SUMMARY OF THE DOCTORAL DISSERTATION

Heat kernels for the Bessel operator

by Kamil Bogus

The dissertation consists of the following author's papers.

[D1] K. Bogus, J. Małecki,
Sharp estimates of transition probability density for Bessel process in half-line,
Potential Analysis 43(1), 1–22 (2015).

[D2] K. Bogus, T. Byczkowski, J. Małecki,
Sharp estimates of Green function of hyperbolic Brownian Motion,
Studia Mathematica 228(3), 197–221 (2015).

[D3] K. Bogus, J. Małecki,
Heat kernel estimates for the Bessel differential operator in half-line,
Mathematische Nachrichten 289, 2097–2107 (2016).

[D4] K. Bogus,
Asymptotic behaviour of the Bessel heat kernels,
preprint arXiv:1709.05796, s. 19 (2017).

In this dissertation I studied Dirichlet heat kernels $p_a^{(\mu)}(t,x,y)$ in half–lines (a,∞), where $a>0$, for the Bessel differential operator

$$L^{(\mu)} = \frac{d^2}{dx^2} + \frac{2\mu+1}{x}\frac{d}{dx}, \quad \mu \in \mathbf{R},$$

with the reference measure $m^{(\mu)}(dy) = y^{2\mu+1}dy$. The main objective was to describe the behaviour of $p_a^{(\mu)}(t,x,y)$, $x,y>a$, $t>0$, by giving its sharp estimates. These results were used in the sequel to provide precise estimates for the λ–Green function of the interior of the horocycle on real hyperbolic spaces.

In the paper [D1] we provide sharp two–sided estimates of the considered heat kernel (the fundamental solution of the heat equation $2L^{(\mu)} = \partial_t$) in the case $\mu \neq 0$ for the whole range of the space variables $x,y>a$ and every $t>0$. The main advantage of this result is that the exponential behaviour is described explicitly, i.e. there are the same constants in the exponential factors in the upper and lower bounds. Such precise results are very exceptional in the theory of partial differential equations as well as in the theory of stochastic processes.

In the article [D3] we deal with the behaviour of $p_a^{(\mu)}(t,x,y)$ in the case $\mu=0$. Obtaining the estimates for $xy \leqslant t$ requires a new approach, since, in comparison to [D1], the heat kernel behaviour near the boundary (level $a>0$) is described by logarithmic factors instead of linear terms given in the case $\mu \neq 0$. On the other hand, for $xy>t$ estimates obtained in [D3] are exactly of the same form as those given in the article [D1]. Once again derived results give a very precise description of the exponential behaviour of the heat kernel for the Bessel differential operator in half–line.

The next part of the dissertation, namely [D2], is devoted to the study of the potential theory on real hyperbolic spaces. In particular, λ–Green function $G^{\lambda}_{\mathbf{H}_a}(x,y)$, $\lambda \geqslant 0$, of a half–space $\mathbf{H}_a = \{x \in \mathbf{H}^n : x_n > a\}$ in hyperbolic space $\mathbf{H}^n = \{x = (x_1,\ldots,x_n) \in \mathbf{R}^n : x_n > 0\}$ was expressed in terms of heat kernels considered in [D1] and [D3]. Then, estimates given therein were used to provide precise two–sided bounds of $G^{\lambda}_{\mathbf{H}_a}(x,y)$.

The last part of the doctoral dissertation (see [D4]) involves some improvements of the results published in [D1] and [D3]. We provided the asymptotic expansions of the heat kernel $p_a^{(\mu)}(t,x,y)$ for $xy/t \to \infty$. This result is more precise than those given in [D1] and [D3], but on limited range of arguments. In particular, the previously obtained estimates are a consequence of those given in [D4], in the mentioned régime.

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