Summary of scientific achievements

I. Name and surname: Piotr Kowalczyk

II. The summary of the citations of dr Piotr Kowalczyk for the period 2002-2019 according to the database Web of Science. As recorded on the 26.02.2019 without self citations. (That is, if at least one of the co-authors cited a given work, this citation is excluded. Details of citations for each scientific work of dr Piotr Kowalczyk are shown in a separate document.)

- **Number of citation:** 1469
- **Number of cited works:** 27
- **Hirsch index:** 10*
- **Impact Factor (sum for all publications): 34.64**

*The Hirsch index is shown according to the database Web of Science (with self-citations, but only according to works cited in Web of Science).

III. Employments at academic institutions

1. From 19.02.2018 employed as a Senior Lecturer of Mathematics in the Department of Mathematics, in an Applied Mathematics Division, at Wroclaw University of Technology.
2. From 01.03.2011 to 31.01.2018: Full time position as a Senior Lecturer of Mathematics, School of Computing, Mathematics and Digital Technology, Manchester Metropolitan University: duties - research and dissemination of research results through publishing, conference attendance and organising of research workshops; teaching, mentoring and curriculum development
3. From 01.02.2008 to 28.02.2011: Full time position as a Research Associate in the Centre for Interdisciplinary Computational and Dynamical Analysis (CICADA) School of Mathematics, The University of Manchester
   Analysis of bifurcations in hybrid systems (systems modelled by sets of differential equations and discrete mappings).
4. From 01.01.2006 to 31.01.2008: Full time lecturer of Applied Mathematics in the School of Engineering, Computer Science and Mathematics, University of Exeter.
5. From 01.01.2003 to 31.12.2005: Full time position as a Research Assistant in the Department of Engineering Mathematics, University of Bristol Analysis and computation of bifurcations in piecewise-smooth (PWS) dynamical systems

IV. Academic Qualifications and Diplomas

1. 7.05.2003 Awarded a PhD degree in Applied Mathematics from the University of Bristol.
2. 7.04.2003 PhD Viva in the Department of Engineering Mathematics at the University of Bristol.
3. 01.01.2000 – 31.12.2002 Department of Engineering Mathematics, University of Bristol, United Kingdom. PhD in numerical and analytical investigation of sliding bifurcations in Filippov type systems
4. 01.10.1993 – 31.10.1999 Wroclaw University of Technology, Five years combined undergraduate and MSc course in Electronics and Telecommunication.

V. Below are listed achievements as required for degrees and honors in Sciences, and degrees and titles in Arts, described in article 16, legislation 2 from the 14th of March 2003 (65th book of legislation, position 595 with amendments).

(a) Title:

Theory and classification of one- and two-parameter Discontinuity Induced Bifurcations of limit cycles in Piecewise-smooth Dynamical Systems

(b) The list of publications which make up the scientific achievement:

[K1] P. Kowalczyk. A novel route to a Hopf-bifurcation scenario in switched systems with dead zone. *Physica D*, 348(8):60-66, 2017.

[K2] P. Glendinning, P. Kowalczyk and A. B. Nordmark. Multiple attractors in grazing-sliding bifurcations in Filippov type flows. *IMA Journal of Applied Mathematics*, 81(4):711-722, 2016.

[K3] P. Glendinning, P. Kowalczyk and A. B. Nordmark. Attractors near grazing-sliding bifurcations. *Nonlinearity*, 25(6):1867-1885, 2012.

[K4] M. di Bernardo, A. R. Champneys, C. Budd and P. Kowalczyk. Piecewise-smooth Dynamical Systems: Theory and Applications. *Springer*, 2008.

[K5] P. Kowalczyk and P. T. Piiroinen. Two-parameter sliding bifurcations of periodic solutions in a dry-friction oscillator. *Physica D*, 237(8):1053-1073, 2008.

[K6] P. Kowalczyk, M. di Bernardo, A. R. Champneys, S. J. Hogan, M. Homer, Yu. A. Kuznetsov, A. B. Nordmark and P. T. Piiroinen. Two-parameter nonsmooth bifurcations of limit cycles: classification and open problems. *Int. Journal of bifurcation and chaos*, 16(3):601-629, 2006.

[K7] A. B. Nordmark and P. Kowalczyk. A codimension-two scenario of sliding solutions in grazing-sliding bifurcations. *Nonlinearity*, 19(1):1-26, 2006.

[K8] P. Kowalczyk and M. di Bernardo. Two-parameter degenerate sliding bifurcations in Filippov systems. *Physica D*, 204:204 - 229, 2005.

(c) Short description of the scientific aim of the habilitation paper and obtained results:

Piecewise-smooth dynamical systems are mathematical models of systems of relevance to applications, which can be described by means of a combination of continuous and discrete evolution. Such a model formulation leads to the presence of discontinuities. The presence of discontinuities, in turn, may trigger qualitative changes in the structure of phase space under some parameter variations, which are not observed in differentiable systems. See for example [95, 11, 21, 18, 19, 17, 37, 99, 46, 33, 12, 13, 25, 98, 2, 26, 15, 20, 63, 64, 94, 51, 24, 29] among other works. These structural changes have been termed as discontinuity induced bifurcations, DIBs for short (see, for example [35, 34, 81] among other works). The work which I present here, which forms the basis of the habilitation paper, concerns the theory of local one- and two-parameter DIBs in piecewise-smooth dynamical systems (PWS). In Chapter 3 of Section VI., where the results which are the subject of this habilitation are presented, I start with the presentation of one-parameter normal form maps for sliding bifurcations, which were first derived in [KO1]. I rederived them using a different technique and presented in the monograph [K4]. There are still open problems for one-parameter DIBs. One of these concerns the classification of the dynamics around a given DIB. In [K2,K3], we present the classification of the dynamics in 3D Filippov type flows around a one-parameter *grazing-sliding* bifurcation. In particular, we show how one can reduce a 3D Filippov type flow to a one-dimensional discontinuous map locally around a grazing limit cycle, and explain bifurcations which lead to the creation of a multiple number of stable limit cycles emanating from a single limit cycle. Such a bifurcation cannot occur in differentiable vector fields. Another question related to one-parameter DIBs, raised in [K1], concerns the type of perturbations which may lead to a DIB. In particular, in [K1], I unfold the dynamics of a one-parameter DIB, different from all other cases of DIBs considered in [K2-K8], in that in [K1], it is the perturbation to the structure of the switching manifold which serves as a variation parameter. In so doing, I unfold a novel class of a supercritical Hopf-like bifurcation in a 3-zone planar Filippov type system.

The results presented in [K6] lay the foundations of the theory for two-parameter DIBs of PWS systems out, which is related to the classification strategy for co-dimension two bifurcations in smooth dynamical systems. One-parameter normal forms for DIBs are then used in [K5,K8] to unfold different cases of two-parameter sliding bifurcations of limit cycles in Filippov systems. In [K5] in [K7] I derived normal forms for other cases of two-parameter DIBs of limit cycles in piecewise-smooth systems (Filippov systems).

(d) My contribution to main results included in publications which make up the habilitation paper:

My contribution (in terms of percentage) to each of the articles, which have more than one author, is indicated in the enclosed documents, and is conformed by the statements from the co-authors. My main specialised contribution concerns DIBs of limit cycles in Filippov systems (it is Chapter 8 in the monograph [K4]). In particular, my entire contribution (unless specified otherwise in terms of percentage) includes:

- Theorem 1 - normal forms for one-parameter sliding bifurcations rederived for the monograph;
- Theorem 2 - normal form derivations for two-parameter degenerate crossing-sliding bifurcation;
- Theorem 3 - unfolding of the two-parameter grazing-sliding bifurcation of nonhyperbolic limit cycles (50% contribution);
- Theorem 4 - unfolding of the two-parameter bifurcation of a simultaneous occurrence of a grazing-sliding and an adding-sliding bifurcation;
- Theorems 5 i 6 - reduction of a three-dimensional Filippov flow to a one-dimensional mapping with discontinuities; classification of a one-parameter grazing-sliding bifurcations leading to multiple attractor bifurcations (40% contribution);
- Theorem 7 - the analysis of a novel Hopf-like bifurcation in a planar Filippov type system with three zones and a symmetry.

VI. A detailed description of the results which make up the habilitation paper as well as other scientific results is presented in the following sections. Description of scientific achievements which make up the habilitation paper: Sections 1-3. Description of other scientific achievements: Section 4.

1 Introduction

On the macroscopic scale, models of systems characterised by an interaction of continuous and discrete evolution - that is hybrid dynamical systems, or piecewise smooth dynamical systems (PWS) - abound in everyday life. To give just a few examples: an aircraft, whose position evolves continuously in time is controlled by microprocessors which operate on discrete inputs; similarly a power plant, a modern car, or a train all are controlled by microprocessors. Another example from outside of the field of engineering is growth and division of biological cells. Growth is a continuous process, but division is a discrete transition. In the control and mechanical engineering literature, the modelling and control design by means of systems with discontinuous nonlinearities has been used for already some decades, see for example [7, 4, 9, 10, 6, 1, 2, 79].

To understand the dynamics of differentiable systems, e.g. vector fields or maps, a highly successful tool, which is commonly used for such purpose, is numerical and analytical bifurcation analysis. Bifurcations, in the context of differentiable vector fields (smooth systems), give information on changes in the structural stability of systems as functions of system parameters, which translates onto changes in the number and stability of invariant sets, such as equilibrium points, limit cycles and chaotic attractors, see for example [54, 83, 62]. Even planar piecewise linear systems with two zones present rich dynamics, e.g. [46, 20], and so it comes as no surprise that much effort has been devoted to introduce some framework to systematically describe and analyse state space transitions in piecewise-smooth systems. It turns out that PWS systems, due to the presence of discontinuous nonlinearities, may exhibit structural changes, under some parameter variations, which do not occur in smooth systems, see for example [95, 11, 21, 18, 19, 17, 37, 99, 46, 33, 12, 13, 25, 98, 2, 26, 15, 20, 63, 64, 94, 51, 24, 29] among other works. These structural changes have been termed as discontinuity induced bifurcations, DIBs for short (see, for example [35, 34, 81] among other works). Discontinuity induced bifurcations may lead to an abrupt change in a system's behaviour. For instance, due to the so-called grazing-sliding bifurcation an abrupt transition from a stable oscillatory motion to a robust chaotic attractor may occur [KO2,KO3]. It was shown that a DIB may cause an emergence of multiple stable states from a single attractor [12, 77], which incidence can occur only in hybrid systems. The state space of systems of relevance to applications is, firstly, multidimensional [65, 100], and secondly, the structure of state space transitions may be understood better if two-parameter bifurcation diagrams are obtained, e.g [71, 76, 91, 73, 97, 59].

The work which I present here, which forms the habilitation document, concerns the theory of local one- and two-parameter DIBs in piecewise-smooth dynamical systems (PWS). It is important to point out that there is no generally agreed definition of a bifurcation or a co-dimension of a bifurcation for PWS systems. Hence, the term one- and two-parameter Discontinuity Induced Bifurcations will be used which often, but not always, may lead to the loss of stability of an invariant set involved in a DIB. For a definition of a DIB and different cases of one-parameter DIBs in planar Filippov systems, that is systems with discontinuous vector fields, see for example [63]. In Sec. 2, I present the definitions which spell out the different classes of nonsmooth systems as well as introduce the basic tools, the so-called Zero time Discontinuity Mapping (ZDM) and Poincaré Section Discontinuity mapping (PDM), which will be used to unfold one- and two-parameter DIBs of limit cycles in different classes of nonsmooth systems.

In Section 3, where the results which are the subject of this habilitaion are presented, I start with the presentation of one-parameter normal form maps for sliding bifurcations, which were first derived in [KO1]. I rederived them using a different technique and presented in the monograph [K4]. These normal forms are used in [K5,K7,K8] to unfold different cases of two-parameter sliding bifurcations of limit cycles in Filippov systems. The results presented in [K6] lay the foundations of the theory for two-parameter DIBs of PWS systems out, which is related to the classification strategy for co-dimension two bifurcations in smooth dynamical systems. Namely, we consider 1. a nondegenarcy in normal form maps for one-parameter DIBs, 2. nonhyperbolicity of a limit cycle exhibiting so-called grazing contact and 3. simultaneous occurrence of two independent grazing contacts. In [K6], numerical examples representing each distinct type of a two-parameter DIB is shown. Except for one scenario, the system models, which are presented, are models of mechanical oscillators. In [K7], we present a methodology for unfolding two-parameter DIBs of nonhyperbolic limit cycles exhibiting the so-called *grazing* contact (see Sec. 2). The results, spelled out in Theorem 3, on the existence of the bifurcation curves emanating from the co-dimension-two point in a two-parameter space, are given. The unfolding is then illustrated using a dry-friction oscillator model, which is an example of a Filippov system, where a two-parameter DIB of type 2 is unfolded and the dynamics around a co-dimension-two point, in a two-parameter space, is rigorously explained. The unfolding strategy presented in [K7] may be used to unfold other cases of DIBs of type 2 in different classes of PWS systems. In [K8], we present conditions for the four cases of two-parameter degenerate sliding bifurcations of limit cycles in Filippov systems. We also uncover a two-parameter degenerate crossing-sliding bifurcation in a model system, which is a Filippov type dry-friction oscillator model (however, other than the one in [K7]). The theoretical unfolding of the two-parameter degenerate crossing-sliding bifurcation is then carried out in [K5] by means of the ZDM mapping. The results are spelled out in Theorem 2. In [K5], we also present a theoretical unfolding of type 3 DIB of a simultaneous occurrence of two one-parameter sliding bifurcations. The theoretical results are again illustrated by numerically unfolding the dynamics around a co-dimension two point in two-parameter space in a Filippov type system of relevance to applications in mechanical engineering. It is important to point out that the framework for developing a theory of two-parameter DIBs, and the unfoldings which illustrate each distinct type of a two-parameter DIB as presented in [K6], allows one to develop efficient numerical schemes for two-parameter continuation of DIB curves, which is an important part of bifurcation theory, e.g. [41, 32, 31, 52, 82].

Setting out the framework for two-parameter DIBs as presented here does not imply that the theory of one-parameter DIBs is complete. This issue is taken up in [K5,K6,K7]. In particular, in [K2,K3], we present the classifcation of the dynamics in 3D Filippov type flows around a one-parameter *grazing-sliding* bifurcation. In particular, we show how one can reduce a 3D Filippov type flow to a one-dimensional discontinuous map locally around a grazing limit cycle, and explain bifurcations which lead to the creation of a multiple number of stable limit cycles emanating from a single limit cycle. Such a bifurcation cannot occur in differentiable vector fields. A first example of a Filippov type flow exhibiting this type of a grazing-sliding bifurcation is then constructed and shown in [K2]. The results in [K2,K3] suggest new research directions. In particular, by means of the ZDM a reduction of the system and the ensuing dynamics, locally about a bifurcating limit cycle, depends on both, the system dimension and its structure at the bifurcation. It then follows that one could attempt to classify DIBs from different classes of PWS systems having different dimensions depending on resulting dynamics. In particular, one may ask a question of developing a unified theory of one-parameter DIBs leading to multiple number of stable limit cycles born from one attractor.

Another question related to one-parameter DIBs, raised in [K1], concerns the type of perturbations which may lead to a DIB. In particular, in [K1], we unfold the dynamics of a one-parameter DIB, different from all other cases of DIBs considered in [K2-K8], in that in [K1], it is the perturbation to the structure of the switching manifold which serves as a variation parameter. In so doing, I unfold a novel class of a supercritical Hopf-like bifurcation in a 3-zone planar Filippov type system of relevance to modelling in control engineering and macroscopic modelling of neromuscular control. Again, one may ask a question whether there is a general theory of certain classes of PWS systems which imply the same type of a supercritical Hopf-like bifurcation. In particular in [K1], I also show numerically that the same type of a DIB takes place when a different perturbation such as an introduction of a time delay in the switching function produces equivalent supercritical Hopf-like bifurcation. Interestingly, these two qualitatively different types of perturbations lead to two different classes of PWS systems, but the dynamics observed under the perturbations appears equivalent. Hence, similarly as in the former case, one may attempt to develop a unified theory of supercritical one-parameter Hopf-like bifurcations in different classes of nonsmooth systems. For different types of Hopf-like bifurcations in Filippov systems, see for example [66, 101, 76, 5].

