Summary of Professional Accomplishmer

1 Name

Grzegorz Serafin

2 Diplomas and degrees, including the name of the institution which conferred the degree, year of degree conferment, title of the PhD dissertation.

2010 **M.Sc. in Mathematics**
Institute of Mathematics and Computer Sciences
Faculty of Fundamental Problems of Technology
Wrocław University of Technology[1]
Master's dissertation:
Selected aspects of the potential theory on hyperbolic spaces
Supervisor: Dr. Sc. Jacek Małecki.

2014 **Ph.D. in Mathematics**
Institute of Mathematics and Computer Sciences
Faculty of Fundamental Problems of Technology
Wrocław University of Technology[1]
Doctoral dissertation:
Potential theory of hyperbolic Brownian motion with drift
Supervisor: Dr. Sc. Tomasz Żak
Auxiliary supervisor: Dr. Sc. Jacek Małecki

3 Information on employment in research institutes or faculties/departments

2010–2016	Wrocław University of Technology[1], Poland Position: Assistant
since 2016	Wrocław University of Science and Technology[1], Poland Position: Assistant Professor
2015–2019	Nanyang Technological University, Singapore Position: Research Fellow Periods of employment: 13.05. – 30.09.2015 02.10.2017 – 18.02.2018 18.02. – 25.08.2019

[1] In 2017, the name of Wrocław University of Technology changed to Wrocław University of Science and Technology

4 Description of the achievements, set out in art. 219 para 1 point 2 of the Act

The scientific achievement:

The cycle of scientific articles titled:
Brownian motion and Bessel process in convex domains.

The list of articles constituting the scientific achievement:

[H1] J. Małecki, G. Serafin, T. Żórawik
Fourier-Bessel heat kernel estimates
Journal of Mathematical Analysis and Applications 439(1), 91–102, 2016.

[H2] G. Serafin
Exit times densities of the Bessel process
Proceedings of the American Mathematical Society 145, 3165–3178, 2017.

[H3] J. Małecki, G. Serafin
Dirichlet heat kernel for the Laplacian in a ball
Potential Analysis 52, 545–563, 2020.

[H4] G. Serafin
Laplace Dirichlet heat kernels in convex domains
Journal of Differential Equations 314, 700–732, 2022.

4.1 Introduction

Heat kernels are basic objects in mathematical analysis, as fundamental solutions to parabolic differential equation (heat equations), as well as in the theory of stochastic processes, playing the role of transition probability densities. They are also, or maybe primarily, important from the point of view of physics, since they describe evolution of particles, temperature and other phenomena. Studies on the behaviour of heat kernels related to various kinds of operators and domains or manifolds have very long history and there is an enormous number of research papers on this topic including many beautiful and general results (see, among others, [62, 15, 22, 23, 28, 39, 67] and the references therein). Nevertheless, it turns out that there are still many open question even in the most classical case, i.e. the one involving the Laplace operator (or, equivalently, the Brownian motion) in Euclidean space, which is the subject of articles [H3] and [H4].

Another operator/process set examined in the achievement is the half of the Bessel operator $\frac{1}{2}\mathcal{L}_\mu$ and the related Bessel process $R^{(\mu)}$, where $\mu \in \mathbb{R}$ denotes its index. For indices of the form $\mu = n/2 - 1$, $n = 1, 2, 3, ...$, $\mathcal{L}_\mu$ is the radial part of the n-dimensional Laplacian and the Bessel process may be described as a norm of n-dimensional Euclidean Brownian motion. The process is also closely related to the geometric Brownian motion by the Lamperti relation. Furthermore, in form of the Bessel-Brownian diffusion, it plays a crucial role in approach to hyperbolic Brownian motion [1, 14] and is used to represent symmetric stable processes as well [45]. Finally, it is exploit to model stock prices and has further applications to mathematical finance [25, 27, 65].

Our goal is to study the behaviour of probability transition densities of the aforementioned diffusions killed when exiting a given set. In other words, we examine Dirichlet heat kernels for the Bessel and Laplace operators. We consider mainly bounded sets, for which long-time behaviour usually follows from general spectral theory, which leads to series representations involving eigenfunctions and eigenvalues of an operator in a given set. Unfortunately, such representations happen to be completely unhelpful when dealing with small times, since the sum is highly oscillating and the cancellations between the terms matter in that case. The first estimates addressing short time behaviour of the heat kernels are associated with the *property of not feeling the boundary* introduced by M. Kac in [37], which says that for points x, y from a domain D such that the interval $\overline{xy}$ is contained in D it holds

