \quad \kappa = 0, \\ e^{-t\widetilde{\psi}(\kappa\theta)+\kappa(\theta\cdot y)} & \text{if} \quad \kappa > 0, \end{cases} \tag{46}$$

where

$$\widetilde{\psi}(\xi) = -\xi \cdot b - \xi \cdot A\xi + \int_{\mathbb{R}^d\setminus\{0\}} \left(1 - e^{\xi\cdot y} + \xi \cdot y\mathbb{1}_{B(0,1)}(y)\right)\nu(y)\,dy$$

is the Laplace exponent of the Lévy process generated by L (when $\kappa > 0$, then the condition (44) and (45) imply that $\widetilde{\psi}(\xi)$ is well defined for $\xi = \kappa\theta$ [P9, Lemma 2]). Moreover, when the limit (45) is uniform in $(\theta, y) \in E \times D$, for every compact set $D \subset \mathbb{R}^d$, then (46) is also uniform in $(t, \theta, y) \in T \times E \times B(0, \varrho)$, for $\varrho > 0$. As a corollary, we get sharp two-sided estimates of p_t in cones Γ_E, away from 0 [P9, Corollary 2]. Similar results hold true for the absolutely continuous parts of the compound Poisson processes [P9, Theorem 2, Corollary 2].

Our method in new and it is based on appliction of the function $K(r)$ which was introduced in [H3] (see (35) above). The condition (44) compactifies a convergence in our proofs. Our approach allows us to compute the limit (46) for processes with exponential tails, e.g. for ralativistic stable semigroups (see discussion of examples in [P9, Section 6]). Known methods, working well in the polynomial case, were not sufficient for such processes.

Stochastic processes on fractals and in random media

In the series of papers [P4] and[P8] we have investigated the spectral properties of generalized random Schrödinger operators H_ω which correspond to subordinate Brownian motions (both diffusive and jump) evolving in random environment induced by the Poisson point process on the Sierpiński gasket $\mathcal{G}$. More precisely, we studied the Feynman-Kac semigroups of such processes with independent random potential $V(x,\omega) = \int_{\mathcal{G}} W(x,y)\mu^\omega(dy)$, where W is a non-negative profile on $\mathcal{G} \times \mathcal{G}$ and $\mu^\omega(dy)$ is the Poisson random measure on $\mathcal{G}$ with intensity $\nu m(dx)$, $\nu > 0$ (here m denotes the $\log 3/\log 2$-dimensional Hausdorff measure on $\mathcal{G}$).

In [P4] we proved the existence of the so-called *integrated density of states* (IDS in short) of the operator H_ω. This was done for a very general class of subordinate diffusions [P4, Assumption 2.1 and Example 2.1] and a family of two-argument profiles W introduced in [P4, (W1)-(W3) on p. 1262] (see also [P4, Section 4]), which includes the profiles of infinite range and even singular ones (such a general class of potentials has not been considered on fractals so far). The density of states l was obtained as a common (non-random) limit of random empirical measures based on spectra of generators of killed and reflected processes on fractal complexes $\mathcal{G}_n \nearrow \mathcal{G}$, $n \to \infty$ [P4, Theorems 3.1-3.2]. Killing and reflecting of the process correspond to imposing the Dirichlet and Neumann conditions on its generator.

In the paper [P8], under some additional assumptions on the subordinator (see (L1), p. 3906 and (U1)-(U3), p. 3913), we established the Lifschitz-type singularity around the bottom of the spectrum of H_ω for the integrated density of states. We have proven that $-\log l([0,\lambda]) \approx \lambda^{-\gamma}$ for some $\gamma > 0$, when $\lambda \to 0^+$ [P8, Theorems 3.3 and 4.4]. As a by-product, we also obtained the large-time estimates for the Feynman-Kac functionals averaged with respect to the process and the environement. Such objects can be interpreted as mean survival probabilities up to time t of processes whose paths are being killed by Poissonian potentials. The behaviours we found depend on the decay rate of $W(x,y)$ as $|x-y| \to \infty$, and on the long range distributional properties of the processes (which are formally described by the behaviour of the Laplace exponent of the subordinator around zero). It should be emphasized that our results identify certain *qualitative transitions* in the asymptotic properties of IDS and averaged functionals which reflect the impact of the process and the potential.