2 Piecewise-smooth systems of interest

2.1 Basic definitions

Definition 1 *A* **piecewise-smooth flow** *is given by a finite set of ODEs*

$$\dot{x} = F_i(x, \mu), \quad \textit{for} \quad x \in S_i, \tag{1}$$

where $\cup_i S_i = \mathcal{D} \subset \mathbb{R}^n$ *and each* S_i *has a non-empty interior. The intersection* $\Sigma_{ij} := \bar{S}_i \cap \bar{S}_j$ *is either an* $\mathbb{R}^{(n-1)}$ *dimensional manifold included in the boundaries* ∂S_j *and* ∂S_i*, or is the empty set. Each vector field* F_i *is sufficiently smooth in both - state* x *and parameter* μ*, and it defines a corresponding smooth flow, say* $\Phi_i(x, t)$*, within any open set* S_i*. However, each flow* Φ_i *is well-defined in all of* $\mathcal{D}$*.*

A non-empty border between two regions Σ_{ij} will be called a **discontinuity set**, **discontinuity boundary** or, sometimes, a **switching manifold**. We suppose that each piece of Σ_{ij} is of codimension-one, i.e. it is an $(n-1)$-dimensional smooth *manifold* (something locally diffeomorphic to $\mathbb{R}^{n-1}$) embedded within the n-dimensional phase space. Moreover, we demand that each such Σ_{ij} is itself piecewise-smooth. That is, it is composed of finitely many pieces that are as smooth as the flow.

Definition 2 *The* **degree of discontinuity** *at some point* x *in switching set* Σ_{ij} *of a piecewise-smooth ODE is the order of the first non-zero partial derivative with respect to* t *of the difference between flows* $\Phi_i(x, t) - \Phi_j(x, t)$ *evaluated at* $t = 0$*.*

Now, consider an ODE locally to a single discontinuity set Σ_{12} that can be written as

$$\dot{x} = \begin{cases} F_1(x, \mu), & x \in S_1, \\ F_2(x, \mu), & x \in S_2, \end{cases} \tag{2}$$

where F_1 generates flow Φ_1 and F_2 flow Φ_2. We have

$$\left. \frac{\partial \Phi_i(x, t)}{\partial t} \right|_{t=0} = F_i(x)$$

$$\left. \frac{\partial^2 \Phi_i(x, t)}{\partial t^2} \right|_{t=0} = \frac{\partial F_i}{\partial t} = \frac{\partial F_i}{\partial \Phi_i} \frac{\partial \Phi_i}{\partial t} = F_{i,x} F_i(x),$$

where subscript x means partial differentiation with respect to x. Similarly

$$\left. \frac{\partial^3 \Phi_i(x, t)}{\partial t^3} \right|_{t=0} = F_{i,xx} F_i^2 + F_{i,x}^2 F_i,$$

etc. So, if vector fields F_1 and F_2 differ in an mth partial derivative with respect to state x, we find that flows Φ_1 and Φ_2 differ in their $(m+1)$st partial derivative with respect to t.

Therefore if $F_1(x) \neq F_2(x)$ we have the degree of discontinuity 1 at point $x \in \Sigma_{12}$. Systems with discontinuity degree 1 are said to be of *Filippov* type.

Definition 3 *A discontinuity boundary* Σ_{ij} *is said to be* **uniformly discontinuous** *in some domain* $\mathcal{D}$ *if the degree of discontinuity of the system is the same for all points* $x \in \Sigma_{ij} \cap \mathcal{D}$*. Furthermore, we say that the discontinuity is* **uniform with degree** $\boldsymbol{m}$ *if the first non-zero partial derivative of* $F_i - F_j$ *evaluated on* Σ_{ij} *is of order* $m-1$*. Furthermore the degree of discontinuity is 1 if* $F_i(x) - F_j(x) \neq 0$ *for* $x \in \Sigma_{ij} \cap \mathcal{D}$*.*

2.2 Filippov systems

The case of systems with uniform degree of discontinuity 1 must be treated with great care since we have to allow the possibility of sliding motion. In order to define sliding, it is useful to think of a piecewise-smooth system local to a single discontinuity boundary between two regions defined by the zero set of some smooth function, say $H(x) = 0$. Specifically, consider now an ODE (2) defined in some region $D \subset \mathbb{R}^n$, which is charcterised by a single discontinuity set, say $\Sigma_{12} = \Sigma$. That is, we have

$$\dot{x} = \begin{cases} F_1(x,\mu), & H(x,\mu) > 0, \\ F_2(x,\mu), & H(x,\mu) < 0, \end{cases} \tag{3}$$

where F_1 generates flow Φ_1 and F_2 flow Φ_2; F_1, F_2 are sufficiently smooth vector functions, and $H(x,\mu)$ is some smooth scalar function, which depend on the system's state $x \in \mathbb{R}^n$ and parameter $\mu \in \mathbb{R}^m$. Region D is split into two subspaces, S_1 and S_2, in which the dynamics is smooth and continuous. We assume that the discontinuity boundary Σ, between S_1 and S_2, is a smooth hyperplane, so that

$$S_1 := \{x \in \mathbb{R}^n : H(x,\mu) > 0\}, \tag{4}$$

$$S_2 := \{x \in \mathbb{R}^n : H(x,\mu) < 0\}, \tag{5}$$

$$\Sigma := \{x \in \mathbb{R}^n : H(x,\mu) = 0\}. \tag{6}$$

The resulting topology admits the possibility of evolution on Σ. That is, depending on the direction of vector fields F_i $(i = 1, 2)$ with respect to Σ, one may construct the system's flow either by concatenating the flow solutions (when the vector fields point in the same direction with respect to Σ) or, in the case when the vector fields point in the opposite direction with respect to Σ, the system's evolution will take place on Σ. In the latter case, a definition of the vector field governing the flow on Σ is required. Define the directional derivative of $H(x)$ in some vector field F as H_xF. Then, $\forall x \in \Sigma$ where the product $(H_xF_1)(H_xF_2) > 0$, the flow solution of Filippov system (3) switches between vector fields F_1 and F_2 upon reaching Σ. On the other hand, $\forall x \in \Sigma$ where the product $(H_xF_1)(H_xF_2) < 0$, the systems's flow follows evolution on Σ. There are different formalisms which allow one to define the flow - termed as *sliding flow* - on Σ. The vector field generating the sliding flow will be termed as *sliding vector field*. We will use so-called Filippov's convex method to define the sliding vector field. Filippov's method takes a convex combination of the two vector fields

$$F_{12} = (1-\alpha)F_1 + \alpha F_2 \tag{7}$$

with $0 \leq \alpha \leq 1$, where

$$\alpha(x) = -\frac{H_xF_1}{H_x(F_2 - F_1)}. \tag{8}$$

Sometimes, when there is no ambiguity, we will write

$$F_{ij} := F_s$$

to represent the sliding vector field.

Using α we may define a region on Σ where sliding is possible. Namely

$$\hat{\Sigma} := \{x \in \Sigma : 0 < \alpha(x) < 1\}$$

defines the sliding region. We may define the boundaries of the sliding region as

$$\partial\hat{\Sigma}^+ := \{x \in \Sigma : \alpha(x) = 1\},$$

and

$$\partial\hat{\Sigma}^- := \{x \in \Sigma : \alpha(x) = 0\}.$$

We assume that on $\hat{\Sigma} \cup \partial\hat{\Sigma}^+ \cup \partial\hat{\Sigma}^-$, $H_x(F_2 - F_1) > 0$, which implies the existence of so-called *attracting sliding*. That is both vector fields, F_1 and F_2, for x in a sufficiently small neighbourhood of $\hat{\Sigma}$ point toward Σ.

2.3 Hybrid systems

Definition 4 *A* **piecewise-smooth hybrid system** *comprises a set of ODEs*

$$\dot{x} = F_i(x, \mu), \quad \text{if} \quad x \in S_i, \tag{9}$$

plus a set of reset maps

$$x \mapsto R_{ij}(x, \mu), \quad \text{if} \quad x \in \Sigma_{ij} := \bar{S}_i \cap \bar{S}_j. \tag{10}$$

Here $\cup_i S_i = \mathcal{D} \subset \mathbb{R}^n$ *and each* S_i *has a non-empty interior. Each* Σ_{ij} *is either an* $\mathbb{R}^{(n-1)}$*-dimensional manifold included in the boundary* ∂S_j *and* ∂S_i*, or is the empty set. Each* F_i *and* R_{ij} *are assumed to be smooth and well defined in open neighborhoods around* S_i *and* Σ_{ij} *respectively.*

A special type of hybrid systems, in the context of mechanical engineering, is used to describe so-called impact oscillators. In such models surfaces Σ_{ij} act as hard constraints, so that the resets map points in Σ_{ij} back to itself.

Definition 5 *An* **impacting hybrid system** *is a piecewise-smooth hybrid system for which* $R_{ij} : \Sigma_{ij} \to \Sigma_{ij}$*, and the flow is constrained locally to lie on one side of the boundary, that is in* $\overline{S}_i = S_i \cup \Sigma_{ij}$*.*

We often refer to the reset map R_{ij} in this context as being the *impact law* or *impact rule*. The discontinuity boundaries Σ_{ij} are referred to as *impact surfaces* and the event of a trajectory intersecting Σ_{ij} as an *impacting event* or just an *impact*.

2.4 Discontinuity map

Definition 6 *The* **discontinuity map** Q *for the transverse crossing of a discontinuity set* Σ_{ij} *in a piecewise smooth flow (or hybrid system) is the extra mapping that the flow maps* Φ_i *and* Φ_j *must be composed with in order to get a description of the piecewise smooth (hybrid) flow. Thus if* Σ *is crossed in the sense of passing from region* S_i *to* S_j *say, the correct flow map is* $\Phi_j \circ Q \circ \Phi_i$*. The Jacobian derivative* Q_x *of* Q *is called the* **Saltation matrix***.*

Example 1 *Crossing a two-zone Filippov system outside the sliding set. For a trajectory in a Filippov system where crossing of the switching manifold takes place outside of the sliding region, from region* S_1 *to* S_2*, the saltation matrix is given by the expression*

$$Q_x = I + \frac{(F_2 - F_1)H_x}{H_x F_1}, \tag{11}$$

where I *is the identity matrix. This expression was first derived in [3].*

Example 2 *Crossing a two-zone Filippov system within the sliding set. Saltation matrices also apply to trajectories of Filippov systems that undergo a transition into sliding. The saltation matrix for the case of switching from* S_1 *into sliding region* $\hat{\Sigma}$ *is given by*

$$I + \frac{(F_{12} - F_1)H_x}{H_x F_1},$$

where F_{12} *is the sliding vector field defined by (7).*

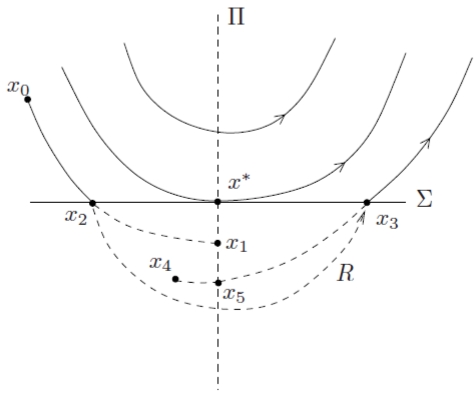

Figure 1: A local illustration of the ZDM and PDM close to a grazing contact. In this figure the solid line represents the actual flow of the hybrid system, and the dashed line the extended flow. The ZDM is the map $x_1 \mapsto x_4$ and the PDM is the map $x_1 \to x_5$.

2.5 Non-transversal (grazing) intersections

The discontinuity mappings in examples 1 and 2 were derived under the condition of tranversality, namely that $H_x F_1 \neq 0$. So-called *grazing* occurs when a trajectory becomes tangent to the discontinuity surface Σ. This occurs precisely when $H_x F_1 \neq 0$ is violated, that is when $v = H_x(x^*)F_1(x^*) = 0$ (see later Definition 10).

We illustrate the situation close to a grazing for a general hybrid system in Fig. 1. In this figure, we show a distinguished trajectory (which we suppose to be part of a periodic orbit) locally lying entirely in S_1. This trajectory we assume to graze with the discontinuity boundary Σ at the point x^* at time t^*.

To construct the discontinuity mapping, in the case of grazing contact, we need to know the fate of two different types of trajectories with initial conditions close to x^*. Some trajectories do not cross Σ locally, for these, the discontinuity mapping is the identity. In contrast, the discontinuity mapping will be nontrivial for the trajectory illustrated in Fig. 1 that passes through the point x_0 close to Σ at time t_0, hits Σ at the point x_2 at time $t_2 = t^* + \delta$, is mapped to the point x_3 by the map $\Phi_2(R(x_2), t_3 - t_2)$ and continues in S_1 from this point. Note that we allow here for both the impacting hybrid system case, in which Φ_2 is the identity, or the piecewise smooth flow case, where R is the identity. In the latter case, $t_3 - t_2$ is the time of flight of the trajectory until its second impact with Σ.

We describe two different ways of defining the nontrivial part of the discontinuity map. These are constructed either, like the DM for transversal trajectories defined above, such that the total elapsed time is zero — a so-called *zero-time discontinuity mapping (ZDM)* — or are defined with respect to a local *Poincaré* section — a *Poincaré-section discontinuity mapping (PDM)*. Our treatment is inspired by the analysis of n-dimensional impacting systems by [45], which extends earlier results in [84, 96, 75, 55]. Both these constructions allow to reformulate the problem of finding the local map which encapsulates the effect of grazing contact on system trajectories, sufficiently close to the grazing trajectory, using a suitably chosen variables which measure the clossness to grazing trajectory.

To explain the difference between the ZDM and the PDM, consider in more detail the trajectory in Fig. 1 that passes through x_0. It intersects Σ at x_2, is mapped to x_3 where it subsequently evolves back to starting point x_0. By extending the smooth vector field $F_1(x)$ defined in the region $H(x) > 0$ (which is the region above Σ in S_1) to the region $H(x) < 0$ (so that the trajectories may be extened below Σ) we may continue the trajectory forward from x_2 under the action of the flow map Φ_1, or equally backward from x_3. As the point x_0 is close to x^*, then the smooth trajectory carried forward from x_2 under the action of Φ_1 will intersect the Poincaré surface

$$\Pi_N = \{x : v = H_x(x)F_1(x) = 0\}$$

at a point x_1 close to $x^* = 0$. Similarly, the backward continuation of the flow from x_3 will intersect the set Π_N at the point x_5. The mapping that takes x_1 to x_5 is the Poincaré Discontinuity Mapping (PDM). Such a reformulation of the problem allows us to use the Implicit Function Theorem to find the functional form for a PDM which depends on the local properties of the vector fields with respect to the switching surface Σ.

Definition 7 *The* **Poincaré-section discontinuity mapping (PDM)** *near a grazing orbit is the discontinuity mapping defined on a suitable surface Π_N transversal to the flow, which contains the grazing set and intersects Σ transversally, that takes initial conditions on Π_N back to themselves. There is no requirement that this map takes zero time.*

The same trajectory starting from x_2 can also be continued forward to x_1 and then backward from x_3 for the same time as it takes to get from x_2 to x_1. Say that the time of flight from x_2 to x_1 under the action of the flow Φ_1 is δ. Then we flow from x_3 using Φ_1 by $-\delta$ until a point x_4 is reached. We then define the ZDM as the map from x_0 to x_4.

