DOCTORAL DISSERTATION

Title: Applications of multiorder in measure-theoretic and topological actions of amenable groups

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Abstract in English

For classical dynamical systems, with actions of an additive group $\mathbb{Z}$, there is a known characterization of zero-entropy topological dynamical systems, as factors of systems with no asymptotic pairs. One part of this characterization was given by F. Blanchard, B. Host and S. Ruette in 2002. They proved the theorem stating that asymptotic pairs exist in every topological dynamical system of positive entropy. In 2010, T. Downarowicz and Y. Lacroix completed this result by showing that every topological dynamical system of entropy zero has an extension with no asymptotic pairs. So far, no analogous characterization, for actions of groups different than $\mathbb{Z}$, has been obtained. In fact, even providing a definition of an asymptotic pair, which fully generalizes the classical one, and is valid for actions of wider classes of groups, such as countable amenable groups, turned out to be problematic. All such definitions, provided so far, either require existence of an invariant order on a group (which is not guaranteed for every countable amenable group) or they are too restrictive and proofs of any result analogous to the Blanchard-Host-Ruette Theorem, valid for wider classes of groups, require making additional assumptions, for instance that the action of group is expansive. In the introductory chapter of the dissertation, hitherto known results concerning existence of asymptotic pairs in dynamical systems with actions of groups other than $\mathbb{Z}$, were discussed.

The problem of existence of an invariant order on a group can be overcome with the aid of the recently invented notion of a *multiorder*. A multiorder on a group G is defined as a collection $\tilde{\mathcal{O}}$ of orders of type $\mathbb{Z}$ on G (i.e. orders isomorphic to the natural order $<$ on $\mathbb{Z}$), such that there exists a measure ν, supported by $\tilde{\mathcal{O}}$, which is invariant under the action of G. It turns out that multiorders exist on all countable amenable groups (even those which do not posses an order which is invariant itself). Hence, it was possible to formulate definitions of asymptotic pairs, which fully generalize the classical one and are valid for actions of countable amenable groups. The main goal of this dissertation was to obtain a characterization of topological dynamical systems with actions of countable amenable groups, of entropy zero, by the criterion of existence of asymptotic pairs, analogous to the one known for actions of $\mathbb{Z}$. Specifically, the aim was to prove the theorem stating that every topological dynamical system (X, G) with an action of a countable amenable group G, has entropy zero, if and only if there exists a topological extension (Y, G) of (X, G), such that there are no pairs in Y, which are asymptotic to any order from a certain multiorder on G. This goal was achieved

The main part of the dissertation begins with formulation of the definition of a multiorder on a countable group and discussion of its important properties. It was shown that multiorders exists on all countable amenable groups. It was also proved that every multiorder $\tilde{\mathcal{O}}$ on such group has so-called *Følner property*, i.e. for almost every order from a multiorder, order intervals of increasing lengths form a Følner sequence. These two properties show that existence of a multiorder on a countable group is equivalent to the condition that the group is amenable. Multiorders were also compared with more general *invariant random orders*, which were introduced by J. Kieffer in 1975. It was also shown

that every multiorder on a group G can be represented as a certain collection of bijections from $\mathbb{Z}$ to G.

Subsequently, some properties of dynamical systems which have multiorders as their factors were discussed. Such systems were called *multiordered.* Special attention was paid to the orbital equivalence between a multiordered system (X, G) and a specific action of $\mathbb{Z}$ on X, given by the iterates of a so-called *successor map*. This equivalence can be used to prove some interesting properties of entropy in multiordered systems. One of the chapters of the dissertation was particularly devoted to presenting and proving such properties. It starts with a formula for entropy calculated along a multiorder. Then, it was proved that the orbit equivalence with the action of $\mathbb{Z}$ given by the iterates of the successor map, preserves the conditional entropy with respect to the multiorder. This result was used to prove the more general Rudolph-Weiss Theorem concerning the preservation of conditional entropy by an orbit equivalence between any two dynamical systems with actions of countable amenable groups, which have a common factor (not necessarily being a multiorder). The chapter is concluded with a theorem which provides a characterization of a Pinsker sigma-algebra in a system with an action of a countable amenable group, using multiorders, which generalizes the classical Rokhlin-Sinai Theorem.

Properties of entropy in multiordered dynamical systems were used to prove one of the two main theorems of the dissertation, which states that in every topological dynamical system (X, G) with an action of a countable amenable group, which has positive entropy, for any multiorder $\tilde{\mathcal{O}}$ on G and almost every order from $\tilde{\mathcal{O}}$, there exist asymptotic pairs in X. It was preceded by formulations of two new definitions of asymptotic pairs in systems with actions of countable groups, which fully generalize the classical one. The theorem was proved in two steps. Firstly, a partial result, concerning only multiordered systems, was obtained. Then, the proper theorem, valid for actions of all countable amenable groups, was derived from the previous one.

In the dissertation, there was also presented a construction of a multiorder arising from a so-called *tiling system.* The construction itself was preceded by an introduction, which familiarizes the reader with the theory of tilings and systems of tilings on countable amenable groups. Such tiling-based multiorder has certain topological properties which played a crucial role in the proof of the version of Downarowicz-Lacroix Theorem for actions of countable amenable groups, and which do not stem from the general definition of a multiorder. Some examples of particular tiling-based multiorders were provided as well.

The dissertation was concluded by the proof of the theorem stating that every topological dynamical system with an action of a countable amenable group G, of entropy zero, has a topological extension which has no asymptotic pairs for any order from a certain (tiling-based) multiorder on G. Similarly as in the proof of the generalization of the Blanchard-Host-Ruette Theorem, a partial result is obtained first. It is valid only for systems which have a tiling system as their topological factor and which fulfil some additional assumptions. Then, this partial result is used to prove the proper one, valid for any topological dynamical systems with actions of countable amenable groups. By combining this theorem with the previous one concerning the existence of asymptotic pairs in any system of positive entropy, the full characterization of zero-entropy systems with actions of countable amenable groups, as factors of systems with no asymptotic pairs, was obtained. Henceforth, the main goal of the dissertation was achieved.