A key tool used in our both papers are the so-called reflected processes in the complexes $\mathcal{G}_n$, $n \geq 1$. In the diffusive case, such a process has been constructed in [56] via folding projection of the free Brownian motion from the unbounded gasket $\mathcal{G}$ to $\mathcal{G}_n$. In jump case, such a process is obtained by the subordination of the reflected diffusion on $\mathcal{G}_n$. Thanks to this specific construction, we were able to establish a clear connection between the properties of reflected processes on subsequent levels $\mathcal{G}_n$ and $\mathcal{G}_{n+1}$. This allowed us to get a monotonicity for averaged Feynman-Kac functionals for reflected processes in the complexes $\mathcal{G}_n$ with potentials that are periodic with respect to the Poissonian medium (the so-called Sznitman periodization; see [P4, inequality (3.8)]). This observation was crucial for the proof of the existence of IDS (the convergence of Laplace trasnforms in [P4, Theorem 3.1]) and in the proof of the upper bounds ([P8, Lemma 4.4] and its further applications). Another important step was to prove a version of the theorem of A.S. Sznitman which gives the estimate of the ground state eigenvalue for random Schrödinger operators [P8, Theorem A.1].

These results and methods essentially depend on the specific geometry of the Sierpiński gasket (e.g. on the existence of the sequence of reflected processes). We now want to get the counterparts of them for much more complicated fractal spaces. In [Pre2] we gave the description of a large class of nested fractals on which a natural projection exists and constructed the desired sequence of reflected processes [Pre2, Definitions 3.1-3.3, Theorems 4.1-4.2].

Non-local Parabolic Anderson Model (PAM)

In the paper [P10] we have studied symmetric and strong Feller Lévy processes with jumps $\{X_t\}_{t\geq 0}$ evolving in random environment in $\mathbb{R}^d$ induced by independent Poissonian potential $V^\omega(x) = \int_{\mathbb{R}^d} W(x-y)\mu^\omega(dy)$. Here W is a non-negative profile on $\mathbb{R}^d$ and $\mu^\omega(dy)$ is the random Poisson measure with intensity ρdx, with parameter $\rho > 0$. We do not require the absolute continuity of the Lévy measure, but we always assume that ψ is such that $e^{-t_0\psi(\cdot)} \in L^1(\mathbb{R}^d)$ for some $t_0 > 0$. Our main results are the estimates of the random variables

$$u^\omega(t,x) := \mathbf{E}_x\left[e^{-\int_0^t V^\omega(X_s)\,ds}\right] \tag{47}$$

as $t \to \infty$, almost surely with respect to ω (the so-called *quenched* behaviour). The asymptotics of averaged variables $u^\omega(t,x)$ (the so-called *annealed* behaviour) for Lévy processes with jumps has been investigated by Donsker and Varadhan, and Okura. The quenched behaviour has been an open problem for almost 40 years. It should be emphasized that $u^\omega(t,x)$ is the survival probability up to time t for the process starting from x, whose paths are being killed with random intensity given by V^ω. It is a probabilistic solution of the parabolic problem $\partial_t u = -H_\omega u$, $u(0,x) \equiv 1$, where $H_\omega := -L + V^\omega$ is the non-local random Schrödinger operator based on generator L of $\{X_t\}_{t\geq 0}$.

Our first main results are [P10, Theorems 3.1 and 4.1], which give general upper and lower estimates. The first result requires some information on decay rates of the tails of random viariables X_t and on the asymptotic behaviour of IDS for the operator H_ω. For the proof of the second theorem we have to impose an appropriate condition on the asymptotic behaviour of the ground state eigenvalue for the processes killed in large balls, and assume that W is of finite range. In [P10, Section 5] we first prove a series of auxiliary results which allow us to check the assumptions of general results mentioned above, and then we apply them to identify the asymptotic profiles $\eta(t)$ and explicit constants $C_1, C_2 > 0$ such that