Definition 8 *The* **zero-time discontinuity mapping (ZDM)** *near a grazing orbit is the discontinuity mapping in a neighborhood of the grazing point, say x^*, which takes zero time. That is, when this map is composed with the flow map of the non-impacting/non-switching system in order to define a trajectory of the full system, the time taken is the same as for the flow map alone.*

The zero time condition allows the ZDM to be incorporated into a natural way as part of the calculation of a fixed time-T Poincaré map, say P_S, sometimes called a *stroboscopic map.* For example, for a grazing periodic orbit that is contained entirely within region S_1, the stroboscopic map can be written as

$$P_S = P_2 o ZDM o P_1,$$

where P_1 describes the evolution with flow Φ_1 through time t_1 and P_2 describes the evolution with flow Φ_1 through time $T - t_1$.

The PDM may be preferable to use as an analytical tool for studying behaviour of grazing limit cycles in autonomous systems. In contrast, it is natural to apply the ZDM for $T-$periodically forced nonautonomous ones.

The leading order terms of the ZDM and PDM, generically, have the same power, but the PDM correction takes non-zero time.

To decribe the dynamics of limit cycles (periodic solutions) exhibiting grazing incidence, locally about grazing, the ZDM or PDM map is composed with an affine map which describes the dynamics of a grazing limit cycle ignoring the presence of grazing. Without loss of generality, we assume that the grazing limit cycle is hyperbolic if grazing contact is ignored.

To simplify the presentation of the results in the following section, we will make use of operator notation. For this purpose let us introduce the Lie derivative.

Definition 9 *The* Lie derivative *is the total derivative of some smooth scalar function h along the direction of the flow governed by some vector field f. Specifically, if f and g are smooth vector fields and h is a smooth scalar function, then we have*

$$\begin{aligned}
\mathcal{L}_f h(x) &= \frac{\partial h}{\partial x} f(x), \\
\mathcal{L}_g \mathcal{L}_f h(x) &= \frac{\partial(\mathcal{L}_f h)}{\partial x} g(x), \\
\mathcal{L}_g \mathcal{L}_f^k h(x) &= \frac{\partial(\mathcal{L}_f^{k-1} h)}{\partial x} g(x).
\end{aligned}$$

For example, using the already introduced notation, we have

$$\mathcal{L}_f h(x) = h_x f(x),$$

and

$$\mathcal{L}_f^2 h(x) = (h_x f(x))_x f(x).$$

3 Description of the results

3.1 One- and two-parameter Discontinuity Induced Bifurcations

A bulk of the results, which we present here, concerns state space transitions which we will term as *Discontinuity Induced Bifurcations* (DIBs for short). DIBs involve a non-trivial interaction between a limit cycle and a switching manifold.

Let us consider a two-zone Filippov system. Let us suppose that there exists a limit cycle that lives either in one of the subspaces S_1 or S_2, or is built of distinct segments generated by vector fields F_1, F_2 and F_s. In the latter case the cycle includes switchings between S_1, S_2 and $\hat{\Sigma}$. Further, let us assume that the points at which switchings occur depend on some parameter μ. Variation of this parameter may cause that an intersection point reaches one of the boundaries $\partial\hat{\Sigma}^{\pm}$. If the parameter μ is varied further the intersection point might move onto $\hat{\Sigma}$ (or outside of it). This scenario describes a one parameter event. Consider some point x where the system switches between two vector fields. Call the flow approaching the point of switching, say x, as the incoming flow, and the flow which generates the trajectory after switching occurs as the outgoing flow. If at x either of the two vector fields generating the outgoing or incoming flow is tangent to Σ then we say that the system exhibits grazing contact at x.

Definition 10 *If at some point x, one of the two vector fields which generate an incoming or an outgoing flow is tangent to Σ, or $\partial\hat{\Sigma}^{\pm}$ when the evolution containing the tangent point is on Σ, we then say that at point x the Filippov system is characterised by the* grazing *contact with the discontinuity set or with the boundary of the sliding region.*

Definition 11 *Suppose that there exists a hyperbolic limit cycle $L(x, \mu)$ in the Filippov system (3), where $\mu \in \mathbb{R}$ is a parameter and x is a point on the limit cycle. If for some μ^* a limit cycle L^* at $x^* = x(\mu^*)$ exhibits a single grazing contact with the boundary of the sliding region $\partial\hat{\Sigma}^-$, and the μ-dependence on x^* is nondegenerate, that is $\langle \frac{dx}{d\mu}, H_x\rangle(x^*, \mu^*) \neq 0$ and $\langle \frac{dx}{d\mu}, (H_x F_1)_x\rangle(x^*, \mu^*) \neq 0$, we say that the limit cycle undergoes a one-parameter Discontinuity Induced Bifurcation of a sliding type, where $\langle , \rangle$ denotes the dot product.*

We should remark here that in phase space the dot product is not defined, but since the theory presented here is local, we consider the dot product on the tangent space. Also, in all the definitions of one and two parameter DIBs presented here, to ensure that μ is an appropriate unfolding parameter, without loss of generality, we assume that in the local coordinates $x^* = 0$.

3.1.1 Normal form maps for one-parameter sliding bifurcations

We start by giving analytical conditions which define each of the four one-parameter sliding bifurcation scenarios, along with appropriate non-degeneracy assumptions. In all four cases, the critical trajectory involved in the bifurcation event has a point of intersection with the boundary of the sliding region $\partial\hat{\Sigma}^-$. Suppose this point of intersection occurs at $x = x^*(\mu^*)$ (where μ is the bifurcation parameter), then in all four cases we have the following *defining* conditions

$$H(x^*) = 0, \qquad H_x(x^*) \neq 0, \tag{12}$$

$$\alpha(x^*) = 0 \quad \text{(which implies} \quad F_s(x^*) = F_1(x^*) \quad \text{and} \quad \mathcal{L}_{F_1} H(x^*) = 0). \tag{13}$$

The first condition (12) states that the point x^* belongs to the switching manifold which is well defined; whereas the second condition (13) states that x^* is on the boundary of the sliding region (which without loss of generality we assume to be $\partial\hat{\Sigma}^-$).

Now let us turn to *non-degeneracy* conditions for each of the four sliding bifurcations. The first is that in a neighbourhood of x^*, the vector field F_2 is not grazing and points towards Σ. That is

$$H_x F_d(x^*) > 0, \tag{14}$$

where $F_d^* = F_2^* - F_1^*$. Other considerations involve the tangency of the sliding flow with respect to the boundary $\partial\hat{\Sigma}^-$.

The crossing-sliding and grazing-sliding cases require the sliding flow to evolve locally *towards* $\partial\hat{\Sigma}^-$ (see Fig. 2). Hence we require

$$\left.\frac{\partial\alpha(\Phi_s(x^*, 0))}{\partial t}\right|_{t=0} < 0.$$

Where Φ_s is the flow operator corresponding to the sliding flow generated by the vector field F_s. However, we have that $F_s = F_1$ at x^* by (13), hence $\Phi_s(x^*, 0) = \Phi_1(x^*, 0)$. Moreover

$$\frac{\partial\alpha(\Phi_1(x^*, 0))}{\partial t} = \alpha_x F_1(x^*) =: \mathcal{L}_{F_1}\alpha(x^*)$$

Therefore the sign of $\mathcal{L}_{F_1}\alpha(x^*)$ determines whether the boundary $\partial\hat{\Sigma}^-$ is attracting or repelling with respect to the sliding flow. Crossing-sliding and grazing-sliding will therefore require the non-degeneracy condition

$$\mathcal{L}_{F_1}\alpha(x^*) < 0, \tag{15}$$

whereas switching-sliding (top sketch in Fig. 3) requires

$$\mathcal{L}_{F_1}\alpha(x^*) > 0, \tag{16}$$

so that the sliding flow points *away* from the boundary.

Adding-sliding (bottom case in Fig. 3) is more subtle. Here we require an additional *defining condition* that there is a point of tangency of the sliding flow with $\partial\hat{\Sigma}^-$ at the bifurcation point. That is

$$\mathcal{L}_{F_1}\alpha(x^*) = 0. \tag{17}$$

Moreover, the sliding flow must reach a local minimum of α at the bifurcation point. Hence, we also require

$$\frac{\partial^2\alpha(\Phi_s(x^*, 0))}{\partial^2 t} > 0,$$

that is

$$\mathcal{L}^2_{F_1}\alpha(x) := \alpha_x F_{1x} F_1 + \alpha_{xx} F_1^2 > 0. \tag{18}$$

We can now state the following Theorem on the form of the ZDM at each of the four sliding bifurcations. The ZDM in this case describes the correction that must be made to trajectories near grazing trajectory b (see Fig. 2 and 3) in order to account for the acquisition (or loss) of an additional short segment making up the limit cycle for μ sufficiently close to μ^*.

Theorem 1 *Suppose a piecewise smooth system of the form (3) undergoes a sliding bifurcation at point x^*, defined by the conditions (12) and (13) under the nondegeneracy assumption (14). Then we have the following four cases:*

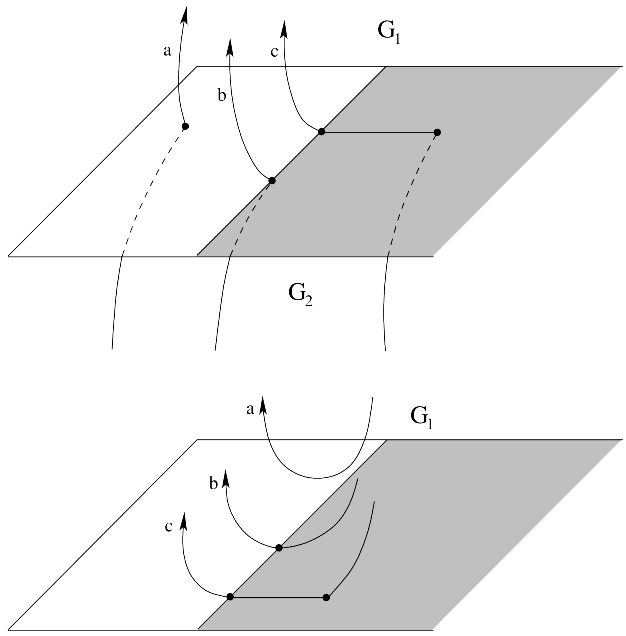

Figure 2: A schematic representation of the crossing-sliding (top) and grazing-sliding (bottom) bifurcations.

crossing-sliding; *under the additional nondegeneracy condition (15) the ZDM for trajectories starting in a sufficiently small neighbourhood of* x^* *may be written as*

$$x \mapsto \begin{cases} x & \text{if } \mathcal{L}_{F_1}H(x) \leq 0, \\ x + (\mathcal{L}_{F_1}H(x))^2 \dfrac{F_2(x) - F_1(x)}{2\mathcal{L}_{F_2}H(x)\mathcal{L}^2_{F_1}H(x)} + O((x-x^*)^3) & \text{if } \mathcal{L}_{F_1}H(x) > 0; \end{cases} \tag{19}$$

grazing-sliding; *also under the additional non-degeneracy condition (15) the ZDM for trajectories starting in a sufficiently small neighbourhood of* x^* *may be written as*

$$x \mapsto \begin{cases} x & \text{if } H(x) \geq 0, \\ x - \dfrac{H(x)(F_2(x) - F_1(x))}{\mathcal{L}_{F_2}H(x)} + O((x-x^*)^{3/2}) & \text{if } H(x) < 0; \end{cases} \tag{20}$$

switching-sliding; *under the additional non-degeneracy assumption (16), the ZDM for trajectories starting in a sufficiently small neighbourhood of* x^* *may be written as*

$$x \mapsto \begin{cases} x & \text{if } \mathcal{L}_{F_1}H(x) \leq 0, \\ x + \dfrac{2}{3} \dfrac{(\mathcal{L}_{F_1}H(x))^3}{(\mathcal{L}_{F_2}H(x))^2(\mathcal{L}^2_{F_1}H(x))^2} Q + O((x-x^*)^4) & \text{if } \mathcal{L}_{F_1}H(x) > 0, \end{cases} \tag{21}$$

where

$$Q = \mathcal{L}_{F_2}H(x)(F_{1x}F_d - F_{dx}F_1) - \mathcal{L}_{(F_{1x}F_d - F_{dx}F_1)}H(x)F_d,$$

and $F_d = F_2 - F_1$*;*

adding-sliding; *under the additional defining conditino (17) and the non-dengeneracy assumption (18), the ZDM for trajectories starting on* $\hat{\Sigma}$ *in a sufficiently small neighbourhood of* x^* *may be written as*

$$x \mapsto \begin{cases} x & \text{if } \mathcal{L}_{F_1}H(x) \geq 0, \\ x - \dfrac{9}{2} \dfrac{(\mathcal{L}_{F_1}H(x))^2}{(\mathcal{L}_{F_2}H(x))^2\mathcal{L}^3_{F_1}H(x)} Q + O((x-x^*)^{5/2}) & \text{if } \mathcal{L}_{F_1}H(x) < 0, \end{cases} \tag{22}$$

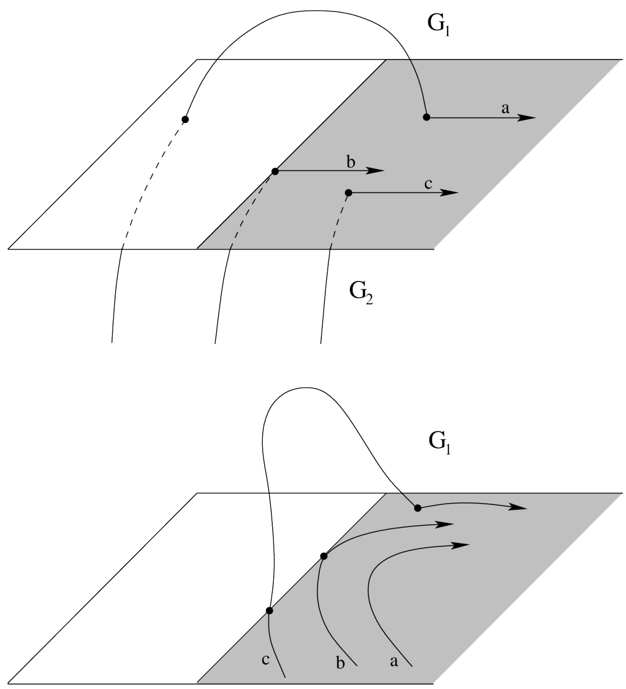

Figure 3: A schematic representation of the switching-sliding (top) and adding-sliding (bottom) bifurcations.

with Q defined as above.

The dynamics of a Filippov system, locally about a limit cycle undergoing a one-parameter sliding bifurcation, may be understood by investigating the dynamics of the related piecewise-smooth ZDM. The type of discontinuity for corresponding PDMs is the same as for the given ZDMs. The functional forms for the PDM for one-parameter sliding bifurcations have been also derived and are presented together with the proofs in [K4]. Later we will show how one can use the ZDMs presented here to unfold two parameter sliding bifurcations in Filippov systems. We will also present the application of the ZDMs/PDMs as a tool for the classification of the dynamics near a sliding bifurcation.

3.2 Two-parameter Discontinuity Induced Bifurcations of limit cycles

3.2.1 Classification

In [K6], we propose that the possible two parameter Discontinuity Induced Bifurcatios of limit cycles of periodic orbits in piecewise-smooth flows can be put into one of the following three types:

Type I **Degenerate DIBs**; i.e. there is a degeneracy of one of the analytical conditions determining the properties of the vector fields local to the grazing event. This is likely to influence the leading order term of the normal form map derived via the discontinuity mapping.

