The modelling of a capillary rise dynamics using a nonlinear differential equation

In the following dissertation thesis, we investigate the nonlinear differential equations that describe the capillary rise phenomenon. Capillary rise is an extraordinary physical process that is ubiquitous in nature and the properties of which can also be used in industry. In particular, when the radius of a narrow tube, called capillary, is sufficiently small, the surface tension (caused by cohesive forces within the liquid) and the forces responsible for the intermolecular interaction between the liquid and the walls of the tube, induce the rise of the level of fluid inside the capillary. The models which we analyze are obtained as a consequence of a force balance for a liquid column inside a thin tube. We provide some theoretical mathematical results concerning the nonlinear governing equations that model the capillary rise phenomenon. Some statements of the thesis might be useful for the experimenters in a better understanding of all properties of this physical phenomenon and help them set new directions for research on capillary action.

The outline of the thesis is organized as follows. At the beginning, we present a series of theoretical results concerning the existence and uniqueness of solutions of the studied nonlinear equations. For the classical model, with some choice of the initial condition, we observe a singularity at the initial stage of the flow. To facilitate analysis of the model, we manage the problem of existence and uniqueness of the solution separately in the neighborhood of the initial stage of the flow and for other stages. Moreover, we also consider another highly nonlinear model of capillary rise that was obtained by establishing certain improvements to the classical model. After introducing the generalization of the discontinuous function responsible for an additional loss of energy, we transform the considered equation into a system of two nonlinear integral equations and then show the appropriate properties of the solution: boundedness, asymptotic behavior for small values of the independent variable, existence and uniqueness of the solution.

Next, for both the classical model and the improved one, we present a condition for changing the character of capillary flow, which agrees with the experimental data, that shows that the function of height of the liquid in a cylindrical capillary may be monotonic for some fluids, while for others it may oscillate around the equilibrium height. For the classical model, we provide an estimate of the basin of attraction for the unique critical point of an equivalent system of two nonlinear differential equations. Furthermore, for the improved model of capillary flow, we find the exact set of initial conditions starting from which the solution attains the equilibrium height. It turns out that the obtained set includes the initial conditions that do not belong to the basin of attraction for the model without improvements.

After those results, we present an asymptotic analysis of the function determining the level of fluid in the capillary at the initial stage of the flow and after a sufficiently long time when the free surface of the fluid inside the tube oscillates around the equilibrium height. Finally, we perform a perturbation analysis with respect to the dimensionless parameter appearing in the classical model, which may be useful for interpreting the experimental measurements in which the value of the analyzed parameter is relatively large for some fluids and fixed tube radius, and small for others fluids.

The dissertation thesis ends with a short summary of the main results and a description of their importance in the analysis of nonlinear differential equations and capillary rise phenomena.

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