$$-C_1 \leq \liminf_{t\to\infty} \frac{\log u^\omega(t,x)}{\eta(t)} \leq \limsup_{t\to\infty} \frac{\log u^\omega(t,x)}{\eta(t)} \leq -C_2,$$

almost surely with respect to ω, for every fixed $x \in \mathbb{R}^d$. These results are amply illustrated with a class of isotropic-unimodal Lévy processes [P10, Table 1]. It is worth noting that we have observed two *qualitative transitions* in the growth of the rates η which depend on the decay rate of the intensity of jumps of the process (for a discussion see [P10, p. 6]). In particular, when the decay rate of this intensity is of order $e^{-c|x|^\beta}$, $c, \beta > 0$, or faster (e.g. relativistic stable processes), then $C_1 = C_2$, i.e. we computed the limit $\lim_{t\to\infty} \frac{\log u^\omega(t,x)}{\eta(t)}$, for almost all ω. For such processes the profile $\eta(t)$ is the same as for the Brownian motion. Interestingly, this need not be true for any Lévy measure with finite second moment.

Ground state transformed processes

In the paper [P11] we have investigated the asymptotic behavior of the ground state transformed processes associated with the non-local Schrödinger operators $H = -L + V$, starting from their stationary distributions $\mu(dx) = \varphi_0^2(x)dx$. A similar problem was studied by Rosen and Simon in one dimension for the classical Schrödinger operators $-\Delta + V$ with confining polynomial potentials [59]. Due to technical limitations, we only look at the trace of paths on the positive integer time. The first result are the estimates of the order of fluctuations: the LIL-type theorem giving the upper envelope for the paths of the process [P11, Corollary

3.1] which follows directly from the corresponding integral test of the Kolmogorov type [P11, Theorem 3.1]. These observations are fairly general (the only condition here is the existence of a non-degenerate ground state of H which guarantees the existence of the transformed process). Another general result is the comparison principle in [P11, Theorem 3.2]. Further chapters, [P11, Sections 4.2-4.3 and 4.4], contain more detailed versions of these results, specialized for analysis of processes with DJP and potentials $V \in \mathcal{K}_{\pm}^{\infty}$ and $V \in \mathcal{K}_{\pm}^{0}$, respectively. Our estimates of ground states obtained in [H1] and [H3] have been crucial in proving these results. They allowed us to establish a clear and direct dependence of asymptotic profiles on the potential and the intensity of jumps of the initial Lévy process.

The paper [P6], published in a physics journal, contains a broader discussion of our results on the localization of ground states for non-local Schrödinger operators in relation to the transformed processes. The aim of this work was to communicate and disseminate these results to specialists working in statistical physics.

Zero-energy bound states and zero-resonances

In the paper [Pre1] we have studied a problem similar to that in [H3], but in a critical situation, i.e. when the eigenvalue $\lambda = 0$. More precisely, we have investigated the localization properties (i.e. the estimates of the rate of decay at infinity) of solutions to the equation $H\varphi = 0$, for non-local Schrödinger operators $H = -L + V$ (L is a generator of the symmetric Lévy process satisfying the assumptions (A1)–(A3) and $V \in \mathcal{K}_{\pm}^{0}$). Due to some technical limitations, we had to additionally assume that the density of the Lévy measure has the doubling property. When $\varphi \in L^2(\mathbb{R}^d)$, then φ is the eigenfunction of the operator H corresponding to zero eigenvalue. However, now 0 lies at the bottom of the essential spectrum of H and it is no longer an isolated eigenvalue. We also consider the solutions $\varphi \notin L^2(\mathbb{R}^d)$ such that $\varphi \in L^p(\mathbb{R}^d)$, for some $p > 2$, which are called the *zero-resonances*. Our argument in [Pre1] is based on combination of a subtle version of Lemma 23 (specialized for the case $\lambda = 0$; see [Pre1, Lemma 4.1]) and some abstract, sufficiently sharp, self-improving estimates [Pre1, Section 3]. This allowed us to obtain the estimates of (X, V)-harmonic functions (and, in consequence, for the eigenfunctions φ corresponding to $\lambda = 0$) which take into account the actual fall-off rate of the potential V at infinity. In some cases, if $\varphi > 0$, then these bounds are sharp. We also identified two *qualitative transitions* in the decay rates of eigenfunctions which depend on the behavior of the function $\Psi(1/|x|)/V(x)$ and the integrability of the ratio $\nu(x)/V(x)$ at infinity (for a discussion based on examples and the heuristic interpretation we refer to [Pre1, Section 6]).