Type II **DIBs of non-hyperbolic cycles**, i.e. bifurcations where the linear part of the PWS normal form map capturing the dynamics of the limit cycle undergoing the bifurcation contains a degeneracy; that is, the cycle is non-hyperbolic. This scenario can be seen as a combination of a smooth and a non-smooth bifurcation occurring at the same parameter values;

Type III **Simultaneous occurrence of two one parameter DIBs** at two different points along the bifurcating orbit.

3.2.2 Two-parameter degenerate crossing-sliding bifurcation. DIB of Type I

We will first introduce two parameter sliding bifurcations which arise as a result of degeneracy of conditions (15), or (16) or (18). Thus, we consider additional degeneracy of the outgoing flow only as this is the flow which exhibits grazing contact in the case of sliding bifurcations. In the case when the incoming flow is generated by the vector fields F_1 or F_s the incoming and outgoing flows have the same properties with respect to Σ and $\partial\hat{\Sigma}$ at the bifurcation point. The results which we present here are expounded in [K3,K4].

First of the degenerate two-parameter sliding bifurcations which we consider is the degenerate crossing-sliding scenario. Degeneracy of the outgoing flow Φ_1 in this case implies additional tangency to the the boundary of the sliding region. We also expect the curvature of the vector field such that the flow Φ_1 leaves the switching manifold. Thus, we arrive at the following set of analytical conditions. Obviously conditions (12) and (13) must hold in this case as well.

Additional tangency of the outgoing flow is expressed by the conidtion

$$\left.\frac{\partial(\alpha(\Phi_1(x,t)))}{\partial t}\right|_{t=0} = \alpha_x F_1 = 0.$$

Expressing condition above in terms of H_x and F_1 yields

$$\left.\frac{\partial^2(H(\Phi_1(x,t)))}{\partial t^2}\right|_{t=0} = H_x F_{1x} F_1 = 0. \tag{23}$$

Non-degeneracy condition for the degenerate crossing-sliding Since we require the trajectory to leave the switching manifold, the vector field F_1 should exhibit local maximum with respect to $\partial\hat{\Sigma}^-$. Thus, the non-degeneracy condition can be written as

$$\left.\frac{\partial^2(\alpha(\Phi_1(x,t)))}{\partial t^2}\right|_{t=0} = \alpha_x F_{1x} F_1 < 0.$$

Expressing the condition above in terms of H_x, F_1 gives

$$\left.\frac{\partial^3(\alpha(H(\Phi_1(x,t)))}{\partial t^3}\right|_{t=0} = H_x (F_{1x})^2 F_1 > 0. \tag{24}$$

Definition 12 *Suppose that there exists a hyperbolic limit cycle $L(x,\mu)$ in the Filippov system (3), where $\mu = (\mu_1,\mu_2) \in \mathbb{R}^2$ is a parameter and x is a point on the limit cycle. If for some μ^* a limit cycle L^* at $x^* = x(\mu^*)$ exhibits a single degenerate grazing contact with the boundary of the sliding region $\partial\hat{\Sigma}^-$, and the μ-dependence on x^* is nondegenerate, that is $\langle \frac{\partial x}{\partial \mu_i}, H_x\rangle(x^*,\mu^*) \neq 0$, $\langle \frac{\partial x}{\partial \mu_i}, (H_x F_1)_x\rangle(x^*,\mu^*) \neq 0$ for $i=1,2$, and $\frac{\partial x}{\partial \mu_1}(x^*,\mu^*)$ and $\frac{\partial x}{\partial \mu_2}(x^*,\mu^*)$ are linearly independent, we say that the limit cycle undergoes a two-parameter degenerate Discontinuity Induced Bifurcation of a sliding type.*

Theorem 2 *Assuming that a Filippov system (3) exhibits a two-parameter degenerate crossing-sliding bifurcation and so it satisfies the defining conditions (12) and (13), the condition (23) for the degenerate grazing contact and the non-degeneracy condition (24), a local normal form map which captures the dynamics of the system in a sufficiently small neighbourhood of the co-dimension two point, to leading order, has the form:*

$$ZDM(x,y,z) = \begin{cases} x & \text{for} \quad V_{min} \geq 0,\ q_{min} \geq 0, \\ x - z^3 \dfrac{2}{\mathcal{L}_{F_d} H(x) \mathcal{L}^3_{F_1} H(x)} F_d + z^3 \mathcal{O}(z) & \text{for} \quad V_{min} < 0,\ q_{min} \leq 0, \\ x - 3y^2 \dfrac{\mathcal{L}^2_{F_1} H(x)}{\mathcal{L}^3_{F_1} H(x) \mathcal{L}_{F_d} H(x)} F_d + y^2 \mathcal{O}(y) & \text{for} \quad V_{min} > 0,\ q_{min} < 0, \end{cases} \tag{25}$$

where

$$q(x,t) = \frac{H(\phi_1(x,t)) - H(x)}{t}$$

and

$$y^2 + q_{min} = 0;$$

q_{min} *denotes a minimum value of the function* q *attained along a trajectory generated by the flow* ϕ_1 *starting at some* x*. In our case* q *is a small quantity. The variable* z *is given by*

$$z^2 + V_{min} = 0,$$

where $V_{min} = \mathcal{L}_{F_1} H(x)$ *on*

$$\{x \in \Sigma \, : \, \mathcal{L}_{F_s} \mathcal{L}_{F_1} H(x) = 0\}.$$

We will elucidate the effect of the ZDM on the existence and stability of the critical limit cycle exhibiting a degenrate crossing sliding contact at $x^*(\mu^*)$. We want to consider the existence and stability $\forall(x,\mu)$ in a sufficiently small neighbourhood of $x^*(\mu^*)$. Let us denote by

$$v = \mathcal{L}_{F_1} H(x), a = \mathcal{L}^2_{F_1} H(x) \text{ and } c = \mathcal{L}^3_{F_1} H(x),$$

where v and a are small quantities and $c = \mathcal{O}(1)$, $\forall(x,\mu)$ in some sufficiently small neighbourhood of $x^*(\mu^*)$. We also use the $\mathcal{O}$ symbol to denote all of the higher order terms starting with the order indicated by the variable in the bracket. In the case of several variables, where the products of small quantities occur, we alternatively specify the order of the next term (and all of the higher order terms) in the expression using $\mathcal{O}(\varepsilon)$ symbol.

Consider now Fig. 4. In the figure T_0 refers to the part of the critical cycle that exhibits the degenerate contact with the boundary of the sliding region $\partial\hat{\Sigma}^-$. The trajectories that are rooted in Σ, in a sufficiently small neighbourhood of x^*, in the region labelled as R_1 in the figure, do not interact with the sliding region (see trajectory T_1), and thus no correction is applied to these trajectories by the ZDM. The Implicit Function Theorem guarantees the existence and the stability type corresponding to that of the degenerate cycle for a limit cycle rooted in R_1 in a sufficiently small neighbourhood of the crossing sliding cycle.

Trajectories that reach the region labelled as R_3 in the figure (cf. T_3) follow the sliding flow for some small time, say s, and then leave the switching surface. For these trajectories the correction term of the ZDM map is of $\mathcal{O}(z^3)$. Since, $z = \mathcal{O}(\sqrt{a})$ and $a = \mathcal{O}(\varepsilon)$ in R_2, it follows by the Impicit Function Theorem that in a sufficiently small neighbourhood of the crossing-sliding cycle, there exists a limit cyle rooted in R_3 with the stability corresponding to that of the degenerate cycle for some μ sufficiently close to μ^*.

Finally, the trajectories that reach the regione labelled as R_2 in the figure (cf. T_2) first follow the flow in G_1 for some small time, say t, and then hit the sliding region and flow in $\hat{\Sigma}$ for some time, to finally leave the switching manifold. This is the most sutble correction of the ZDM. Although the ZDM correction is of the linear order it is bounded by $\mathcal{O}(\varepsilon^2)$ correction term. The details are explained in [K5]. Hence, it follows by the Implicit Function Theorem, that in a sufficiently small neighbourhood of the crossing-sliding cycle, there existsts a limit cyle rooted in R_2 with the stability corresponding to that of the degenerate cycle for some μ sufficiently close to μ^*.

The analysis presented here was used in [K5] to unfold the dynamics of a dry-friction oscillator model which was shown to exhibit a two-parameter degenerate crossing-sliding bifurcation. Analytical derivations of two-parameter bifurcation curves with the focus on analysing two-parameter sliding bifurcations of the same dry-friction oscillator model as in [K5] were undertaken in[71].

The unfolding of the four different degenerate two-parameter sliding bifurcations of limit cycles allows one to develop numerical continuation techniques with appropriate branch switchings as discussed in [K5].

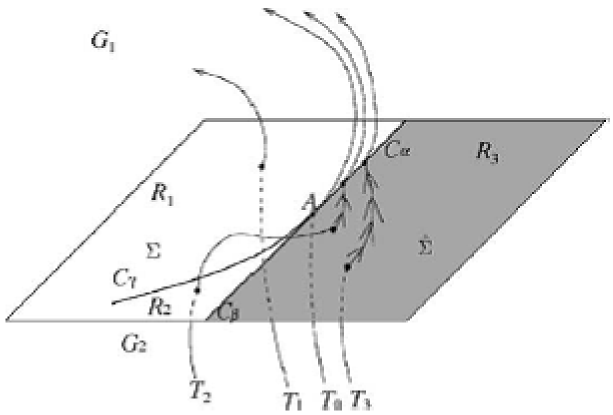

Figure 4: Phase space topology around the codimension-two degenerate crossing-sliding point (point labelled A in the figure). Trajectory labelled as T_0 denotes the part of the critical limit cycle that interacts with the boundary of the sliding set in the two-parameter degenerate crossing-sliding scenario. Small parameter perturbations result in the existance of limit cycles rooted in regions R_1, R_2 or R_3. This is schematically depicted by trajectories T_1, T_2 and T_3 (rooted correspondingly in R_1, R_2 and R_3) which differ from T_0 by the number of segments which make up the perturbed limit cycle.

3.2.3 Two parameter DIB of non-hyperbolic cycle. DIB of Type II

We shall now consider a general Filippov system with two zones, that for a parameter value μ^* has a grazing orbit of period $T^* > 0$ and where $H > 0$ along the cycle except at the grazing point x^*.

Define the flow of the vector field F_1 to be $\phi_1(x, t, \mu)$, that is

$$\begin{aligned} \phi_{1t}(x,t,\mu) &= F_1(\phi_1(x,t,\mu),\mu) && (26)\\ \phi_1(x,0,\mu) &= x. && (27) \end{aligned}$$

By the above we have

$$\begin{aligned} \phi_1(x^*,T^*,\mu^*) &= x^* && (28)\\ H(x^*,\mu^*) &= 0 && (29)\\ \mathcal{L}_{F_1}(H)(x^*,\mu^*) &= 0 && (30)\\ \mathcal{L}^2_{F_1}(H)(x^*,\mu^*) &> 0 && (31) \end{aligned}$$

Define the local projection mapping $P(x,\mu)$ that maps x the shortest way along a trajectory of F_1 into the surface given by $\mathcal{L}_{F_1}(H) = 0$. We study the dynamics of the system close to the grazing orbit through the Poincaré like mapping

$$PM(x,\mu) = \begin{cases} P(\phi_1(x,T^*,\mu),\mu) & \text{when } H(P(\phi_1(x,T^*,\mu),\mu),\mu) \geq 0 \\ g(P(\phi_1(x,T^*,\mu),\mu),\mu) & \text{when } H(P(\phi_1(x,T^*,\mu),\mu),\mu) \leq 0. \end{cases} \quad (32)$$

For trajectories that do not involve sliding, the first expression gives the regular Poincaré mapping. For trajectories involving sliding, the second expression corrects for the presence of sliding through the zero-time discontinuity mapping given by equation (20). However, for our purposes, we define the non-identity part of the ZDM for one-parameter grazing-sliding here in the form

$$g(x,\mu) = x + \beta(x,\mu,y)y^2, \quad (33)$$

where $y = \sqrt{-H(x,\mu)}$ and β, H are smooth functions of their arguments. In this form we have the exact expression for the non-identity part of the ZDM and the leading order term of $\beta(x,\mu,y)y^2$ expanded in x about (x^*,μ^*) is exactely the expression given in (20).

Note that the second expression for PM does not necessarily map points into the section $\mathcal{L}_{F_1}(H) = 0$, so PM is not the Poincaré mapping using this section. However, it is locally equivalent to any Poincaré mapping using a section that does not pass through the grazing point, through a smooth change of coordinates.

Stability properties of the grazing cycle Let us first note that the cycle of period T^* at the grazing point can be viewed as a cycle that does not contain sliding segments or as a cycle of period T^* that contains a zero length sliding segment (as each graze may or not be regarded as sliding, it may of course as well be regarded as a cycle of period nT^* with m zero-length sliding segments in some given pattern of sliding/non-sliding). In the former case the linear stability properties of the cycle can be determined by solving the variational equations. In the later the linear stability can be determined by combining the variational equations with the discontinuity mapping techniques. In other words, the stability of these orbits can be determined by considering the limits leading to grazing contact. Both these limits which determine the stability of grazing cycles without and with zero length sliding are well-defined but different. (Note that we are only discussing the stability of these limit orbits, not that of the grazing orbit itself, which is a more involved problem.)

To be more specific, we observe that x^* is fixed by all of ϕ_1, P, and g. The Jacobians of the mappings are (as will be shown below)

$$\begin{aligned} \phi_{1x}(x^*,T^*,\mu^*) &= J^* && (34)\\ P_x(x^*,\mu^*) &= I - \frac{F^*V^*}{V^*F^*} && (35)\\ g_x(x^*,\mu^*) &= I - B^*C^*, && (36)\end{aligned}$$

where

$$\begin{aligned} F^* &= F_1(x^*,\mu^*) && (37)\\ V^* &= \mathcal{L}_{F_1}(H)_x(x^*,\mu^*) && (38)\\ B^* &= \frac{F_d(x^*,\mu^*)}{\mathcal{L}_{F_d}(H)(x^*,\mu^*)} && (39)\\ C^* &= H_x(x^*,\mu^*). && (40)\end{aligned}$$

When viewed as a cycle without sliding, we have that the linear stability of the cycle is given by

$$A^* = (I - (F^*V^*)/(V^*F^*))J^* \qquad (41)$$

(apart from the trivial eigenvalue 0). When viewed as a cycle with a zero-length sliding we have the linear stability determined by the non-trivial eigenvalues of

$$A_s^* = (I - B^*C^*)A^*. \qquad (42)$$

If, while following the branch of the grazing orbits in two-parameter space, we monitor the eigenvalues of A^*, and an eigenvalue lies on the unit circle we come across a two parameter grazing-sliding event. If, in turn, we monitor the eigenvalues of A_s^*, and we detect that it is characterised by an eigenvalue lying on the unit circle we again encounter a different type of a two parameter event.

In [K7], we unfolded a two parameter grazing-sliding bifurcation in the case where one of the eigenvalues of A_s^* was equal to 1, but the eigenvalues of A^* were not.

Existence of limit cycles about the co-dimension two point First consider a branch of non-sliding periodic orbits emanating from the grazing orbit. All points on an orbit return if we integrate the vector field F_1 for the period time $\bar{T}$. To pick out a particular point on an orbit, we pick the point $\bar{x}$ where the function H attains its minimum along the orbit. We should also require this minimum to be non-negative, if it is to correspond to a real orbit of the system.