$$\frac{p_D(t,x,y)}{p(t,x,y)} \xrightarrow{t\to 0} 1, \tag{1}$$

where $p(t,x,y)$ and $p_D(t,x,y)$ are the global and Dirichlet heat kernels, respectively. Later on, this property has been proven and generalized in many other settings. Nevertheless, its weakness is that the space arguments x and y are fixed or, at least, far away from the boundary of D. On the other hand, there are numerous results on the estimates of the heat kernels which take into consideration the boundary behaviour. However, they share another weakness - lower and upper bounds are completely incomparable for some range of argument, which does not enable one to catch the precise behaviour of the Dirichlet heat kernels, or their ratio to the global heat kernel. The achievement consisting of the articles [H1]–[H4] is devoted to solving this problem in the case of the sets $(0,1)$ and $[0,1)$ for the Bessel process and of a large class of $C^{1,1}$ domains for the Brownian motion. Namely, estimates are derived such that the the lower and upper bounds are, up to a multiplicative constant, the same. We will call such estimates sharp. Some results for the distribution of the first exit time and place are established as well. We refer the reader to [6, 7, 8, 31, 48, 49] for some other recent articles focused on sharp estimates of heat kernels in other settings. Such precise estimates for Dirichlet heat kernels are very rare. In the case of the Brownian motion they have been known only in such basic cases as a half-line and an interval (and their multidimensional extensions and products) as they are given by simple explicit formulae. For the Bessel process the Dirichlet heat kernels for a half-line have been recently estimated as well [8, 9], but it required some substantial effort. After analyzing the literature, one can observe that most approaches to the heat kernel estimates are purely analytical, which usually leads to different constants in exponents in Gaussian-type estimates and this causes a significant gap between lower and upper bounds. In comparison, our approach relies on probabilistic methods which yield some additional representations of the heat kernels. Furthermore, they provide us with many intuitions and help us with understanding of the behaviour of the studied functions, which plays an important role in proper approximation of the heat kernels. Nevertheless, despite of the strength of the probabilistic methods, they still require to be complemented with rough analysis and this mixture turned out to be very efficient in the topic.

The description of the results is organized as follows. In Section 4.2 we collect some preliminary material. Section 4.3 is devoted to the Bessel process, which is studied in the articles [H1] and [H2]. Section 4.4 corresponds to results from [H3] and [H4], that concern the Dirichlet heat kernels estimates for Brownian motion. Note that in the case of articles [H1] and [H3] not only general methods are mentioned, but sketches of the proofs are given as well, since these articles contain only one main result each. Finally, Section 4.5 specifies my contribution to the articles included in the achievement and in Section 4.6 one can find the summary of the results and their significance.

4.2 General definitions and notation

In this subsection we gather basic facts and introduce notation related to the general theory of stochastic processes and semigroups.

Let $X = \{X(t)\}_{t\geqslant 0}$ be a n-dimensional diffusion and denote by $\mathbb{P}_x$ and $\mathbb{E}_x$ the probability law and the expected value of the process X starting from $x \in \mathbb{R}^n$, respectively. The transition probability density of the process is a function $p(t,x,y)$ satisfying

$$\mathbb{P}_x(X(t) \in A) = \int_A p(t,x,y)m(y)dy, \qquad t \geqslant 0, A \in \mathcal{B}(\mathbb{R}^n), \tag{2}$$

where $m(dy) = dy$ is the Lebesgue measure in the case of the Brownian motion and the speed measure $m(y)dx = 2y^{2\mu+1}dy$ in the case of the Bessel process. Note that in Section 4.3 we use notation $p^{(\mu)}$ instead of p in order to indicate the index $\mu \in \mathbb{R}$ of the Bessel process (see Section 4.3.1) and to distinguish its transition probability density from the density of the Bessel process.

The definition (2) ensures the symmetry $p(t,x,y) = p(t,y,x)$. Furthermore, the following Chapman - Kolmogorov identity (semigroup property) holds for $s,t \geqslant 0$ and $x,y \in \mathbb{R}^n$:

$$\int_{\mathbb{R}^n} p(s,x,z)p(t,z,y)m(z)dz = p(s+t,x,y). \tag{3}$$

The infinitesimal generator of the process is given by

$$Af = \lim_{t\to 0^+} \frac{1}{t}\left(\int_{\mathbb{R}^n} p(t,x,y)f(y)m(y)dy - f(x)\right), \qquad f : \mathbb{R}^n \to \mathbb{R}.$$

Note that the transition probability density $p(t,x,y)$ is a solution to the problem

$$\begin{cases} \dfrac{\partial}{\partial t}u(t,x,y) = Au(t,x,y), \quad t > 0, \quad x,y \in \mathbb{R}^n, \\ \lim\limits_{t\to 0^+} u(t,x,y)m(y) = \delta_x(y). \end{cases} \tag{4}$$

The limit is understood in the distributional sense and δ_x stands for the Dirac delta distribution. The first equation is known as the heat equation and therefore $p(t,x,y)$ is called the heat kernel.

