SUMMARY OF THE DOCTORAL DISSERTATION 'TREES, IDEALS AND CARDINAL INVARIANTS OF THE REALS'

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The aim of this work is to study the cardinal invariants associated to various types of trees and of definable ideals on countable sets. The work consists of three main chapters. The first chapter contains all types of tree we will be interested in, together with their basic properties and calculation of cardinal invariants related to Borel ideals associated to these trees. These trees are generalizations of Sacks. Miller and Laver trees as well as some pseudo-trees related to the ideal of closed sets of measure zero. The second chapter contains a study of the cardinal invariants of (non-Borel) Marczewski-like ideals related to our tree types. We focus on additivities. covering numbers, cofinalities, incompatibility shrinking numbers and antichain numbers. The third chapter contains a study of cardinal invariants related to ideals on countable sets. We focus on the +-covering numbers and antichain numbers. We also study the splitting ideal. This work continues some investigations of Brendle, Hrušák, Khomskii, Sabok, Spinas, Steprāns, Rosłanowski, Wohofsky and Zapletal.

This thesis is made of two parts, first one about cardinal invariants of (Marczewski-like) ideals related to trees (chapter 2 and 3), second one about cardinal invariants related to ideals on countable sets (chapter 4). Mostly, these two parts can be read independently. However, sometimes each part needs certain invariants, definitions, or some simple facts from the other.

In chapter 1, we review the main classical definitions and notions that appear throughout this thesis. In particular we discuss classical cardinal invariants of the reals and relation systems, basic definitions regarding ideals on countable sets, and, basic notions regarding trees, descriptive set theory or forcing.

Chapter 2, contains basic definitions and properties regarding trees and the ideals related to them. In the first section, we discuss the notion of a tree ideal $\mathcal{I}(\mathbb{T})$, its Borel part $\mathcal{I}_{\mathcal{B}}(\mathbb{T})$ and their basic properties. In the second section, we go through the tree types in which we are interested in this thesis.

- two tree types for adding bounded infinitely equal real - $\mathbb{IE}_F$ and $\mathbb{S}_F$.
- two tree types for adding infinitely equal real on ω^ω - $\mathbb{IE}$ and $\mathbb{FM}$,
- Miller and Laver trees which split positively according to an ideal $\mathcal{J}$ - $\mathbb{M}_{\mathcal{J}}$ and $\mathbb{L}_{\mathcal{J}}$,
- the tree type $\mathbb{SP}$ for adding splitting real and $\mathbb{ED}_F$ for adding bounded eventually different real,

- Hurewicz schemas - a pseudo-tree notion for the ideal $\mathcal{E}$ of closed measure zero sets.

For each of these tree types we discuss fusion, continuous reading of names, related perfect set dichotomy and cardinal invariants of the Borel ideal $\mathcal{I}_{\mathcal{B}}(\mathbb{T})$ related to them.

Chapter 3, contains a study of cardinal invariants of the tree ideals. In the first section, we deal with covering numbers. We prove a general theorem saying that a tree type adds a real of a certain kind (dominating, Cohen etc.) in continuous way. naturally gives an upper bound for the covering number. Later, we discuss which types of reals, each of the tree types add and which they do not. In particular. we show that $\mathbb{L}_{\mathcal{J}}$ never adds random reals and that $\mathbb{L}_{\mathcal{ED}_{fin}}$ has Laver property. We also show that, for a meager ideal $\mathcal{J}$ both $\mathbb{L}_{\mathcal{J}}$ and $\mathbb{M}_{\mathcal{J}}$ increase the groupwise dense number which, as a corollary, gives the consistency of $\mathrm{cov}(\mathbb{T}) < \mathrm{add}(\mathcal{M})$ for the two. In the second section, we discuss the incompatibility shrinking number $is(\mathbb{T})$ i.e. the cardinal invariant which measures how many small sets do we need to intersects every positive set. This number will later play an important role on the discussion of additivities, cofinalities and antichains of tree ideals. In this regard, we show that:

- $is(\mathbb{T}) = \mathrm{add}(\mathcal{M})$ for the tree types that add Cohen reals,
- $is(\mathbb{T}) \geqslant \mathrm{cov}(\mathcal{M})$ for $\mathbb{IE}_F$ and $\mathbb{S}_F$,
- $is(\mathbb{T}) \geqslant \min\{\mathfrak{b}, \mathrm{cov}^*_+(\mathcal{J})\}$ for $\mathbb{M}_{\mathcal{J}}$ and $\mathbb{L}_{\mathcal{J}}$ and
- $is(\mathbb{T}) \geqslant \min\{\mathfrak{b}, \mathrm{cov}(\mathcal{I}_{\mathcal{B}}(\mathbb{T}))\}$ for $\mathbb{M}_{\mathcal{J}}$.