Estimates of Feller and Lévy semigroups

In the paper [P1] we have found the upper estimate for the integral kernels of Feller semigroups which describe the evolution of the spatially non-homogeneous jump Markov processes in $\mathbb{R}^d$, the so-called *Lévy-type processes* [17]. Intensities of jumps $f(x, y)$ of the processes under consideration are required to satisfy certain regularity conditions (including some symmetry properties). The most important assumption is the existence of a sufficiently smooth profile function ϕ and constants $M, c > 0$ such that

$$f(x,y) \leq M \frac{\phi(|x-y|)}{|x-y|^{d+\alpha}}, \quad x \neq y, \quad \text{and} \quad \inf_{x \in \mathbb{R}^d} \int_{|x-y|>r} \frac{f(x,y)}{\phi(|x-y|)} dy \geq c r^{-\alpha}, \quad r \in (0,1),$$

for some $\alpha \in (0,2)$. We assume that the function ϕ is sufficiently smooth, constant around zero, and that it decays sufficiently regularly at infinity. In particular, $\phi(r)r^{-d-\alpha}$ is supposed to satisfy the convolution condition as in (A1.c).

The main result is the following estimate for the integral kernels (transition probability densities) of the considered semigroups: there are constants $C_1, C_2 > 0$ such that

$$p(t,x,y) \le C_1 e^{C_2 t} \min\left(t^{-d/\alpha}, \frac{t\phi(|x-y|)}{|x-y|^{d+\alpha}}\right), \quad x,y \in \mathbb{R}^d, \quad t > 0.$$

Interestingly, this result also covers the situation when $\phi(r)$ is of order $e^{-cr}r^{\gamma}$, for some $c > 0$ and $\gamma \ge 0$ (e.g. spatially non-homogeneous relativistic and tempered-type processes). Under our assumptions, the above bound is sharp for small t's in the sense that the function on the right hand side gives sharp two-sided estimates for the density of the isotropic Lévy process with the Lévy measure $\nu(dx) = C\phi(|x|)|x|^{-d-\alpha}dx$ which dominates the Markov process in question.

The paper [P5] includes upper and lower estimates of the transition probability densities and the upper estimates of their spatial derivatives for a fairly general class of jump Lévy processes. The first result, [P5, Theorem 1], gives the upper bound for the densities under certain assumptions on the Lévy-Khintchine exponent and on the Lévy measure. This result can be applied to a wider class of processes than the results in [H2] discussed above, but it typically gives less sharp estimates. The second theorem, [P5, Theorem 2], gives the lower estimate under the assumption that the Lévy measure of balls can be controlled from below. This result is local. Upper estimates for derivatives of transition densities are obtained in [P5, Theorem 3].

Weyl asymptotics for non-local operators

The paper [P2] is concerned with spectral properties of the Hamiltonian describing the energy of a one-dimensional quasi-relativistic particle in an infinite potential well. The main goal of this research was to describe the structure of the spectrum of the Dirichlet square-root Klein-Gordon operator $(-\hbar^2c^2d^2/dx^2 + m^2c^4)^{1/2}$ on the interval. Here c denotes the speed of light, m represents the mass of a particle, and $\hbar$ is the reduced Planck constant. We obtained the Weyl-type asymptotic formulas for eigenvalues λ_n of this operator as $n \to \infty$ (from which we derive that all λ_n's are simple), as well as uniform estimates of L^∞ and L^2 norms of all eigenfunctions, depending on the length of the interval. In the paper [P7] these results have been generalized to the case of operators of the form $\phi(-\Delta)$, where ϕ is a complete Bernstein function such that $\lambda\phi(\lambda) \to \infty$, when $\lambda \to \infty$. The methods of proofs in these papers are based on new techniques proposed by Kwaśnicki [49], which rely on an approximation of the eigenfunctions by the so-called *generalized* eigenfunctions.

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