The equations for this fixed point are

$$\begin{aligned} \mathcal{L}_{F_1}(H)(\bar{x},\mu) &= 0 && (43)\\ \bar{x} - \phi_1(\bar{x},\bar{T},\mu) &= 0. && (44) \end{aligned}$$

At the codimension-two point, when $\mu = \mu^*$ we know that $\bar{x} = x^*$, $\bar{T} = T^*$ is a solution, and if the matrix

$$L^* = \begin{pmatrix} V^* & 0 \\ I - J^* & -F^* \end{pmatrix}, \qquad (45)$$

where

$$\begin{aligned} J^* &= \phi_{1x}(x^*,T^*,\mu^*) && (46)\\ F^* &= \phi_{1t}(x^*,T^*,\mu^*) = F_1(x^*,\mu^*) && (47)\\ V^* &= \mathcal{L}_{F_1}(H)_x(x^*,\mu^*), && (48) \end{aligned}$$

is non-singular, the implicit functions theorem defines $\bar{x}(\mu)$ and $\bar{T}(\mu)$ uniquely for μ near μ^*. The determinant of this matrix is

$$\det(L^*) = V^*F^* \det(I - A^*) \qquad (49)$$

where

$$\begin{aligned} A^* &= (I - \frac{F^*V^*}{V^*F^*})J^* && (50)\\ V^*F^* &= \mathcal{L}^2_{F_1}(H)(x^*,\mu^*) > 0. && (51) \end{aligned}$$

Since we have assumed that the grazing orbit, when viewed as non-sliding, does not have an eigenvalue equal to 1, we find that L^* is non-singular. Defining

$$\nu_0(\mu) = H(\bar{x}(\mu),\mu) \qquad (52)$$

it is clear that $\bar{x}$ corresponds to a unique non-sliding orbit of period one, if and only if $\nu_0 > 0$, but the function $\bar{x}(\mu)$ is nevertheless well defined and smooth for all μ close to μ^*.

Note that A and J have the same eigenvectors, and the same eigenvalues except that the trivial eigenvalue 1 of J is changed to a trivial eigenvalue 0 of A.

The equations for a branch of sliding orbits of period one emanating from the grazing orbit are

$$\begin{aligned} \mathcal{L}_{F_1}(H)(x',\mu) &= 0 && (53)\\ x' - \phi_1(x'',T'',\mu) &= 0 && (54)\\ x'' - x' - \beta(x',\mu,y,0)y^2 &= 0 && (55)\\ y^2 + H(x',\mu) &= 0. && (56) \end{aligned}$$

Disregarding (56) for a moment, and viewing y as an independent variable, we find that $x' = x'' = \bar{x}$, $T'' = \bar{T}$ is a solution when $y = 0$ for all μ close to μ^*, and under the same conditions as for the existence of $\bar{x}(\mu)$ we find that $x'(\mu,y)$ is a well defined function with

$$x'(\mu,y) = \bar{x}(\mu) + \left[\bar{A}(I-\bar{A})^{-1}\left(\bar{B} + y\bar{B}_1\right) + \mathcal{O}(y^2)\right]y^2, \qquad (57)$$

where

$$
\begin{aligned}
\bar{A}(\mu) &= \left(I - \frac{\bar{F}\bar{V}}{\bar{V}\bar{F}}\right)\bar{J} &(58)\\
\bar{J}(\mu) &= \phi_{1x}(\bar{x}(\mu), \bar{T}(\mu), \mu) &(59)\\
\bar{F}(\mu) &= F_1(\bar{x}(\mu), \mu) &(60)\\
\bar{V}(\mu) &= \mathcal{L}_{F_1}(H)_x(\bar{x}(\mu), \mu) &(61)\\
\bar{B}(\mu) &= \beta(\bar{x}(\mu), \mu, 0, 0) &(62)\\
\bar{B}_1(\mu) &= \beta_y(\bar{x}(\mu), \mu, 0, 0) &(63)\\
\bar{C}(\mu) &= H_x(\bar{x}(\mu), \mu). &(64)
\end{aligned}
$$

Substituting this into (56) and expanding in y gives

$$
\begin{aligned}
\nu_0(\mu) + \nu_2(\mu)y^2 + y^3\left(\nu_3(\mu) + \mathcal{O}(y)\right) &= 0 &(65)\\
y &\geq 0. &(66)
\end{aligned}
$$

where we have introduced

$$
\begin{aligned}
\nu_2(\mu) &= 1 + \bar{C}\bar{A}(I - \bar{A})^{-1}\bar{B}, &(67)\\
\nu_3(\mu) &= \bar{C}\bar{A}(I - \bar{A})^{-1}\bar{B}_1. &(68)\\
& &(69)
\end{aligned}
$$

Note that at the co-dimension two point μ^*, $\nu_0(\mu^*) = \nu_2(\mu^*) = 0$ but we will assume $\nu_3(\mu^*) \neq 0$. Further we will assume that the μ parameter dependence is non-degenerate, that is $\nu_{0\mu}(\mu^*)$ and $\nu_{2\mu}(\mu^*)$ are linearly independent.

Theorem 3 *Let $\mu = (\nu_0, \nu_2)$. The limits $\nu_0 \to 0$ and $y \to 0$ give the grazing-sliding branch in the two-parameter space $\forall\mu$ in a sufficiently small neighbourhood of μ^*. About μ^* there is a curve of fold bifurcations given by the expression*

$$
\begin{aligned}
\nu_0(\mu) &= \nu_2(\mu)^3\left(-\frac{4}{27\nu_3(\mu)^2} + \mathcal{O}(\nu_2)\right) &(70)\\
\frac{\nu_2(\mu)}{\nu_3(\mu)} &< 0, &(71)
\end{aligned}
$$

which is one-sided, and cubically tangent to the grazing curve.

We note that it makes sense to compare ν_0 for a non-sliding orbit with $-y^2$ for a sliding orbit, as they both measure the minimum value of H along the incoming trajectory. Calling this quantity $H_{\min}$, we find from (52) and (65) that

$$
\frac{H_{\min}(\text{sliding})}{H_{\min}(\text{non sliding})} = \frac{1}{\nu_2} \tag{72}
$$

in the limit as $\nu_0 \to 0$ whenever $\nu_2 \neq 0$. Since $H_{\min}$ must be positive for a non-sliding orbit and negative for a sliding, a positive value of ν_2 means that the non-sliding orbit exists for $\nu_0 > 0$ and the sliding for $\nu_0 < 0$, whereas a negative value of ν_2 means that both orbits exists for $\nu_0 > 0$ and not for $\nu_0 < 0$.

The condition for a saddle-node bifurcation on the sliding branch is that (65) should have a double root $y > 0$. This means

$$
2\nu_2(\mu) + 3y\left(\nu_3(\mu) + \mathcal{O}(y)\right) = 0. \tag{73}
$$

Eliminating y from (65) and (73) gives the fold curve in parameter space.

The above unfolding was presented in [K7]. In this work, we applied the analytical unfolding to explain the dynamics of a dry-friction oscillator model exhibiting a two-parameter DIB discussed here. The unfolding allows for numerical continuation of limit cycle solutions in two-parameter space of Filippov systems of relevance to applications, which was also shown in [K7] using the dry-friction oscillator model. Other cases of unfoldings of two-parameter DIBs of limit cycles of type 2 in nonsmooth systems may be found, for example, in [22, 59, 93].

3.2.4 DIB of Type III

In the current section, we will discuss another type of two-parameter sliding bifurcations, one of which was unfolded in [K5]. In particular, we will present an unfolding of a two-parameter sliding bifurcation scenario of limit cycles which is characterised by the occurrence of two independent simultaneous one parameter sliding bifurcations at two distinct points along a limit cycle.

Consider a limit cycle in Filippov system (3) which at some parameter $\mu = \mu^* = (\mu_1^*, \mu_2^*) \in \mathbb{R}^2$ exhibits two independent grazing contacts at x_1^* and x_2^*, and such that $x_1^* \neq x_2^*$. We can now introduce the following definition.

Definition 13 *Suppose that there exists a hyperbolic limit cycle $L(x_1, x_2, \mu)$ in the Filippov system (3), where $\mu = (\mu_1, \mu_2) \in \mathbb{R}^2$ is a parameter and x_i $(i = 1, 2)$ are points on the limit cycle such that $x_1 \neq x_2$. If for some μ^* a limit cycle L^* at $x_1^* = x_1(\mu^*)$ and $x_2^* = x_2(\mu^*)$ exhibits two isolated grazing contacts with the boundary of the sliding region $\partial\hat{\Sigma}^-$, and the μ dependence on x_i^* is nondegenerate, that is $\langle \frac{\partial x_1}{\partial \mu_1}, H_x \rangle (x_1^*, \mu^*) \neq 0$, $\langle \frac{\partial x_1}{\partial \mu_1}, (H_x F_1)_x \rangle (x_1^*, \mu^*) \neq 0$, $\langle \frac{\partial x_2}{\partial \mu_2}, H_x \rangle (x_2^*, \mu^*) \neq 0$, $\langle \frac{\partial x_2}{\partial \mu_2}, (H_x F_1)_x \rangle (x_2^*, \mu^*) \neq 0$, and $\frac{\partial x}{\partial \mu_1}(x^*, \mu^*)$ and $\frac{\partial x}{\partial \mu_2}(x^*, \mu^*)$ are linearly independent, we say that the limit cycle undergoes a two-parameter Discontinuity Induced Bifurcation of simultaneous sliding type.*

To describe the dynamics of the system $\forall \mu$ in a sufficiently small neighbourhood of μ^* one may use the ZDMs/PDMs for one-parameter sliding bifurcations. Let us suppose that the two-parameter DIB we are considering is that of a simultaneous occurrence of a grazing-sliding and an adding-sliding bifurcation as this is the cases unfolded in [K5]. Let us now choose a Poincaré section, say Π, that is transversal to the vector field generating the critical cycle. In particular, for the purposes of the analysis, we choose Π such that it is transversal to the limit cycle at either of the two contact points x_i^* $(i = 1, 2)$.

We define Π, $\forall x$, in a sufficiently small neighbourhood of x_1^* as the zero level set of $H_x F_1(x, \mu) = 0$, where we assume x_1^* to be the grazing contact point of one parameter grazing-sliding bifurcation.

Suppose now that $\bar{x} \in \Pi$ is a periodic point of a limit cycle that exists in a sufficiently small neighbourhood of the periodic point of the critical cycle x_1^*. Then $\bar{x}$ satisfies

$$\bar{x} = f_2 \circ as \circ f_1 \circ gs(\bar{x}, \mu), \tag{74}$$

where $gs(x, \mu)$ and $as(x, \mu)$ are the zero-time discontinuity mappings for the GS and the AS bifurcations respectively (see eqs. (20) and (22)), whereas f_1 and f_2 are some smooth mappings that describe the dynamics along the segments of trajectory between the points of contact of the limit cycle with the boundaries of the sliding region $\partial\hat{\Sigma}^{\pm}$.

Note that the effect of the zero time correction due to the AS is contained in higher order terms and hence $f_T = f_2 \circ as \circ f_1$ is a C^1 differentiable mapping. On the other hand, gs is piecewise linear to leading order. This implies that it is the grazing-sliding bifurcation which is detrimental to system dynamics. Obviously the critical limit cycle has to satisfy the conditions for the grazing-sliding and adding-sliding bifurcations, respectively, at x_1^* and x_2^*.

Existence of limit cycles around the codimension-two point We can now consider a branch of orbits that emanate from the critical cycle at the codimension-two point. We first note that a cycle at grazing can be viewed as a cycle that either contains a zero-length sliding segment or does not. Since the gs map is piecewise linear the Floquet multipliers of the critical cycle have distinct values depending if we consider the critical cycle as characterised by a zero-length sliding segment or not. We assume, without loss of generality, that the Floquet multipliers of the critical cycle do not lie on the unit circle of the complex plane, regardless whether we consider the cycle as containing the zero-length sliding segments or not. Under such assumption, a continuous parameter variation leads to the creation of cycles with and without 'short' sliding segments. Let us first consider the existence of the family of cycles which do not contain short sliding segments. Without loss of generality, we assume for these cycles the periodic point $\bar{x} \in \Pi$ to be such that $H(\bar{x}, \mu) > 0$.

Equations for the fixed point are

$$H_x F_1(\bar{x}, \mu) = 0, \tag{75}$$
$$\bar{x} - f_T(\bar{x}, \mu) = 0. \tag{76}$$

Note that for the cycles which do not acquire a sliding portion the gs map is the identity map. If, at the codimension-two point, where $(\bar{x}, \mu) = (x^*, \mu^*)$ is a solution of (75) and (76), the matrix

$$L^* = \begin{pmatrix} V^* & 0 \\ I - J^* & -F^* \end{pmatrix} \tag{77}$$

is non-singular, the implicit function theorem defines $\bar{x}(\mu)$ uniquely for all μ near μ^*. The determinant of the matrix L^* is non-singular provided that f_{T_x} does not have any multipliers on the unit circle (which is our assumption). Let us define $\nu_0(\mu) = H(\bar{x}(\mu), \mu)$. From our definition $\nu_0 > 0$ corresponds to a unique orbit with no short sliding segments that emanate from the codimension-two point.

To obtain equations for a branch of sliding orbits of period-one (that is characterised by one iteration of $f_T \circ gs$) emanating from the critical orbit we have to include the effect of the $gs(x, \mu)$ map. This yields two additional equations. Therefore, we have that the equations for a branch of these orbits are

$$H_x F_1(x', \mu) = 0, \tag{78}$$
$$x' - f_T(x'', \mu) = 0, \tag{79}$$
$$x'' - b(x', \mu, y)y^2 = 0, \tag{80}$$
$$y^2 + H(x', \mu) = 0, \tag{81}$$

where b denotes the correction terms of all orders of the discontinuity map (20). From (78)–(81) and under the same condition as for the existence of $\bar{x}(\mu)$ we find that $x' = x'' = \bar{x}$ is the solution when $y = 0$ for all μ close to μ^*. From the linearisation of (78)–(80), we find that

$$x'(\mu, y) = \bar{x}(\mu) + \bar{A}(I - \bar{A})^{-1}\bar{B}y^2 + \mathcal{O}(y^3), \tag{82}$$

where

$$\bar{A}(\mu) = \left(I - \frac{\bar{F}\bar{P}}{\bar{P}\bar{F}}\right)(f_T)_x, \quad \bar{B} = b(\bar{x}(\mu), \mu, 0), \quad \bar{C}(\mu) = H_x(\bar{x}(\mu), \mu).$$

We note here that $\bar{B}$ is the leading order term of the ZDM given in (20). Expanding (81) in y finally gives

$$\nu_0(\mu) + \nu_2(\mu)y^2 + \mathcal{O}(y^3) = 0, \tag{83}$$

with $\nu_2 = 1 + \bar{C}\bar{A}(I - \bar{A})^{-1}\bar{B}$.

Grazing-sliding and adding-sliding branches We may treat ν_0 and ν_2 as unfolding parameters and determine the character of the adding-sliding and the grazing-sliding curves in ν_0 and ν_2 two-parameter space.

Theorem 4 *Let $\mu = (\nu_0, \nu_2)$. The limits $\nu_0 \to 0$ and $y \to 0$ give the grazing-sliding branch in the two-parameter space $\forall \mu$ in a sufficiently small neighbourhood of μ^*. If $\nu_2 > 0$ then the adding-sliding curve, existing $\forall \mu$ in a sufficiently small neighbourhood of μ^*, crosses the grazing-sliding branch at x^*. On the other hand, if $\nu_2 < 0$, two branches of adding-sliding emanate from the co-dimension two point and lie on the same side of the grazing-sliding curve in two-parameter space, in which case one of the two limit cycles exsiting in the neighbourhood of (x^*, μ^*) must be unstable.*

For further details we refer to [K2,K4]. The above unfolding is further explained below.

Grazing-sliding occurs when $\nu_0 \to 0$ for orbits with no short sliding segments or when $y \to 0$ for orbits with short sliding segments. Both these limits give the same grazing-sliding (GS) branch in the two-parameter space. From our discussion we determined that from the codimension-two point at least two cycles emanate. Since both these cycle at the codimension-two point exhibit the adding-sliding contact by the continuity argument these cycles must still exhibit the AS contact for ν_0 and ν_2 in some neighbourhood of $\nu_0 = 0$ and $\nu_2 = \nu_2^*$, where '$*$' denotes the value of ν_2 at the codimension-two point. If both of these cycles exist on one side of the grazing-sliding curve in the two-parameter space then the adding-sliding (AS) curves terminate at the codimension-two point. On the other hand if the two orbits exist on either side of the grazing curve then the AS curves that emanate from the codimension-two point lie on the opposite sides of the grazing curve. In both cases, if we trace a curve of the AS bifurcations at the codimension-two point the AS curve exhibits a corner type singularity or we might say that one branch of the AS terminates and we switch continuously to another AS branch.

