Summary

Dissertation:

Gradient Perturbation of Unimodal Lévy Processes

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Let X_t be a pure-jump, isotropic, unimodal Lévy process in $\mathbb{R}^d$, $d \in \mathbb{N}$. That is, X_t is a Lévy process with rotationaly invariant density function $p_t(x)$ on $\mathbb{R}^d \setminus \{0\}$, where $p_t(x)$ is radially nonincreasing. The characteristic exponent is given as

$$\psi(x) = \int_{\mathbb{R}^d} \left(e^{i\langle x,z\rangle} - 1 - i\,\langle x,z\rangle\, \mathbf{1}_{|z|<1}\right) \nu(|z|)dz, \quad x \in \mathbb{R}^d,$$

where $\nu(|x|)\,dx$ is a Lévy measure, ie. $\int_{\mathbb{R}^d}(1 \wedge |z|^2)\nu(|z|)\,dz < \infty$, and ν is nonincreasing. For $d \geqslant 2$ we assume

$$\psi \in \mathrm{WLSC}(\underline{\alpha}_1,1,\underline{c}) \cap \mathrm{WUSC}(\overline{\alpha},\vartheta_d,\overline{C}), \tag{1}$$

where WLSC and WUSC stand for weak lower and upper scaling condition respectively, $2 > \overline{\alpha} \geqslant \underline{\alpha}_1 > 1$, and $\vartheta_d = 1$ if $d > 2$ and 0 otherwise. For $d = 1$, we assume

$$\psi \in \mathrm{WUSC}(\overline{\alpha},0,\overline{C}) \cap \mathrm{WLSC}(\underline{\alpha}_1,1,\underline{c}_1) \cap \mathrm{WLSC}(\underline{\alpha},0,\underline{c}), \tag{2}$$

where $2 > \overline{\alpha} \geqslant \underline{\alpha}_1 > 1$ and $\underline{\alpha}_1 \geqslant \underline{\alpha} > 0$. Now, we let

$$\mathcal{L}f(x) = \int_{\mathbb{R}^d} \left(f(x+z) - f(x) - \mathbf{1}_{|z|<1}\,\langle z,\nabla f(x)\rangle\right) \nu(|z|)dz\,, \quad f \in C_b^2(\mathbb{R}^d),$$

be a generator of the process X_t. We are interested in a process X_t perturbed by gradient. For this, let us take a vector field $b \in \mathcal{K}_d$, where $\mathcal{K}_d$ stands for a proper Kato class in the dimension d. We consider the perturbed operator

$$\tilde{\mathcal{L}} = \mathcal{L} + b(x) \cdot \nabla,$$

which is the generator of the perturbed process $\tilde{X}_t$. Let $D \subset \mathbb{R}^d$ be a $C^{1,1}$ set and G_D its Green function with respect to process X_t. To obtain the estimate for the respective Green function $\tilde{G}_D$ of perturbed process, we have constructed a perturbation series. In the process, it was essential to have an estimate of the gradient of G_D. Moreover, with proper estimates for ∇G_D, we have also shown the estimate of gradient of $\mathcal{L}$-harmonic function.

In the thesis, we establish general notations and facts essential for effective proofs. After that, in Chapter 4, we start to focus on the case $d \geqslant 2$. For small set D, we prove

$$|\nabla_x G_D(x,y)| \leqslant C_1 \frac{G_D(x,y)}{\delta_x \wedge |x-y|}, \quad x, y \in D,$$

where C_1 depends on the process only via scaling, and on the set D via its $C^{1,1}$ characteristic (but here the definition of "smallness" is also a consequence of scaling). As a result for

$$\psi \in \mathrm{WLSC}(\underline{\alpha}_1,1,\underline{c}) \cap \mathrm{WUSC}(\overline{\alpha},0,\overline{C}),$$

we get

$$C_2^{-1} G_D(x,y) \leqslant \tilde{G}_D(x,y) \leqslant C_2 G_D(x,y). \tag{3}$$

Chapter 5 is devoted to the case $d = 1$. Let D be any bounded $C^{1,1}$ set. Now, we denote by K a compensated potential of X_t. We show that

$$|\partial_x G_D(x,y)| \leqslant C_3 \frac{G_D(x,y) \wedge K(|x-y|)}{|x-y| \wedge \delta_x}, \quad x, y \in D,$$

where C_3 depends on the process only via scaling, and on the set D via its $C^{1,1}$ characteristic. In consequence, we obtain the comparability of the Green functions G_D and $\tilde{G}_D$, that is inequality (3).

In the last chapter (namely Chapter 6) we study a gradient of harmonic functions. Let f be a nonnegative and regularly harmonic function on D. Under assumptions (1) and (2), we prove that

$$|\nabla_x f(x)| \leqslant C_4 \frac{f(x)}{\mathrm{dist}(x, \partial D) \wedge 1}, \quad x \in D,$$

where C_4 depends on the process via scaling, and on the set D via its $C^{1,1}$ characteristic.