In the third section, we study the antichain numbers $\mathfrak{a}(\mathbb{T})$ for our tree types i.e. the smallest cardinality of an uncountable maximal antichain in $\mathbb{T}$. We prove the following inequalities:

- $\mathrm{cov}(\mathcal{M}) \leqslant \mathfrak{a}(\mathbb{S}_F), \mathfrak{a}(\mathbb{IE}_F)$,
- $\min\{\mathrm{is}^*(\mathbb{ED}_F), \mathrm{cov}(\mathcal{M})\} \leqslant \mathfrak{a}(\mathbb{ED}_F)$,
- $\min\{\mathfrak{b}, \mathrm{cov}(\mathcal{I}_{\mathcal{B}}(\mathbb{M}_{\mathcal{J}}))\} \leqslant \mathfrak{a}(\mathbb{M}_{\mathcal{J}})$
- $\mathfrak{p} \leqslant \mathfrak{a}(\mathbb{P}_{\mathcal{E}})$.

In the fourth section, we discuss the additivities of tree ideals. The results of this section are generalizations of known methods. Under the assumption $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$, we prove a general theorem saying that an existence of a projection of a tree onto a relational system $\mathcal{R}$, naturally gives an upper bound for $\mathrm{add}(\mathcal{I}(\mathbb{T}))$ of the form $\mathfrak{b}_\sigma(\mathcal{R})$. Then we show that:

- $\mathbb{IE}_F$ and $\mathbb{S}_F$ have both $\mathfrak{b}$-projection and $\mathfrak{s}$-projection,
- $\mathbb{M}_{\mathcal{J}}$ and $\mathbb{L}_{\mathcal{J}}$ have both $\mathfrak{b}$-projection,
- $\mathbb{M}_{\mathcal{J}}$ has an $\mathfrak{s}$-projection.

We also mention related results from the literature and corollaries. In the fifth section, we establish more bounds regarding the additivities of our tree types. All of them are generalizations of the known arguments for classical tree types like Sacks, Miller or Laver. First, we show that $\mathfrak{t} \leqslant \mathrm{add}(\mathbb{L}_{\mathcal{J}})$ for F_σ-ideals $\mathcal{J}$. Then we show that nowhere-distributivity of $\mathbb{T}$ gives a natural upper bound for $\mathrm{add}(\mathcal{I}(\mathbb{T}))$ and, we show that

- $\mathbb{IE}_F$ and $\mathbb{S}_F$ are $\mathfrak{b}$-nowhere distributive,
- $\mathbb{M}_{\mathcal{J}}$ and $\mathbb{L}_{\mathcal{J}}$ are $\mathfrak{h}_{\mathcal{J}}$-nowhere-distributive for a meager ideal $\mathcal{J}$ with a base tree.

We apply these (and results from the third section) to show the consistency of $\mathrm{add}(\mathcal{I}(\mathbb{T})) < \mathrm{cov}(\mathcal{I}(\mathbb{T}))$ for various tree types. In the last, sixth section, we use the methods developed by Brendle, Khomskii and Wohofsky, to analyse the cofinalities of the tree types. In particular, we mention related results, prove the constant or 1-1 property for $\mathbb{L}_{\mathcal{J}}$ where $\mathcal{J}$ is nowhere-maximal P-ideal and, as a corollary, we obtain $\mathrm{cof}(\mathcal{I}(\mathbb{T})) > \mathfrak{c}$ for such trees.

Chapter 4 contains a study of cardinal invariants of ideals on countable sets. In the first section, we gather all ideals important to this work. We will focus on Borel and analytic ideals with a special emphasis on the F_σ-ideals. We will collect known results on cardinal invariants add^*, cov^*, non^*, and cof^* as well as their ω-versions. We also discuss the Katetov order between our examples. In the second and third section, we investigate two ideals on ω, the splitting ideal and the infinitely equal ideal. We study their cardinal invariants and the position in the Katetov order. As a result, we obtain a new upper bound of the covering number of the density zero ideal. In the fourth section, we investigate the cardinal invariant cov^*_+ and continue the study of covering numbers for selected ideals. In the fifth section, we study the antichain numbers of the algebras $\mathcal{P}(\omega)/\mathcal{J}$. As a result, we obtain a strengthening of Steprāns lower bound of the antichain number for the nwd ideal.

The main contribution of this work is:

- the isolation and investigation of the incompatibility shrinking numbers, a calculation of which gives a partial answer to the problem of Brendle, Khomskii and Wohofsky,
- the study of the cardinal invariants of the tree types $\mathbb{M}_{\mathcal{J}}$ and $\mathbb{L}_{\mathcal{J}}$,
- the study of the cardinal invariants of Marczewski-like tree ideals,
- the study of ω-invariants of ideals on ω,
- the investigation of antichain numbers of the Boolean algebras $\mathcal{P}(\omega)/\mathcal{J}$ for a wide class of ideal. In particular we show that $\mathfrak{a}(\mathrm{nwd}) \geqslant \mathrm{add}(\mathcal{M})$. This improves a result of Steprāns and give another proof of the inequality than the one of Cancino Manríquez,
- the study of the splitting ideal, which leads to a new upper bound of the covering number of density zero ideal.