For a fixed set $D \subset \mathbb{R}^n$ (if D is connected and open we call it a domain) we define the first exit time from D by

$$\tau_D = \inf\{t > 0 : X(t) \notin D\}.$$

The density of this random variable will be denoted by $q_D(t,x)$. Next, we define $q_D(t,x,y)$ as we the density of exit time and place of the process X from D, i.e. for Borel sets $A \subset \partial D$ and $T \subset [0,\infty)$ we have

$$P_x(\tau_D \in T; X(\tau_D) \in A) = \int_A \int_T q_D(t,x,y)dt\, \sigma_D(y)dy,$$

where $\sigma_D(y)dy$ is the surface measure on ∂D generated by m. It clearly holds $q_D(t,x) = \int_{\partial D} q_D(t,x,y)dy$.

The process X^D is defined as the process X killed when exiting the domain D:

$$X^D(t) = \begin{cases} X(t), \text{ for } t < \tau_D, \\ \partial, \text{ for } t \geqslant \tau_D, \end{cases}$$

where ∂ is an additional state called cemetery. The transition probability density of the process X^D will be denoted by $p_D(t,x,y)$, i.e. for a Borel set $A \subset \mathbb{R}^n$ we have

$$\mathbb{P}^x\left(X^D(t) \in A\right) = \mathbb{E}^x[t < \tau_D; X(t) \in A] = \int_A p_D(t,x,y)m(y)dy.$$

If D is regular enough (of $C^{1,1}$ type, for example), the function $p_D(t,x,y)$ satisfies the heat equation (4) with additional Dirichlet condition $u(t,x,y) = 0$ for $y \in \partial A$, and is also called the Dirichlet heat kernel.

Furthermore, the Strong Markov property leads to the representation

$$\begin{aligned} p_D(t,x,y) &= p(t,x,y) - \mathbb{E}^x\left[\tau_D < t; p(t-\tau_D, X(\tau_D), y)\right] \\ &= p(t,x,y) - \int_0^t \int_{\partial D} p(t-s,z,y) q_D(s,z,y)dz\,ds. \end{aligned} \tag{5}$$

known in the literature as the Hunt formula [34]. Note that the function $p_D(t,x,y)$ also satisfies the Chapman-Kolmogorov identity (3) with $x,y \in D$. One of the consequences of the Hunt formula is the following monotonicity property: for $A_1 \subset A_2 \subset \mathbb{R}^n$ it holds

$$p_{A_1}(t,x,y) \leqslant p_{A_2}(t,x,y), \qquad x,y \in A_1,\ t > 0. \tag{6}$$

Throughout the whole description $|x-y|$ denotes the Euclidean distance between $x,y \in \mathbb{R}^n$. Furthermore, since many expressions depend on a distance of a point to the boundary of a given set D, we define for $x \in D \subset \mathbb{R}^n$

$$\delta_D(x) = \inf\{|x-z| : z \in \partial D\}.$$

By $B(x,r)$ we denote a ball of radius $r > 0$ centered at $x \in \mathbb{R}^n$. If the dimension of a ball in consideration is different than n, we indicate it in the subscript, i.e. $B_k(x,r)$ is a k-dimensional ball.

Finally, for two nonnegative functions f and g we denote $f \lesssim g$ whenever there exists a constant $c > 0$ such that $f \leqslant cg$ holds in an indicated range of arguments. If $f \lesssim g$ and $g \lesssim f$ we write $f \approx g$ and estimates of that type will be called sharp. Additionally, if the constant c depends on some parameters, we put these parameters over the sign $\lesssim$ or $\approx$. We also use the notation $a \wedge b := \min\{a,b\}$ and $a \vee b := \max\{a,b\}$ for $a,b \in \mathbb{R}$.

4.3 The Bessel process

4.3.1 Basic properties

We denote by $R^{(\mu)}(t)$ the Bessel process with index $\mu \in \mathbb{R}$ and starting from the point $x > 0$. It is a diffusion on $[0,\infty)$ with generator given by

$$\frac{1}{2}\mathcal{L}_\mu = \frac{1}{2}\frac{d^2}{dx^2} + \frac{2\mu+1}{2x}\frac{d}{dx}. \tag{7}$$

Another common parameter of the Bessel process is its dimension given by $\rho = 2\mu + 2$.

One of the difficulties to deal with is the behaviour of the Bessel process at zero, which depend on the value of the index μ. For $\mu \geqslant 0$ the process never riches zero, and for $\mu \leqslant -1$ we impose that the point zero is killing. In the case of $\mu \in (-1,0)$ we may consider one of both: killing or reflecting condition at zero. In the case when $\mu \geqslant 0$ or $\mu \in (-1,1)$ and zero is reflecting one can start the process from $x = 0$ as well, however, all the presented results come then by continuity, so we assume always $x > 0$ in order to simplify the presentation of results and proofs.

