Abstract of the Doctoral Dissertation "Hyperbolic Bessel Process"

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Hyperbolic Bessel process is one of many diffusion processes that are used to model different physical and economic phenomena (e.g. Asian options pricing). This process has been recently investigated by Jakubowski and Wiśniewolski, earlier its properties were investigated by Gruet and by Borodin. These authors defined hyperbolic Bessel process for parameters $\nu > -\frac{1}{2}$ as a solution of a stochastic differential equation or a diffusion generated by a differential operator. They also gave some formulas for the transition density. However, all of these formulas are very complicated, except only for $\nu = 1/2$, when hyperbolic Bessel process is a radial part of a hyperbolic Brownian motion in three-dimensional space.

In the thesis we obtain formulas for the transition densities of hyperbolic Bessel processes of integer dimension, that is for the radial parts of hyperbolic Brownian motions. It is a diffusion with generator $\frac{1}{2}\Delta_{\mathbb{D}^n}$, where

$$\Delta_{\mathbb{D}^n} = \frac{\partial^2}{\partial r^2} + (n-1)\,\mathrm{cth}\,(r)\frac{\partial}{\partial r} + \frac{1}{\mathrm{sh}^2 r}\Delta_S, \tag{1}$$

Using the connection between hyperbolic Brownian motion and hyperbolic Bessel process we describe the transition density of a Bessel process as an integral of a Brownian density. Moreover, application of the so-called Millson formula for hyperbolic Brownian density gives much simpler formulas for hyperbolic Bessel densities than all formulas known so far, namely the following formulae hold:. for $m-1 = \frac{n-3}{2}$

$$p_n^{\mathrm{Bess}}(t;a,b) = \frac{\Omega_{n-2} e^{-\frac{m^2 t}{2}}\,\mathrm{sh}\,b}{t(2\pi)^m\sqrt{2\pi t}\,\mathrm{sh}^{n-2} a} \times \tag{2}$$
$$\times \int_{b-a}^{b+a} \left(\frac{\partial}{\partial r}\frac{1}{\mathrm{sh}\,r}\right)^{m-1} \left\{\mathrm{sh}\,r\,[(\mathrm{ch}\,(b+a) - \mathrm{ch}\,r)(\mathrm{ch}\,r - \mathrm{ch}\,(b-a))]^{m-1}\right\} \frac{r}{\mathrm{sh}\,r} e^{-\frac{r^2}{2t}}\,dr,$$

while for $n = 2m+2$ we have:

$$p_n^{\mathrm{Bess}}(t;a,b) = \frac{\Omega_{n-2} e^{\frac{-(2m+1)^2 t}{8}}\,\mathrm{sh}\,b}{(2\pi)^m 2(\pi t)^{\frac{3}{2}}\,\mathrm{sh}^{n-2} a} \times \tag{3}$$
$$\times \int_{b-a}^{b+a} \left[\left(\frac{\partial}{\partial r}\frac{1}{\mathrm{sh}\,r}\right)^{m} \left\{\mathrm{sh}\,r\,[(\mathrm{ch}\,(b+a) - \mathrm{ch}\,r)(\mathrm{ch}\,r - \mathrm{ch}\,(b-a))]^{m-\frac{1}{2}}\right\} \int_r^\infty \frac{s e^{\frac{-s^2}{2t}}}{\sqrt{\mathrm{ch}\,s - \mathrm{ch}\,r}}\,ds\right] dr.$$

What is more, in the case of a Bessel process of odd dimension, the above transition density can be expressed as a sum of elementary functions and error function Erf. We also give some estimates for the case $n = 5$ and 7.

The second stochastic process considered in the thesis is (Z_t) the diffusion with a generator

$$\Delta_\mu = \frac{1}{2}\frac{d^2}{dx^2} + \mu \coth(\mu x)\frac{d}{dx}. \tag{4}$$

known as a Bessel process of drifting Brownian motion. If $\mu = 1$ then (Z_t) is a radial part of a hyperbolic Brownian motion in three-dimensional hyperbolic space.

The transition density function of this process is well-known but to our best knowledge, distribiution of different functionals of this process have not been investigated yet. In the thesis we investigate process (Z_t) killed on exiting interval $(0, r_0)$ and give a formula describing its transition density and the distribution of $M_t = \sup_{s\le t} Z_s$, the supremum of the process (Z_t). We have

$$\mathbb{P}^x(M_t < r_0) = \mathbb{P}^x(\tau_{r_0} > t) = \tag{5}$$

$$\sum_{n=1}^{\infty}\left[\left(\frac{(-1)^{n+1}2\pi n \,\mathrm{sh}\,(\mu r_0)\sin(n\pi x/r_0)}{\mathrm{sh}\,(\mu x)(n^2\pi^2+\mu^2 r_0^2)}\right)\exp\left(-\frac{(n^2\pi^2/r_0^2+\mu^2)t}{2}\right)\right].$$

Because the formula is given as an infinite series, we give its exact estimate using elementary functions. Moreover, our method of estimation applied to a function $ss_y(v,t)$ used in a handbook by Borodin and Salminen give very precise estimate of this function.