From (83) we have that if $\nu_2 > 0$ then ν_0 must be negative for the orbit with the short sliding segment (note that the orbit with no short sliding segments by the definition exists only for positive values of ν_0). Then it follows that the curves of the adding-sliding bifurcations lie on either side of the curve of the grazing-sliding bifurcations in the two-parameter space. On the other hand, if $\nu_2 < 0$ then ν_0 must be positive for the orbit with the short sliding segment and hence the cycles that emanate from the grazing curve lie on one side of the grazing-sliding curve. It can be further shown that if $\nu_2 < 0$ one of the two orbits that emanate from the codimension-two point must be unstable.

We can further determine the dynamics around the codimension-two point by inspecting the eigenvalues of A^* and $(I - B^*C^*)A^*$. If all the eigenvalues of both matrices and matrix products lie within the unit circle then the two aforementioned limit cycles are stable for all ν_0 and ν_2 in a sufficiently small neighbourhood of $\nu_0 = 0$ and ν_2^*, which is the case encountered and unfolded in a dry-friction oscillator model in [K5].

3.2.5 Discussion on classification and unfolding of two-parameter DIBs

From the results presented and discussed in Sec. 3.2, it follows that the classification and unfolding of two- parameter DIBs of limit cycles in piecewise smooth systems allows one to determine the existence of bifurcation curves which emanate from a co-dimension two point, and classify the dynamics related to the critical limit cycles under crossing of these bifurcation curves in a sufficiently small neighbourhood of the co-dimension two point in the parameter space. However, we should note here two facts. Firstly, there is no theory which allows one to determine all the bifurcations curves which emanate from a particular co-dimension two point in general $n-$dimensional PWS systems. Secondly, the classification of the dynamics will depend on the dimension of a particular system under consideration. So it follows that the unfoldings discussed here hold for general n-dimensional Filippov systems, but the full dynamics will depend on the dimension of a particular system. We should also note that in the case of systems of relevance to applications, often, a given system violates some genericity assumptions. The classiffication startegy and methodology discussed here, even though applied to Filippov systems, may be applied to other classes of PWS systems. Unfoldings of other types of two-parameter DIBs in PWS systems are presented in [90, 88, 44, 53].

We will present now the results of numerical unfolding, shown in [K6], of an impact oscillator of two-parameter DIBs of type II. That is, we consider an example of a system with the degree of discontinuity 0.

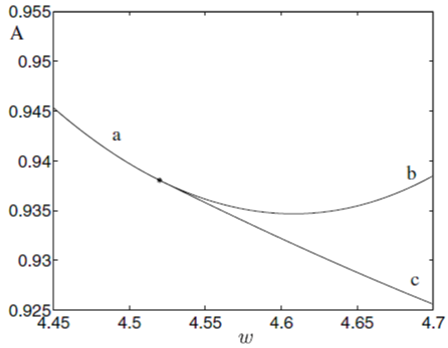

Figure 5: Two parameter bifurcation diagram around the co-dimension two grazing point at which one of the Floquet multipliers of the bifurcating cycle is $\lambda = 1$ (star). Curve of grazing periodic orbits, with $\lambda < 1$ on "a" and $\lambda > 1$ on "b". Curve of $\lambda = 1$ for periodic orbit with no low velocity impacts on "c".

Example: Forced oscillations in an impacting system Consider a model of a periodically forced impact oscillator

$$\begin{cases} \dot{x_1} &= v \\ \dot{x_2} &= -\frac{1}{w^2}x_1 - \frac{2d}{w}x_2 + a\left(\left[\frac{1}{w^2} - 1\right]\cos(x_3) - \frac{2d}{w}\sin(x_3)\right) \\ \dot{x_3} &= 1 \end{cases} \tag{84}$$

with impact at $x_1 = -1$ and with an impact law

$$x_2^+ = -rx_2^-. \tag{85}$$

Here x_1 is the position, x_2 is the velocity $x_3(\text{mod}2\pi)$ is the driving phase, d is the nondimensional damping, w is the driving frequency divided by undamped natural frequency, r is a coefficient of restitution, and a is a particular solution amplitude. When $0 < a < 1$ the system admits the non-impacting periodic solution

$$x_1 = a\cos(x_3) \tag{86}$$
$$x_2 = -a\sin(x_3) \tag{87}$$

and the solution is stable if $d > 0$ and $w > 0$. Besides this solution, the system may have additional impacting periodic solutions and chaotic attractors.

Non-hyperbolic grazing solution with ($\lambda = 1$) For the parameter values

$$d = 0.6,\ w = 4.519798,\ a = 0.938042,\ r = 1$$

there exists a periodic orbit with one non-grazing impact and one grazing, and with one eigenvalue $\lambda = 1$ (if the system is linearised ignoring the grazing impact). One point on the orbit is

$$(x_1, x_2, x_3) = (-1, 0.315637, 2.787732).$$

The other eigenvalue is positive and close to 0. The period of the orbit is 8π.

In a parameter diagram where w and a are varied, (Figure 5) we find that several one-parameter curves meet at the two-parameter point. There is a curve of grazing periodic orbits, and a curve of saddle-node bifurcations. For this system, there is a stable periodic orbit similar to the two-parameter orbit in the region between curves "b" and "c", and at "c" it undergoes a grazing bifurcation leading

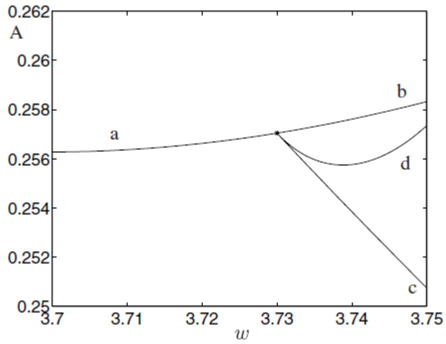

Figure 6: Parameter diagram around the grazing, $\lambda = -1$ point (star). Curve of grazing periodic orbit, with $\lambda > -1$ or complex on "a" and $\lambda < -1$ on "b". Curve "c" of $\lambda = -1$ for periodic orbit with no low velocity impacts. Curve "d" of grazing periodic orbits of the double period.

to a chaotic attractor similar to the two-parameter orbit in a small region above curve "b". Further increases of the parameter a make the attractor disappear in a boundary crisis. The curve where this happens is not included in the diagram. Below "c" and below and above "a" there is no attractor close to the two-parameter orbit.

Non-hyperbolic grazing solution ($\lambda = -1$) For the parameter values

$$d = -0.3, \ w = 3.729986, \ a = 0.257040, \ r = 0.15,$$

there exists a periodic orbit with one non-grazing impact and one grazing, and with one eigenvalue $\lambda = -1$ (if the system is linearised ignoring the grazing impact).

One point on the orbit is

$$(x_1, x_2, x_3) = (-1, 0.170869, 5.186351).$$

The other eigenvalue is negative and about -0.47. The period of the orbit is 6π.

In a parameter diagram where w and a are varied, (Figure 6), we find that several one-parameter curves meet at the two-parameter point. There is a curve of grazing periodic orbits, and a curve of supercritical period-doubling bifurcations. For this system, there is a stable periodic orbit similar to the two-parameter orbit in the region below curves "a" and "c". At "c" there is a supercritical period-doubling bifurcation, branching off a stable orbit of twice the period. At "d" this orbit becomes grazing. The stability characteristics of this orbit changes quite rapidly along the curve "d". Near the two-parameter point one eigenvalue must be close to 1, but at the right edge of the diagram, the eigenvalues are already complex. Similar changes take place on the curve "a". In this system, the grazing bifurcation at "d" does not continuously create an attractor.

The numerical resualts presented here can be efficiently generated numerically thanks to the analytical unfolding which gives the conditions on the existence of particular bifurcation curves. The development of efficient numerical continuation tools for DIBs in different classes of nonsmooth systems is an ongoing research work. For some available software, see for example [30, 27].

3.3 Multiple attractors in grazing-sliding bifurcations. Classification problems

In Sec. 3.2, we have presented a methodology which we developed to unfold two parameter DIBs. In particular, we have presented an example of the unfolding for each of the three types of two parameter DIBs. However, the beauty and nightmare of analysing DIBs stems from the fact that depending on the dimension of phase space different dynamic outcomes may be observed. In the current section, we present the classification results, which show that a one-parameter grazing-sliding bifurcation in 3-dimensional Filippov type flows may produce a bifurcation that leads to the creation of a multiple number of attractors, originating from a single attractor, which bifurcation may not occur in differentiable vector fields. The possibility of such a scenario in piecewise-smooth systems was first demonstrated for piecwise smooth maps, e.g. [38], and multistability itself has been observed in different types nonsmooth systems [100].

Consider a Filippov systems for which the evolution of variable x in some region $\mathbb{D} \subseteq \mathbb{R}^3$ is determined by the equations

$$\dot{x}(t) = \begin{cases} F_1(x(t), \mu) & \text{if} \quad H(x(t), \mu) > 0 \\ F_2(x(t), \mu) & \text{if} \quad H(x(t), \mu) < 0, \end{cases} \tag{88}$$

where F_1, F_2 are sufficiently smooth vector functions, F_1, $F_2 : \mathbb{D} \times \mathbb{R} \mapsto \mathbb{R}^3$, and $H : \mathbb{D} \times \mathbb{R} \mapsto \mathbb{R}$ is some smooth scalar function.

Let us define the boundary Σ as

$$\Sigma := \{x \in \mathbb{D} : H(x, \mu) = 0\}. \tag{89}$$

The region $\mathbb{D}$ is then divided by Σ into two subspaces. Namely, we define

$$S_1 := \{x \in \mathbb{D} : H(x,\, \mu) > 0\},$$

and

$$S_2 := \{x \in \mathbb{D} : H(x,\, \mu) < 0\},$$

where the dynamics is smooth as it is governed by the differentiable vector field F_1 or F_2 respectively. Although, F_1 and F_2 are admissible, respectively, in S_1 and S_2 only, they are well-defined everywhere in $\mathbb{D}$. Depending on the direction of the vector fields with respect to Σ those trajectories starting in S_1 and S_2 that reach Σ in finite time will either cross or evolve along Σ. (In the latter case following the sliding flow.) Let $\sigma(x,\, \mu) = \langle H_x, F_1\rangle(x,\, \mu)\langle H_x, F_2\rangle(x,\, \mu)$, where $\langle H_x, F_1\rangle$ denotes the directional derivative of H with respect to the vector field F_1. (Note that the subscript 'x' denotes a differential operator, and x in other contexts denotes a point in the state space.) The switching surface Σ can be divided into subsets, say Σ_c and Σ_s, defined as

$$\Sigma_c := \{x \in \Sigma : \sigma(x,\, \mu) > 0\}, \qquad \Sigma_s := \{x \in \Sigma : \sigma(x,\, \mu) \le 0\}.$$

We will require

$$\langle H_x, (F_2 - F_1)\rangle > 0 \tag{90}$$

on Σ_s. When a trajectory generated by F_1 (or F_2) reaches Σ_c it switches to F_2 (or F_1) on Σ_c. Note that such a trajectory is continuous, and it is built of segments generated by F_1 and F_2. If, on the other hand, Σ_s is reached from S_1 or S_2, then the motion follows the sliding flow along Σ_s, and the vector field that generates this motion is defined as

$$F_s = \alpha_s F_1 + (1 - \alpha_s)F_2, \tag{91}$$

where $\alpha_s = \dfrac{\langle H_x, F_2\rangle}{\langle H_x, (F_2 - F_1)\rangle}$ on Σ_s, and $0 \le \alpha_s \le 1$.

The function α_s can be used to define the boundaries of a region where sliding is possible, namely

$$\partial\Sigma_s := \{x \in \Sigma : \alpha_s(x,\, \mu) = 1\}, \qquad \partial\Sigma_s^0 := \{x \in \Sigma : \alpha_s(x,\, \mu) = 0\}.$$

The condition (90) implies that the interior of Σ_s is an *attracting* sliding region, that is the vector fields point toward Σ from either sides of the switching surface.

3.3.1 Analytical conditions and Poincaré return map

Let us ignore for the moment the presence of switching and sliding, that is we consider the system defined by F_1 in all of $\mathbb{D}$. Assume that this system for a parameter value $\mu = \mu^*$ has a periodic orbit $x(t)$ of period T^* where $H(x(t), \mu) > 0$ for all points on the orbit except at $x = x^*$ where clearly

$$H(x^*, \mu^*) = 0, \tag{92}$$

$$\langle H_x, F_1\rangle(x^*, \mu^*) = 0, \tag{93}$$

and the second directional derivative is non-negative. We will restrict ourselves to the case when the second directional derivative is positive, that is

$$\langle\langle H_x, F_1\rangle_x, F_1\rangle(x^*, \mu^*) > 0. \tag{94}$$

We will now define a Poincaré section Π, containing x^*, as

$$\Pi := \{x \in \mathbb{D} : \langle H_x, F_1\rangle(x^*, \mu^*) = 0\}.$$

From (94) it follows that Π is transversal to the periodic orbit and thus defines a first return Poincaré mapping $P : \Pi \times \mathbb{R} \mapsto \Pi$ in a neighbourhood of (x^*, μ^*). The point x^* is a fixed point for the map P when $\mu = \mu^*$. Assume that the Jacobian of P at (x^*, μ^*) has no eigenvalue equal to 1. By the Implicit Function Theorem there exists a unique family of fixed points $\hat{x}(\mu)$ with $\hat{x}(\mu^*) = x^*$, $\forall\mu$ in some neighbourhood of μ^*. Assume further the unfolding condition

$$\left.\frac{dH(\hat{x}(\mu))}{d\mu}\right|_{\mu=\mu^*} \neq 0, \tag{95}$$

which implies that the fixed point $\hat{x}$ moves with respect to Σ. If $H(\hat{x}(\mu), \mu) \geq 0$ then $\hat{x}(\mu)$ lies on a periodic orbit of the original Filippov system. On the other, if $H(\hat{x}(\mu), \mu) < 0$ then $\hat{x}(\mu)$ does not correspond to a periodic solution of the switched Filippov system. At μ^* we have a grazing periodic orbit and the systems satisfies the conditions (90), (92), (93), (94) and (95) for a grazing-sliding bifurcation.

3.3.2 Reduction to a one-dimensional map

Theorem 5 *Consider the Filippov system (88) which undergoes a one-parameter grazing-sliding bifurcation. Assuming that the inequalties $a > 0$, $b > a^2$ and $\gamma = -C > 0$ are satisfied, the system dynamics decribing the trajectories which include a sliding segment, $\forall x \in \partial\Sigma_s$ in a sufficiently small neighbourhood around the grazing contact x^*, is described by a one-dimensional map $\partial\hat{\Sigma}_s \mapsto \partial\hat{\Sigma}_s$ of the form*

$$w_{n+1} = \begin{cases} -\gamma w_n - 1 & \text{if } w_n < 0 \\ -(\gamma a + b)w_n + \gamma - 1 & \text{if } 0 < w_n < 1/a \\ (\gamma(b - a^2) - ab)w_n + \gamma(a + 1) + b - 1 & \text{if } w_n > 1/a. \end{cases} \tag{96}$$

The reduction of a 3-dimensional flow to the above 1-dimensional discontinuous map is shown in detail in [K3]. We should note here that such a reduction is possible due to the fact that the above map describes the dynamics about one-parameter grazing-sliding when all trajectories contain a sliding segment. However, such a map can exhibit a plethora of dynamical scenarios. In particular, in [K3] we have proven, for the first time, that the one parameter grazing-sliding bifurcation may lead to the onset of multiple attractors under the conditions specified in the following theorem. Other types of attractors, observed in continuous but non-differentiable discrete time maps, which can be born in grazing-sliding bifurcations and are related to the underlying dimension of the Filippov flow are analysed in [50]. In the reduced map, the notion of the co-dimension of the so-called border-collision bifurcation corresponding to grazing-sliding in a flow plays an important role in the number of multiple stable states involved in the bifurcation, as discussed in [57, 89].