We write $\mathbb{P}_x^{(\mu)}$ and $\mathbb{E}_x^{(\mu)}$ for the probability law and the expected value of a Bessel process with an index $\mu \in \mathbb{R}$ on the canonical path space with starting point $x > 0$. The natural filtration of the process $R^{(\mu)}(t)$ is denoted by $\mathcal{F}_t = \sigma\left\{R^{(\mu)}(s) : s \leqslant t\right\}$. The laws $\mathbb{P}_x^{(\mu)}$ and $\mathbb{P}_x^{(\nu)}$ of Bessel processes with indices $\mu \in \mathbb{R}$ and $\nu \in \mathbb{R}$, respectively, are absolutely continuous and the corresponding Radon-Nikodym derivative is described by (see [55], p. 450, and [44], p. 314)

$$\left.\frac{d\mathbb{P}_x^{(\mu)}}{d\mathbb{P}_x^{(\nu)}}\right|_{\mathcal{F}_t} = \left(\frac{R^{(\nu)}(t)}{x}\right)^{\mu-\nu} \exp\left(-\frac{\mu^2-\nu^2}{2}\int_0^t \frac{ds}{\left[R^{(\nu)}(s)\right]^2}\right), \tag{8}$$

where $x > 0$, and the above given formula holds $\mathbb{P}_x^{(\nu)}$-a.s. on $\{\tau_{(0,\infty)} > t\}$. Taking $\nu = -\mu > 0$ and assuming $R^{(\mu)}$ is killed at zero, one can see that potential theory of processes with opposite indices are closely related. In particular, transition probability densities are equal up to a simple factor $(y/x)^{2\mu}$.

Let us note that the Bessel process with index $\mu = n/2 - 1$ (which corresponds do the dimension n), $n = 1, 2, 3, \dots$, may be represented as follows

$$R^{(\frac{n-2}{2})} \stackrel{d}{=} \sqrt{W_1^2 + \dots + W_n^2},$$

where $(W_1, \dots, W_n)$ is a n-dimensional Brownian motion (i.e. a diffusion generated by $\frac{1}{2}\Delta$) starting from $z \in \mathbb{R}^n$ satisfying $|z| = x$. Additionally, for $n = 1$ we may impose a killing condition at zero and then $R^{(-1/2)}$ represents simply the Brownian motion killed at 0.

By $p^{(\mu)}(t, x, y)$ we denote the transition density function (with respect to the speed measure $m(dx) = m(x)dx = 2x^{2\mu+1}dx$) of the process $R^{(\mu)}(t)$ with $\mu > -1$ and reflecting condition at zero. We have (Section 21 in Appendix 1 in [10])

$$p^{(\mu)}(t, x, y) = \frac{1}{2t}(xy)^{-\mu} \exp\left(-\frac{x^2+y^2}{2t}\right) I_\mu\left(\frac{xy}{t}\right), \quad x, y, t > 0, \tag{9}$$

where I_μ denotes the modified Bessel function of the first kind. Using its asymptotic behaviour we get

$$p^{(\mu)}(t, x, y) \stackrel{\mu}{\approx} \frac{e^{-(x-y)^2/2t}}{(yx+t)^{\mu+1/2}\sqrt{t}}, \qquad x, y, t > 0. \tag{10}$$

4.3.2 The Fourier-Bessel heat kernel

Consider now the Bessel process with index $\mu > -1$ starting from $0 < x < a$, reflected at zero and killed when hitting the level a (when exiting the interval $[0, a)$, equivalently). We denote its density by $p_a^{(\mu)}(t, x, y)$. Due to the scaling property $p_a^{(\mu)}(t, x, y) = a\, p_1(t/a^2, x/a, y/a)$ we narrow our attention to the case $a = 1$. Note that the absolute continuity property (8) allows one to easily extended the result presented in this section onto the case when $\mu < 0$ with killing condition at zero.

It turns out that $2p_1^{(\mu)}(2t, x, y)$, $\mu > -1$, is known also as the Fourier-Bessel heat kernel $G_t^\mu(x, y)$, which is represented in terms of the Bessel functions of the first kind $J_\mu(z)$ and its successive positive zeros $\lambda_{n,\mu}$ in the following way

$$G_t^\mu(x, y) = 2(xy)^{-\mu} \sum_{n=1}^{\infty} \exp\left(-\lambda_{n,\mu}^2 t\right) \frac{J_\mu(\lambda_{n,\mu}x) J_\mu(\lambda_{n,\mu}y)}{|J_{\mu+1}(\lambda_{n,\mu})|^2}, \quad x, y \in (0, 1), \quad t > 0. \tag{11}$$

Unfortunately, this explicit representation can be only used to examine the behaviour of the heat kernel for large times. Indeed, it is well-known that $G_t^\nu(x,y)$ behaves then like the first term of the series. The description of the behaviour of $G_t^\nu(x,y)$ for small times is very difficult to obtain from the above-given series representation, since the sum is highly oscillating and the cancellations between the terms matter in that case.

