Abstract of the Doctoral Dissertation
"Inverted elastic pendulum as a mathematical model of running"
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In this doctoral thesis, the spring mass model for running has been conceptually analyzed. A runner has been reduced to a point of mass on a weightless spring, which alternates between the flight phase and the support phase. This model, known as the Spring-Loaded Inverted Pendulum (SLIP), may be thought of as a reductionist model of the essential mechanical properties of running. Consequently, an inverted elastic pendulum has been analyzed, with the attachment point situated at the foot-ground contact location. The physical description of motion during the support phase aligns with the mathematical one that can be expressed using dimensionless variables:

$$\begin{cases} L'' = L(\theta')^2 + K(1-L) - \cos\theta, \\ (L^2\theta')' = P' = L\sin\theta, \end{cases}$$

where L and θ represent the length of the leg and the angle between the leg and the vertical axis.

Analytically, this problem cannot be solved in a closed form. However, approximate solutions can be derived. The initial conditions are established, involving a landing with an attack angle of $-\alpha$ and a leg length of 1. The angular velocity θ_d and radial velocity $-L_d$ at the moment of landing are also fixed. By transitioning to a fast time scale, that is putting $\tau^+(t) = \epsilon^{-1}\widetilde{\omega}(\epsilon)t$, where $\widetilde{\omega}(\epsilon) = 1 - \theta_d^2/2\epsilon^2$ and utilizing perturbation theory, approximations for $\widetilde{L}(\tau^+)$ and $\widetilde{\theta}(\tau^+)$ are arrived at the moment of landing. The error terms for these approximations, in the fast time scale, are of order $O(\epsilon^3)$, which decrease for large K, where $\epsilon = 1/\sqrt{K}$.

Given the requirement that the leg should return to 1 during the contact phase, an approximation is obtained for the contact time of the foot with the ground, denoted as t_c, as well as the angle swept during the support phase, denoted as $\Delta\theta$. Subsequently, by solving the boundary-value problem We obtain two analytical approximation of the leg stiffness: $\widetilde{K_1}$ and $\widetilde{K_2}$. The $\widetilde{K_2}$ approximation is valid for a single boundary value problem, while $\widetilde{K_1}$ is an asymptotic approximation in the line of order of the dorect solution of boundary-value problem. The error of both approximations is compared to the numerically determined, exact value K^*. The approximation $\widetilde{K_2}$ is more accurate, with its error being twice as small for values of attack angles appearing in real running. Both approximations perform well for small values of α.

By utilizing certain assumptions, such as the conservation of energy and momentum and by appropriately concatenating asymptotic solutions for the two phases it is possible to reduce the dynamics to a one-dimensional apex to apex return map.

$$Y_{i+1}(Y_i) = \cos(\alpha - \Delta\theta_i) + \left[\sin(2\alpha - \Delta\theta_i)\sqrt{E_s - Y_i} + \cos(2\alpha - \Delta\theta_i)\sqrt{Y_i - \cos\alpha}\right]^2,$$

For this one-dimensional return mapping to have a fixed point, the model parameters must satisfy certain conditions. For instance, the system's energy must exceed the minimum energy, given by $(\theta_d^*)^2/2\cos^2\alpha + \cos\alpha$. It will turn out that fixed points exist only for symmetric solutions. However, in the case of asymmetry, these fixed points occur only when the angles at take-off are large and exceed the model's range of application.

The natural continuation is to investigate the stability of fixed points for symmetric support phases. The parameter sets for which stability conditions are met are derived from solving

$$|f'(Y_i)|_{Y_i = Y^*} < 1.$$

Which is expressed as constraints on derivative of $\Delta\theta$ at fix point.

Since such a constructed model exhibits both stable and unstable states, it is possible to proceed with the study of bifurcations. A numerical and analytical proof of the occurrence of a transcritical bifurcation in the previously constructed mapping has been presented. This means that conditions on the first and second derivatives at the hyperbolic fixed point are examined. For instance, it is necessary to verify that when passing through the bifurcation point, two branches of fixed points are obtained.

Mathematical modeling is a crucial aspect of biomechanics. Therefore, at the end of this study, an experimental approach is also employed. Based on the collected data, both approximations for K are calculated and

compared to the stiffness calculated directly as the ratio of spring - leg deformation to the ground reaction force. It turns out that also in this case, $\widetilde{K_2}$ is more accurate. After verifying the model's ability to estimate real values, we can move on to the study of running energetics. This involves comparing the energy generated by the runner with the minimum energy required for stable running, predicted by the model. This allows for an interesting conclusion that the runner aims to achieve stable running with minimal energy expenditure.