3.3.3 Classification of the dynamics

Theorem 6 *Consider the map (96) subject to the constraints given in Theorem 5. For each* (a, γ) *with*

$$\gamma > 1, \quad a < 1, \quad 0 < a < \frac{\gamma - 1}{\gamma^2 - \gamma + 1},$$

there is a non-trivial interval of $b-$*values such that the map has two stable fixed points, one in* $(0, 1/a)$ *and the other in* $(1/a, \infty)$*. For each* (a, γ) *with*

$$\gamma > 1, \quad \frac{\gamma - 1}{\gamma^2 + 1} < a < \frac{\gamma - 1}{\gamma^2 - \gamma + 1},$$

there is a non-trivial interval of $b-$*values such that the map has a chaotic attractor in* $[-1, \gamma - 1]$ *and a stable fixed point in* $(1/a, \infty)$*.*

The work in [K3] allowed us to create a 3-dimensional Filippov type system, which exhibits one-parameter grazing-sliding bifurcations leading to multiple attractor grazing-sliding bifurcations. In particular in [K2], we present a constructed explicit example of a three-dimensional Filippov type flow where we show the birth of multiple attractors in grazing-sliding bifurcations. To the best of our knowledge, it is the first such an example of a Filippov type flow where grazing-sliding bifurcation is shown to trigger birth of multiple attractors, reported in the literature. Three qualitatively different scenarios are shown; namely, birth of period-two and period-three stable orbits with one sliding segment, chaotic attractor coexisting with stable period-three orbit characterised by a segment of sliding motion, and a coexistence of a period-three sliding orbit with two sliding segments and a limit cycle with no sliding segments. Our work reveals an important feature of the normal form map used to construct the Filippov flow that would produce the desired dynamics. Namely, due to the fact that the normal form that we use is valid only locally around the grazing-sliding bifurcations, the scale of the variation of the bifurcation parameter past the grazing-sliding had to be carefully chosen to see the dynamics predicted by the map. In other words, sufficiently small neighbourhood where the normal form is valid, in the context of nonsmooth bifurcations, seems to mean a different order of magnitude in the range of the bifurcation parameter variation than in the context of smooth bifurcations. We should add here that even in 3D flows the full dynamics about the grazing-sliding is not understood as yet. For example in [92], the link between grazing-sliding and the occurrence of Arnold's tongues in the bifurcation is explored.

3.4 Diverse new classes of one-parameter Discontinuity Induced Bifurcations

The methodology to classify one- and two-parameter DIBs presented here clearly points out to the difficulties in presenting a unified theory for DIBs. Firstly, from the results shown here it follows that depending on the class of a nonsmooth system a definition of a one and two parameter bifurcation has two be given as well as appriopriate manner of determining the unfolding parameters. Secondly, the outcomoe of the dynamics depends on the dimension of phase space of the system. Finally, it remains an open question what type of perturbations are permitted which lead to a DIB in a PWS system, see for example [23] where a divers types of DIBs are illustared by means of examples. In the current section, we present a novel class of one-parameter DIB of relevance to application presented in [K1].

In particular, consider a class of systems given by

$$\dot{x} = A_I x \quad \text{for} \quad |Cx| \leq \phi, \tag{97}$$

$$\dot{x} = A_O x \quad \text{for} \quad |Cx| > \phi, \tag{98}$$

where $A_I \in \mathbb{R}^2 \times \mathbb{R}^2$ is a non-singular matrix with the eigenvalues corresponding to a saddle-node equilibrium point, and $A_O \in \mathbb{R}^2 \times \mathbb{R}^2$ is a non-singular matrix with the eigenvalues corresponding to a stable equilibrium point of the focus type. The product of the state vector $x \in \mathbb{R}^2$ and the constant control row vector $C \in \mathbb{R}^2$ determines the switching between the two linear vector fields for some fixed and positive value of ϕ. In what follows, we consider a novel Hopf bifurcation scenario in the above class of systems under the variation of the control vector from $C = C_0$ to $C = C_0^\varepsilon$.

3.4.1 Planar switched systems with dead-zone

Consider switched systems where the bifurcation parameter, say β, is increased from 0 and implies a change of the control vector C_0 from $C_0 = [-1 \quad 0]$ to $C_0^\varepsilon = [-1 \quad \beta]$, where $\beta = \mathcal{O}(\varepsilon)$. This variation is a change from purely positional feedback control law to position-velocity feedback law. Matrices A_I, A_O, the state vector $\mathbf{x}$ and the width of the dead-zone are given by

$$A_I = \begin{pmatrix} 0 & 1 \\ A & 0 \end{pmatrix}, \quad A_O = \begin{pmatrix} 0 & 1 \\ A - K_p & -K_d \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} \theta \\ \dot{\theta} \end{pmatrix}, \quad |C_0 \mathbf{x}| \leq \theta^*, \text{ or } |C_0^\varepsilon \mathbf{x}| \leq \theta^*, \tag{99}$$

where $K_p > A > 0$, $K_p - A > K_d^2/4$, $K_d > 0$ and $\theta^* > 0$. In this set up the eigenvalues of A_I correspond to the system's equilibrium point of a saddle type and the eigenvalues of A_O correspond to a stable focus, as assumed earlier. Matrix A_I is expressed in so-called controllable canonical form [14, 16] and this structure can be assumed without loss of generality

3.4.2 Hopf-like one parameter bifurcation

Theorem 7 *Consider the Filippov system (97)-(98) with A_I and A_O as above. Under the $\mathcal{O}(\varepsilon)$ perturbation applied to the switching line by means of perturbation from $C_0 = [-1 \quad 0]$ to $C_0^\varepsilon = [-1 \quad \beta]$, where $\beta = \mathcal{O}(\varepsilon)$, the system undergoes a supercritical Hopf-like bifurcation scenario where a stable pseudo-equilibrium loses its stability and a stable limit cycle, surrounding the now unstable pseduo-equlibrium, is born in the bifurcation. The amplitude of the limit cycle scales as $\mathcal{O}(\sqrt{\varepsilon})$ of the perturbation.*

Theorem 7 shows that switched linear systems with dead-zone and purely positional feedback under a small parameter perturbation, from position to position-velocity control, may loose a stable pseudo-equilibrium state (an equilibrium of the switched system which lies on the switching manifold) and produce a limit cycle in a Hopf-like scenario. Using asymptotic method we analyze this novel bifurcation and show the loss of stability of the pseudo-equilibrium and a birth of stable limit cycles with the amplitude, say $|x|$, growing as the square root of the bifurcation parameter ($|x| = \mathcal{O}(\sqrt{\beta})$, where β refers to the bifurcation parameter).

In [K1] we also consider switched systems with dead-zone and purely positional feedback law, but with the switching decision function that contains time delay. We investigate this system numerically for small values of delay time $\tau = \mathcal{O}(\varepsilon)$. We find that the system, considering the same parameter values as in the non-delayed case, exhibits a Hopf-like bifurcation scenario under the variation of τ, which not only qualitatively but also quantitatively matches the Hopf bifurcation in the switched system with no time delay. In control literature, it has been suggested that delays in positional feedback laws may serve as approximation of velocity components since $v \approx (x(t+\tau) - x(t))/\tau$. However, in our case the time delay is included only in threshold detection and so the agreement of the qualitative and quantitative nature in the case of the two types of novel Hopf-bifurcation scenarios reported in the current work is somehow surprising. A theoretical explanation for this agreement follows from the fact that for sufficiently small time delays the delayed switching line can be approximated as a position-velocity switching line provided that the time of evolution between switchings is greater than the delay time τ, which is, indeed, the case.

We should note here that similar systems have been analyzed in [8, 87] and there an onset of small scale limit cycles born from a pseudo-equilibrium have been also reported. However, these systems have been characterized by the presence of time delays in the position and velocity state variables as well.

Bifurcations which lead to the creation of limit cycles in switched systems, due to changes in the control strategy, have been observed in different contexts, for example due to an introduction of small hysteresis, see Sec. 2.1 in [72]. In some instances, these bifurcations can be seen as bifurcations from infinity; from the point of view of perturbations applied not to the switching, but to a state variable, see Sec. 2.2 in [72] and [36, 66] for further details. From these works, it is clear that limit cycles can be born from a pseudo-equlibrium in switched systems in a variety of scenarios and their exhaustive and unified classification seems extremely difficult. Perhaps one of the ways in which one could attempt the classification of distinct Hopf-like bifurcations in nonsmooth systems would be by considering scaling laws. That is, we should note that the bifurcation reported in the current section differs, for example, from the Hopf-like bifurcation scenario analyzed in [46], by the scaling of the amplitude as a function of bifurcation parameter β. In the other work, the amplitude of the limit cycle born in the bifurcation grows linearly as a function of the bifurcation parameter. The question now arises whether one could divide the Hopf-like bifurcations in switched systems into classes characterized by different scaling laws.

4 Other research and results

4.1 Systems with time delays in switching function

Switched relay control systems, which are modelled by means of an idealised on/off relay control, or a switch between two states, one positive one negative - termed as negative feedback control - often feature hysteretic behaviour, symmetry as well as time delays in control inputs. As it will be discussed later, one such an example is provided by the neuromuscular control in human subjects. For instance, during quiet standing of healthy human subjects, the control input is sent by means of muscle spindles with a delay time of around 150ms triggering jerk-like muscle movements. In [KO4,KO7], we study dynamical systems that switch between two different vector fields, a hysteresis and with a delay in the switching decision function.

In particular, the main results of [KO4] paper can be summarised as follows: 1. We show that, in the case of negative feedback, the system with relay feedback, but no delay can only exhibit symmetric, unimodal cycles (that is limit cycles charcterised by two switching events and one maximum) if they exist. 2. We prove that positive feedback gives rise, instead, to the possibility of aperiodic trajectories and chaotic behaviour in the nondelayed hysteretic relay system. 3. In the case of small delays (in a sense that will be clarified when we discuss the results in [KO7]), we prove that the dynamics of the delayed relay system are qualitatively the same as those of an appropriately selected nondelayed relay system. 4. We derive analytically a diagram of the system's behaviour in the two-parameter space, showing the intricate interplay between the width of the hysteretic region and the amount of delay acting on the system. 5. We describe a novel discontinuity-induced bifurcation event, named an event-collision, that is unique to piecewise smooth systems with delayed switchings which is the subject of investigations in [KO7].

When the delay reaches a problem dependent critical value so-called event collisions occur. The work in [KO7] classifies and analyses event collisions, a special type of discontinuity induced bifurcations, for periodic orbits. Our focus is on event collisions of symmetric periodic orbits in systems with full reflection symmetry, a symmetry that is prevalent in applications. We derive an implicit expression for the Poincaré map near the colliding periodic orbit (Lemma 8 and Theorem 9 in Sec. 4.2). The Poincaré map is piecewise smooth, finite-dimensional, and changes the dimension of its image at the collision. In the second part of the paper, we apply this general result to the class of unstable linear single-degree-of-freedom oscillators where we detect and continue numerically collisions of invariant tori. Moreover, we observe that attracting closed invariant polygons emerge at the torus collision. The derivation of Theorem 9 can be generalised in a straightforward manner to other discrete symmetries and other than binary switches (with more notational overhead) as long as the symmetry can be reduced locally near the periodic orbit. The reduction extends the applicability of theory and numerical methods that have been developed for smooth and piecewise smooth low-dimensional maps to systems with delayed switches. On the numerical side this includes direct continuation of periodic orbits and their bifurcations and discontinuity induced events (such as grazing and collision) and the continuation of smooth invariant curves. Robust and universal methods for continuation and detection of discontinuity induced bifurcations for periodic orbits have been developed, among others, by Piiroinen [78] and Dercole and Kuznetsov [31]. Methods for the continuation of closed invariant curves have been also developed. See for example [28, 82] among other works. There is, however, still a large gap between recent developments of numerical methods for closed invariant curves and piecewise smooth systems and their actual availability in the form of software. Due to this gap the investigation of the oscillator presented in Section 6 of the paper could not rely on generally available tools. The analysis of the oscillator shows that dynamical phenomena of hybrid systems with delayed switches can be systematically discovered with the help of numerical continuation and the reduction theorem.

4.2 Dynamics of singularly perturbed PWS systems

An important question for the theory of one- and two-parameter DIBs is how these bifurcations are affected by perturbations. In particular, in [KO8] we consider how sliding in a Filippov system with two vector fields is affected by perturbations. If we add a small perturbation to the vector fields or to the switching decision function (the derivative of the perturbation is also assumed to be small) then any exponentially stable periodic orbit or equilibrium of a Filippov system persists, as we show in [KO8], and remains stable. This also applies to pseudo-equilibria (equilibria of the sliding flow, sitting exactly on the switching manifold) and to periodic orbits that have sliding segments. This persistence mirrors the results of classical bifurcation and invariant manifold theory for smooth dynamical systems [43]. Another typical perturbation arising in the modelling process are stable singular perturbations. In a simple model one has replaced rapidly converging parts of the dynamics with their equilibrium, making the assumption that this equilibrium follows the slow dynamics quasi-statically. In a more complex case of the same system (or in reality) the equilibrium of the fast dynamics is not attained perfectly, which constitutes a small perturbation. Practical examples of this type of perturbation are small capacitances and inductances in electrical circuits, imperfect rigidity in mechanical systems, or fast chemical reactions (or other processes) in biological systems. Again, for smooth dynamical systems classical theory [43] proves that all hypberbolic equilibria, periodic orbits and, more generally, normally hyperbolic invariant manifolds persist. That is, for example, an exponentially stable equilibrium or periodic orbit (and any of its bifurcations) observed in a simple model obtained by making quasi-static assumptions is also present when the fast dynamics is taken into account, as long as the difference in time scale is sufficiently large. In general, in smooth dynamical systems any phenomenon that persists under regular perturbations (perturbations of the right-hand-side) also persists under stable singular perturbations. Fenichel's Theorem reduces hyperbolic singular perturbations to regular ones by proving the existence of a normally hyperbolic invariant manifold [43].

In [KO8], we prove that stable singular perturbations have a much stronger influence on the dynamics in Filippov systems than in smooth dynamical systems. We demonstrate that stable pseudo-equilibria and stable periodic orbits with sliding do not necessarily persist. We study periodic orbits with an infinitesimally small sliding segment, that is, close to a grazing-sliding bifurcation. We found two generic cases depending on the geometry: the local return map around the grazing periodic orbit develops a discontinuity if the condition on the existence of an attracting sliding region is violated. Otherwise, the continuity of the return map persists, but the asymptotic slope may have a change of order 1 (uniformly for $\varepsilon \to 0$).

The qualitative change of the local return map induces qualitative changes to the dynamics on a small scale. A piecewise discontinuous map with a square root singularity of the slope on one side of the discontinuity, as occurs when a parameter determining the slope satisfies the inequality $\theta < 0$ in the minimal example, shows inverted period-adding cascades of periodic orbits if one varies the parameter through its critical value [39]. The parameter range where these cascades can be observed is of order ε. In the other case, $\theta > 0$, the observed dynamics depends strongly on the one-sided derivative $s(\theta;0;0)$ defined in Lemma 3. It can be chaotic if $s(\theta;0;0) < -1$, which is possible for small θ. Our analysis is valid on a scale of order ε^2 in phase space and parameter space.