The Fourier-Bessel heat kernel $G_t^\nu(x,y)$ and its counterparts has been studied for a long time in many different contexts, such as the study of the fundamental operators associated with the Fourier-Bessel expansions (see [17], [18], [19], [20], [21]) or the related Hardy spaces [26] (see [46] for more references). The estimates of $G_t^\nu(x,y)$ has been recently studied in [46] and [47], where the provided two-sided estimates of $G_t^\nu(x,y)$ were quantitatively sharp, i.e. the different constants appear in the exponential terms of the lower and upper bounds. It makes the estimates not sharp, whenever $|x-y|^2 >> t$. In the main result of the article [H1] given in Theorem 1, the exponential behaviour of the kernel is described explicitly, i.e. the exponential terms in the lower and upper bounds are exactly the same. Note that in the papers [9, 8] the sharp two-sided estimates for the Dirichlet heat kernel of the half-line (a,∞) associated with the Bessel differential operator has been obtained.

Theorem 1 (Theorem 1 in [H1])**.**
For every $\mu > -1$ we have

$$p_1^{(\mu)}(t,x,y) \overset{\mu}{\approx} \frac{(1+t)^{\mu+2}}{(t+xy)^{\mu+1/2}} \left(1 \wedge \frac{(1-x)(1-y)}{t}\right) \frac{1}{\sqrt{t}} \exp\left(-\frac{|x-y|^2}{4t} - \lambda_{1,\mu}^2 t\right). \tag{12}$$

whenever $x,y \in (0,1)$ and $t > 0$.

Remark. Note that by (10), the estimate (12) is equivalent to

$$\frac{p_1^{(\mu)}(t,x,y)}{p^{(\mu)}(t,x,y)} \overset{\mu}{\approx} \left(1 \wedge \frac{(1-x)(1-y)}{t}\right) (1+t)^{\mu+2} \exp\left(-\lambda_{1,\mu}^2 t\right), \tag{13}$$

which shows the relation between the Dirichlet and global heat kernels.

Sketch of the proof of Theorem 1.
Since estimates for large times are known (see [47]), we narrow our attention to $t < t_0$ for some $t_0 < 1/4$ to be fixed later (this, by Chapman-Kolmogorov identity, implies estimates in the range $t < T$ for any $T > 0$). Under this assumption, the assertion is equivalent to (see (13))

$$p_1^{(\mu)}(x,y) \overset{\mu}{\approx} \left(1 \wedge \frac{(1-x)(1-y)}{t}\right) p^{(\mu)}(t,x,y).$$

Furthermore, due to the symmetry $p_1^{(\mu)}(t,x,y) = p_1^{(\mu)}(t,y,x)$, we assume $y > x$.

<u>Lower bound.</u>
Let us consider first $x,y \geqslant 1/32$. Since both of the space arguments are bounded from zero, so for small times the behaviour of $p_1^{(\mu)}(t,x,y)$ is expected to be similar to the classical heat kernel for Laplacian. To obtain the proper lower bound, we use first the inequality $p_1^{(\mu)}(t,x,y) \geqslant p_{(x/4,1)}^{(\mu)}(t,x,y)$ (see (6)). Then, by the absolute continuity property (8) we get

for a set Borel $A \subset (x/4,1)$

$$\int_A p^{(\mu)}_{(x/4,1)}(t,x,y)m^{(\mu)}(dy) = \mathbb{E}^{(\mu)}\left[t<\tau_{(x/4,1)}; R(t)\in A\right]$$

$$= \mathbb{E}^{(-1/2)}\left[t<\tau_{(x/4,1)}; R(t)\in A; \left(\frac{R(t)}{x}\right)^{\mu+1/2}\exp\left(-\frac{\mu^2-\left(\frac{1}{2}\right)^2}{2}\int_0^t \frac{ds}{[R(s)]^2}\right)\right].$$

One can show that the last factor which appeared under the expectation is bounded and bounded away from zero on $\{t<\tau_{(x/4,1)}\}$ with $x>1/32$ and consequently

$$p^{(\mu)}_{(x/4,1)}(t,x,y) \overset{\mu,t_0}{\approx} p^{(-1/2)}_{(x/4,1)}(t,x,y)$$

$$\overset{\mu,t_0}{\approx} \left(1\wedge\frac{(1-x)(1-y)}{t}\right)p(t,x,y), \qquad x,y\in(1/32,1),\ t<t_0, \tag{14}$$

where the last estimate follows from (31) and (10), as $p^{(-1/2)}_{(x/4,1)}(t,x,y)$ is simply the transition probability density function of the Brownian motion killed on exiting the interval $(x/4,1)$.

Assume now $x,y\leqslant 1/4$. In this case both of the space argument are bounded away from the killing point and for small times the behaviour of $p^{(\mu)}_1(t,x,y)$ is expected to be similar to the behaviour of $p^{(\mu)}(t,x,y)$. The Hunt formula (5) takes the form

$$p^{(\mu)}_1(t,x,y) = p^{(\mu)}(t,x,y) - \int_0^t q_{[0,1)}(s,x)p^{(\mu)}(t-s,1,y)ds. \tag{15}$$

We will show that the subtrahend is significantly smaller then the minuend. Indeed, using the estimate (10) one can show that for $t_0>0$ sufficiently small it holds

$$p^{(\mu)}(t-s,1,y) \overset{\mu}{\lesssim} \frac{e^{-9/32t}}{t^{\mu+1}} \overset{\mu}{\lesssim} p^{(\mu)}(t,x,y)\left(\frac{xy+t}{t}\right)^{\mu+1/2} e^{-9/32t+|x-y|^2/2t}$$

$$\overset{\mu,t_0}{\lesssim} p^{(\mu)}(t,x,y)\frac{e^{-1/4t}}{t^{\mu+1}} \overset{\mu}{\lesssim} p^{(\mu)}(t,x,y)e^{-1/8t}.$$

Thus we get

$$\int_0^t q_{[0,1)}(s,x)p^{(\mu)}(t-s,1,y)ds \overset{\mu,t_0}{\lesssim} p^{(\mu)}(t,x,y)e^{-1/8t}\int_0^t q_{[0,1)}(s,x)ds \leqslant p^{(\mu)}(t,x,y)e^{-1/8t},$$

as required.

In the remaining case $x<1/32$, $y>1/4$ we employ the Chapman-Kolmogorov identity (3) narrowing suitably the integration interval as follows

$$p^{(\mu)}_1(t,x,y) \geqslant \int_{1/32}^{1/4} p^{(\mu)}_1(t/8,x,z)p^{(\mu)}_1(7t/8,z,y)dm^{(\mu)}(dz).$$

Note that the choice of the time moments $t/8$ and $7t/8$ is crucial and reflects the intuition that after the time $t/8$ the process is expected to be around $\frac{7}{8}x+\frac{1}{8}y$ which should be included in the integration interval. Next, since $x,z\leqslant 1/4$ and $y,z\geqslant 1/32$ we may take advantage of the previous cases and, after careful estimation of the integral, conclude the required bounds.

Upper bound.

Due to the inequality $p_1^{(\mu)}(t,x,y) \leqslant p^{(\mu)}(t,x,y)$ (see e.g. (6)), it is enough to consider the case $(1-x)(1-y)/t < 1$ only. We will show then the bound

$$p_1^{(\mu)}(t,x,y) \overset{\mu,t_0}{\lesssim} \frac{(1-x)(1-y)}{t} p^{(\mu)}(t,x,y).$$

Furthermore, by the assumptions $y > x$ and $t < 1/4$, we obtain $1 - y < \sqrt{t} < 1/2$, and consequently $y > 1/2$. The main idea is to rewrite the Hunt formula (15) into the following manner

$$\begin{aligned}&\int_A p_1^{(\mu)}(t,x,y)dy = \mathbb{E}_x^{(\nu)}[t < \tau_{[0,1)}; R(t) \in A]\\ &= \left(\mathbb{E}_x^{(\nu)}[R(t) \in A] - \mathbb{E}_x^{(\nu)}[R(t) \in 2 - A]\right) + \left(\mathbb{E}_x^{(\nu)}[R(t) \in 2 - A] - \mathbb{E}_x^{(\nu)}[t \geqslant \tau_{[0,1)}; R(t) \in A]\right),\end{aligned}$$

where $A \subset (0,1)$. This gives us

$$p_1^{(\mu)}(t,x,y) = k_1(t,x,y) + k_2(t,x,y),$$

where

$$k_1(t,x,y) = p(t,x,y) - \left(\frac{2-y}{y}\right)^{2\mu+1} p(t,x,2-y)$$

and

$$\begin{aligned}k_2(t,x,y) &= \left(\frac{2-y}{y}\right)^{2\mu+1} p(t,x,2-y) - \int_0^t q_{[0,1)}(s,x) p^{(\mu)}(t-s,1,y)ds\\ &= \int_0^t q_{[0,1)}(s,x) \left(\left(\frac{2-y}{y}\right)^{2\mu+1} p(t-s,1,2-y) - p^{(\mu)}(t-s,1,y) \right) ds.\end{aligned}$$

This representation is supposed to mimic the reflection principle for Brownian motion (and other symmetric diffusions), as $2-y$ is a reflection of y with respect to the killing point 1. Normally, $k_2(t,x,y)$ vanishes. In this case it does not happen, but we will show that it is significantly smaller than $k_1(t,x,y)$. Let us also note that estimates of the function $q_1(s,y)$ were not known at the time (but have been derived later on in [H2] from Theorem 1), which would be very useful here.