The results of Lemma 3 can be generalized to higher-dimensional slow-fast systems in a straightforward manner as long as the dimension of the fast subsystem is 1. There are, however, some technical difficulties to generalizing the expressions of Lemma 3 for higher dimensional fast subsystems ($y \in \mathbb{R}^m$, $m > 1$); a trajectory following the dynamics inside the stable fibres (following a linear stable ODE) may intersect the switching hyperplane several times. In $\mathbb{R}$ every trajectory in a stable linear system approaches the origin in a monotone (increasing or decreasing) fashion, which is not true in $\mathbb{R}^2$ in the Euclidean norm. Furthermore, the Poincaré-section Discontinuity Mapping is only implicitly given as the root of a nonlinear equation. In general, this implicit expression is determined by the intersection of a trajectory following a stable linear system with a hyperplane.

In [KO10] we study the qualitative dynamics of piecewise-smooth slow-fast systems (singularly perturbed systems) which are everywhere continuous. We consider phase space topology of systems with one-dimensional slow dynamics and one-dimensional fast dynamics. The slow manifold of the reduced system is formed by a piecewise-continuous curve, and the differentiability is lost across the switching surface. In the full system the slow manifold is no longer continuous, and there is an $\mathcal{O}(\varepsilon)$ discontinuity across the switching manifold, but the discontinuity cannot qualitatively alter system dynamics, which is a standard result which can be shown using directly Fenichel's theory. The main results of the paper is the description of phase space topology which is used to unfold qualitative dynamics of planar slow-fast systems with an equilibrium point on the switching surface. In this case the local dynamics corresponds to so-called boundary-equilibrium bifurcations, and four qualitative phase portraits are uncovered. Our results are then used to investigate the dynamics of a box model of a thermohaline circulation, and the presence of a boundary-equilibrium bifurcation of a fold type is shown. In the context of applications of this work to the box model, the presence of multiple stable states may indicate the possibility of flow reversal in thermoaline circulation.

The four scenarios observed in planar slow-fast systems will be observed in higher dimensions (with n-dimensional slow dynamics), and it is very likely that additional dynamics will arise as well; boundary-equilibrium bifurcations in three-dimensional piecewise-smooth flows lead, for instance, to a non-smooth equivalent of a Hopf bifurcation with a limit cycle growing linearly in amplitude from an equilibrium colliding with the switching manifold [35]. This rises the question whether additional dynamics can be triggered by the presence of more than one fast dimension.

4.3 Dynamics of diverse classes of hybrid systems

The use of a digital computer as a controller device has grown in the past decades leading to a widespread application of digital control systems [61, 80]. Nowadays digital control systems occur in a plethora of applications ranging from chemical processes, aircraft and traffic control to process control in industries such as machine tools production [47, 40]. The control, design and analysis of these control systems involve understanding the interaction between continuous and discrete dynamics. For example, the automated control of a car moving on a road is implemented by digital computer but the motion of a car is continuous in time [56]. Computers that are used in such a system send a digital signal which is then converted to an analogue signal and can be fed into the actuator. The digital signal which corresponds to a finite sequence of numbers leaves or enters the computer at some time intervals, say τ , which we will term a sampling time.

In [KO6,KO10,KOCf9] we investigate the dynamics of Filippov systems where the information on switching is not a continuous function but is given at discrite time intervals.

In [KO6], we studied the dynamics of a simple one-dimensional on/off control system where the control variable was given at discrete time intervals. In the case when the system evolution was assumed linear in the on and off states we were able to obtain a re-scaled circle map that captures the system dynamics. We have shown that depending on the system parameters we might encounter a family of periodic orbits, quasi-periodic oscillations or a banding structure of quasi-periodicity. Using equivalent methodology to that which allowed us to study the linear case we extended the analysis to the case when the system evolution is governed by generic non-linear functions. In particular, we have shown that a fixed point attractor and chaotic dynamics are present in this more general case. Some results that link the sampling time with the width of the interval on which the asymptotic dynamics might settle have been also presented (Lemma 6.3).

In [KO10] we study Filippov type systems with digital sampling. It is shown that digital sampling may lead to the onset of chaotic dynamics. A simple example is studied in detail to reveal the mechanism leading to chaotic dynamics. The existence of at least one chaotic attractor is proved rigorously, but we have not excluded the possibility that other attractors exist. Thus uniqueness (or topological transitivity) of the attractor in the bounded region is still an open problem. We conjecture that the chaotic attractor is indeed unique. It is shown that the size of the chaotic attractor, measured as the distance between the chaotic attractor and a stable cycle of continuously sampled orbit is linear for sufficiently small values of the sampling time τ. The results are generalized to planar Filippov type systems with digital sampling. Using planar systems, we also show that in the limit when the sampling time $\tau \to 0$, the Filippov's method that gives the sliding flow of continuously sampled system, converges to the vector field of the discretely sampled system. This result can then be used to approximate the time of evolution along the switching surface which in turn can be utilized to determine if the expansion/contraction due to the zig-zag evolution along the switching surface can qualitatively alter system dynamics. Based on our finding we believe that the analyzed dynamics will also occur in n-dimensional Filippov type systems when the variable that determines switchings between different vector fields is sampled at discrete time intervals. To justify this conjecture we investigate a third order relay feedback system and introduce digital sampling to continuously sampled control variable. It turns out that, indeed, in certain instances we observe a transition from a stable orbit with a segment of sliding motion, existing in a continuously sampled system, to a chaotic attractor of digitally sampled system. Two scenarios are considered. In the first case the introduction of digital sampling destroys the sliding segment – instead of the sliding flow the system switches along the switching surface until it leaves off the switching plane. In the second case the introduction of digital sampling leads to the creation of chaotic dynamics. It was shown in [79] that discrete control typically creates a chaotic attractor in the vicinity of an unstable equilibrium. There are certain similarities that lead to the onset of chaos in our case and in the case investigated in [79]. In our case we deal with periodic orbits and if we wish to stabilize an unstable orbit that might correspond to some desired oscillatory behavior of a control system the application of the digital control may quite likely lead to the creation of a chaotic orbit, whereas control provided in continuous time will lead to a stable orbit with a segment of sliding. In [79] it is claimed that artificial neural networks with reinforcement learning are known to be able to learn such a control scheme. It would be interesting to investigate further the link between our findings and neural networks with reinforcement learning. We should also mention that the onset of chaotic dynamics triggered by this mechanism is similar to an abrupt transition from a stable periodic orbit with sliding to a small scale chaotic dynamics that might occur in Filippov type systems under an introduction of an arbitrarily small time delay in the switching function [85].

4.4 Modelling and analysis of Filippov systems in the context of neuromuscular human balance control

In recent years, much of research effort has been spent on understanding the character of control strategy during different tasks performed by human neuromotorcontrol system. For example, there is currently an ongoing controversy whether human quiet standing can be better described by linear continuous time models or intermittent control models. This controversy has led, broadly speaking, to the use of two classes of control models. First class includes linear, continuous time systems, see for instance [58, 60]. These models exclude thresholds, instantaneous switchings and time variant processes such as open loops. However, impulsive like muscle movements have been detected during quiet standing [70] leading to the use of switched and/or intermittent control models [48, 49, 68, 69, 67], which are examples of hybrid (switched) systems. Another context in which intermittent control plays an important role is target tracking. Consider a manual control task in which, using a joystick, a person has to place a beam on a computer screen at a specific location. In the experimental set-up, the motion of the beam is electronically generated by an output from an unstable system, which is driven by noisy disturbances, with a human acting as a controller in the loop. It seems that the best strategy to place the beam in a neighbourhood of the desired location is by gently tapping the joystick. Thus it is not a continuous time control, but an intermittent control which is used by the neuromuscular system. It is worth noting that in certain instances, it is impossible to stabilise a system using a continuous time control, but an intermittent (switched/hybrid) control action may stabilise a system. In particular, it was shown in [86] that continuous time extended time-delayed feedback control cannot be applied to stabilise an unstable periodic orbit. In contrast, an intermittent extended time-delayed feedback control does stabilise an unstable periodic orbit. This result signifies that to achieve stability, in certain conditions, it may be necessary to apply an intermittent (switched) control, not a continuous time control.

Research in works [KO11,KO12,KO13,KO14,KO15] deals with mathematical modelling and investigations of the character of neuromuscular control during tasks such as quiet standing or the aforementioned target tracking experiment. In particular in [KO11], we introduce a model of human balance during quiet standing following the idea that a human body, on the macroscopic scale, can be modelled by a single-link inverted pendulum, and balance is achieved using linear feedback control with time delay in the proportional and derivative error signals. We assume a threshold value of the angle of the sway below which the human neuromotorcontrol system cannot detect any sway motion. We obtain a planar switched (hybrid) model. We find that to achieve stabilization, which is seen as 'small' oscillations about an upright equilibrium, it is necessary that both the proportional and derivative signals of the control system are used. These stable oscillations seem to represent closer to observation stable state for upright standing than the equilibrium points [74]. Therefore, we study the effects of parameter variations on their existence. Our parameter study leads to the detection of a multiple number of stable oscillatory states existing for the same parameter values, and for a wide range of the control parameters corresponding to the derivative term of the PD controller. We also find a homoclinic bifurcation that gives birth to a stable symmetric orbit with a long period. In particular, we show, using a numerical experiment, that close to a homoclinic bifurcation white noise introduced additively may result in the system switching between the two regions where symmetric stable solutions exist in the deterministic switched system leading to an apparent bi-stability; in other words, the switched system with added noise evolves for some time in the neighbourhood of each one of the two stable asymmetric limit cycles (present in the deterministic system) by switching between their regions of existence. This scenario can explain switchings between a pair of stable asymmetric attractors observed in the first-order model in [42], which in turn was used to explain different scaling patterns that could be detected in human postural sway data.

This initial research, which was conducted in collaboration with experimentalists working on human balance conctrol, led to my further work aimed at 1. investigating the dynamics of switched models of relevance to modelling human nueromusucular control system during tasks such as quiet standing or target tracking; 2. using different techniques to determine some measure of time series data which can then be used for the purposes of determinig whether, indeed, systems with discontinuous nonlinearities are better descriptors for human neurmuscluar control in the case of aforementioned tasks. In particular, the aforementioned collaboration led to a successful grant entitled "Abrupt changes in the behaviour of hybrid systems in discontinuity induced multiple attractors bifurcations" funded by the EPSRC under First Grant Scheme program, reference EP/K001353/1, where I was the Principal Investigator. The value of the grant was 125 000£ (about 600 000 PLN) and the grant commenced on the first of February 2013 and finished on the 31 August 2014. The grant led to 4 journal publications [KO12-KO15] (all published) and one conference proceedings. One of the important contributions of the research conducted during the grant was the development of an algorithm for the detection of discontinuous nonlinearites in switched systems with noise. In particular, the algorithm was used on experimental data and the results support the existence of intermittent control action of the neuromuscular system of humans during quiet stance [KO15]. The research conducted during the grant has led to a number of further research questions such as what is the link between control strategy and discontinuity induced bifurcations in hybrid (switched) systems. Namely work [K1].

In [KO12], we examine whether an ARMA model can be fitted to a process characterised by switched nonlinearities. In particular, we conduct the following test: we generate data from known LTI and nonlinear (threshold/dead-zone) models of human balance and analyse the output using ARMA. We show that both these known systems can be fitted, according to standard criteria, with low order ARMA models. To check if there are some obvious effects of the dead-zone, we compare the power spectra of both systems with the power spectra of their ARMA models. We then examine spectral properties of three posturographic data sets and their ARMA models and compare them with the power spectra of our model systems. Finally, we examine the dynamics of our model systems in the absence of noise to determine what is the effect of the switching threshold (dead-zone) on the asymptotic dynamics

When we compare the power spectrum of the switched system with the power spectrum of its ARMA model as well as when we compare the power spectrum of the linear system with the power spectrum of its ARMA model, the power spectra show a good fit in the lower frequency range (up to around 2 Hz) with a mismatch for higher frequencies. This agrees with the theory because low order ARMA models match low frequency bands. Moreover, there are no obvious qualitative differences in both cases. We then compare power spectra of three ARMA models of three representative experimental posturographic data sets with ARMA models of the switched and linear systems. A reasonably good fit, to both models, is observed in the lower frequency range (up to 1.5 Hz) with a growing mismatch for larger frequencies. Thus, AMRA model fitting may lead to misinterpretation of the results; that is, a good fit of a time series data with an ARMA model does not imply that the underlying process is linear and time invariant. We also investigated the dynamics of underlying models in the absence of noise. We found qualitative differences in the asymptotic dynamics of both systems for parameter values used in our investigations; the dynamics of the linear system represents the fluctuations of a noisy equilibrium, whereas the dynamics of the switched system is dominated by switchings, due to noise, between attractors formed by two coexistent limit cycle oscillators (in the absence of noise).

In [KO13], we propose a method based on wavelet and frequency analysis for detecting abrupt changes in the process data of the van der Pol oscillator model with noise. The advantage of this technique is that it allows us to decompose the data into a set of independent coefficients (mono-components) corresponding to orthogonal basis functions. These mono-components are then analysed and recombined into a signal that contains the instantaneous frequency reflections, but not the system main response or noise. The output is a series of peaks, each of which occurs when the system trajectory exhibits abrupt change. The effectiveness of the developed algorithm in detecting such changes arises from the fact that abrupt changes manifest themselves as spikes in the time-frequency plane. Thus, the proposed method is promising in detecting not only features such abrupt changes but also discontinuities in time series data. The discontinuities which we seek to identify in such systems occur due to switchings between differentiable vector fields.

In [KO14], we numerically investigate a switched system of human balance control during quiet standing. Three analytical tools are applied to the time series data generated by the model system. We start our analysis by investigating the dynamics of the underlying model in the absence of noise for the parameter values corresponding to human balance control of quiet stance. The system dynamics settles on limit cycle attractors. We then use the instantaneous frequency method to detect the discontinuous nonlinearities present in the signal generated by our model system. We find spikes in the time-frequency plot of the analysed simulated data which indicate the presence of discontinuities. Moreover, for shorter delay times the spikes have smaller amplitude than for longer time delay. Longer time delay can be seen as increased dead-zone. Thus there is a correlation between the computed instantaneous frequency and the size of the dead zone. We then employed the entropy analysis for the purpose of investigating the complexity of the system.

We found that the complexity of the switched model system is increased when the noise signal is switched off. We conjecture that a certain level of noise, by smoothing out discontinuities, decreases the system's complexity. We also find that there is a positive correlation between increasing the length of the time delay (which can be viewed as increasing the width of the dead zone) and system's complexity. We also use fluctuation analysis as a measure of complexity in the model system. The results demonstrate that the complexity increases when the the size of the time delay (or the width of the dead-zone) increases. Finally, we perform entropy and fluctuation analyses for the collected experimental data of healthy human subjects. Our results show that the complexity measures of the experimental data correspond to Brownian noise and are similar to the complexity measures of the switched system with noise when the value of the neuro-muscular delay in the model is set to $\tau = 150$ms, which is physiologically feasible.

We believe that there is a direct link between the size of the dead zone of the switched system and the corresponding complexity measures. Such analysis may, therefore, reflect important properties of the sway motion of healthy human subjects.

External Collaborations - dynamical systems and control

I have an ongoing collaboration with Prof. Paul Glendinning from the University of Manchester on mathematical modelling and analysis of switched human balance control systems. Prof. Glendinning is an expert in the analysis of bifurcations in discrete maps, which naturally arise in the process of analytical reduction of switched flows. I also collaborate with Dr Arne Nordmark from the Royal Institute of Technology (Sweden, Stockholm) on the analysis of bifurcations in nonsmooth systems. The explorations of links between bifurcations and control, which is also an area of my interest, will be conducted in collaboration with Dr Jan Siber from the University of Exeter.

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