Using the formula (9), monotonicity of the modified Bessel function $I_\mu(\cdot)$ and the bound

$$\frac{I_\mu(y)}{I_\mu(x)} \leqslant e^{y-x} \left(\frac{y}{x}\right)^{\mu+1}, \quad \mu > -1, \quad y > x > 1,$$

derived in Lemma 2.1 in [H1], we get for $w \in (0,1]$ and $t < t_0$

$$1 - y \overset{\mu,t_0}{\lesssim} 1 - \left(\frac{2-y}{y}\right)^{2\mu+1} \frac{p(t,w,2-y)}{p(t,w,y)} \overset{\mu,t_0}{\lesssim} \frac{1-y}{t},$$

which gives us for $x \in (0,1/2)$

$$k_1(t,x,y) \overset{\mu,t_0}{\lesssim} \frac{(1-x)(1-y)}{t} p^{(\mu)}(t,x,y)$$

as well as

$$k_2(t,x,y) \overset{\mu,t_0}{\lesssim} (1-y)\int_0^t q_{[0,1)}(s,x)p^{(\mu)}(t-s,1,y)ds \leqslant 2(1-x)(1-y)p^{(\mu)}(t,x,y),$$

as required. Note that we simply bounded $\int_0^t q_{[0,1)}(s,x) \leqslant 1$ and therefore we omitted the problem of unknown form of $q_{[0,1)}(s,x)$.

In the case $x \in (1/2,1)$ we apply strong Markov property in the following manner

$$p_1^{(\mu)}(t,x,y) = p_{(x/4,1)}^{(\mu)}(t,x,y) + \int_0^t q_{(x/4,1)}^{(\mu)}(s,x,x/4)p_1^{(\mu)}(t-s,x/4,y)ds, \tag{16}$$

where $q_{(2/4,1)}^{(\mu)}(s,x,x/4)$ is the density of hitting time at $x/4$ before hitting 1. The first term represents the paths from x to y that are included in the interval $(x/4,1)$, while the other one represents the paths that are included in the interval $[0,1)$ but have left $(x/4,1)$ at some point. The first term of the right-hand side in (16) has been estimated in (14). Regarding the other term, one can show by some nontrivial analysis and using the already proven upper and lower bounds of $p_1^{(\mu)}(t,x,y)$ for $x<1/2$ that

$$p_1^{(\mu)}(t-s,x/4,y) \overset{\mu}{\lesssim} e^{-1/64t}p_1^{(\mu)}(t-s,x/2,y).$$

Since additionally

$$q_{(x/4,1)}^{(\mu)}(s,x,x/4) \leqslant q_{(x/2,1)}^{(\mu)}(s,x,x/2),$$

by strong Markov property we estimate the integral in (16) as follows

$$\begin{aligned}&\int_0^t q_{(x/4,1)}^{(\mu)}(s,x,x/4)p_1^{(\mu)}(t-s,x/4,y)ds\\ &\overset{\mu}{\lesssim} e^{-1/64t}\int_0^t q_{(x/2,1)}^{(\mu)}(s,x,x/2)p_1^{(\mu)}(t-s,x/2,y)ds\\ &\leqslant e^{-1/64t}p_1^{(\mu)}(t,x,y).\end{aligned}$$

Applying this and (14) to (16) we arrive at

$$p_1^{(\mu)}(t,x,y) \leqslant c_\mu\left(\left(1\wedge\frac{(1-x)(1-y)}{t}\right)p(t,x,y) + e^{-1/64t}p_1^{(\mu)}(t,x,y)\right),$$

for some constant $c_\mu > 0$. Taking t_0 such that $c_\mu e^{-1/64t} < 1/2$ for $t<t_0$, we obtain the required bound.

□

4.3.3 Exit time of the Bessel process from the intervals $(0,1)$ and $[0,1)$

Another important object to study when dealing with the Bessel process on the interval $(0,1)$ (or $[0,1)$) is the distribution of the first exit time (and place, if the point 0 is killing). In particular, for $\mu=(n-2)/2$, $n=1,2,3,...$, it is equivalent to the distribution of the first exit time of n-dimensional Brownian motion from a unit ball. Our aim is to provide asymptotics and uniform estimates of the first exit time (and place) density in whole range of index $\mu\in\mathbb{R}$. Note that analogous results, for $\mu\in\mathbb{R}$, are already known in case of the half-line $(1,\infty)$ [13, 32, 60], but they are obtained by usage of completely different methods.

Although in our case the domain is bounded, we need to deal with its singular behaviour at zero.

In the case when zero is reflecting, the only exit point is 1. When zero is killing, there are two exit points: 0 and 1, and therefore those cases are not that similar to each other as in the case of the transition density functions. For this reason we introduce the following notation. For $\mu > -1$ and reflecting condition at zero we define

$$q_1^{(\mu)}(t,x) = \mathbb{P}_x^{(\mu)}\left(\tau_{[0,1)} \in dt\right), \quad x \in (0,1), t > 0.$$

Furthermore, when $\mu < 0$ and zero is killing we denote

$$q_{(0,1)}^{(\mu)}(t,x,y) = \mathbb{P}_x^{(\mu)}\left(\tau_{(0,1)} \in dt; R^{(\mu)}\left(\tau_{(0,1)}\right) = y\right),$$

where $x \in (0,1)$, $y \in \{0,1\}$ and $t > 0$. Note that the absolute continuity property (8) we have the following relation for $\mu > 0$

$$q_{(0,1)}^{(\mu)}(t,x,1) = x^{-2\mu} q_1^{(-\mu)}(t,x), \quad x \in (0,1),\ t > 0. \tag{17}$$

It is also worth mentioning that in 1980, J.T. Kent yielded in his paper [38] the following series formula

$$q_1^{(\mu)}(x,t) = x^{-\mu} \sum_{n=1}^{\infty} j_{\mu,n} \frac{J_\mu(j_{\mu,n}x)}{J_{\mu+1}(j_{\mu,n})} e^{-j_{\mu,n}^2 t/2}, \tag{18}$$

which can not be directly use to study the short time behaviour.

The first main result from the article [H2] shows how densities of first exit times might be represented by means of the transition density function of the killed process.

Theorem 2 (Theorem 3.1 in [H2])**.**
Let $x \in (0,1)$. For $\mu > -1$ we have

$$q_1^{(\mu)}(t,x) = -\frac{\partial_-}{\partial y} p_1^{(\mu)}(t,x,y)\bigg|_{y=1}, \tag{19}$$

where $\partial_-/\partial y$ denotes the left-sided derivative. Furthermore, for $\mu < 0$

$$q_{0,1}^{(\mu)}(t,x,1) = -\frac{\partial_-}{\partial y} p_{0,1}^{(\mu)}(t,x,y)\bigg|_{y=1}, \tag{20}$$

$$\begin{aligned} q_{0,1}^{(\mu)}(t,x,0) &= \lim_{y\to 0} y^{2\mu+1} \frac{\partial}{\partial y} p_{0,1}^{(\mu)}(t,x,y) \\ &= -2\mu x^{-2\mu}\, p_{0,1}^{(-\mu)}(t,x,0). \end{aligned} \tag{21}$$

The proof of Theorem 2 starts with justifying identities

$$\begin{aligned} q_1^{(\mu)}(t,x) &= -\frac{\partial}{\partial t} \int_0^1 p_1^{(\mu)}(t,x,y) m(y) dy \\ &= -\int_0^1 (\tfrac{1}{2}\mathcal{L}_\mu p_1^{(\mu)})(t,x,y) m(y) dy \\ &= -\int_0^1 \frac{\partial}{\partial y}\left(m(y)\frac{\partial}{\partial y} f(y)\right) dy, \end{aligned}$$

where properties of the Bessel functions of the first kind J_ν were employed. This allows us to conclude (19). The remaining formulas were derived similarly, but required more calculations and manipulations.

Theorem 2 together with Theorem 1 allow us to obtain the estimates for the first exit times densities presented in Theorem 3. Note that generally estimates of a given function do not imply estimates of its derivative, however, it is possible to deduce them at the points where the function vanishes. For instance, we have

$$q_1^{(\mu)}(t,x,1) = -\frac{\partial_-}{\partial y} p_1^{(\mu)}(t,x,y)\bigg|_{y=1} = \lim_{y\to 1^+} \frac{p_1^{(\mu)}(t,x,y)}{1-y}, \tag{22}$$

since $p_1^{(\mu)}(t,x,1)=0$.

Theorem 3 (Theorem 3.3 in [H2]).
For $\mu > -1$ we have

$$q_1^{(\mu)}(t,x) \overset{\mu}{\approx} \frac{(1-x)(1+t)^{\mu+2}}{(x+t)^{\mu+1/2}t^{3/2}} \exp\left(-\frac{(1-x)^2}{2t} - \frac{1}{2}\lambda_{\mu,1}^2 t\right),$$

where $x\in[0,1), t>0$. For $\mu<0$ we have

$$q_{0,1}^{(\mu)}(x,t,1) \overset{\mu}{\approx} \frac{x^{-2\mu}(1-x)}{(x+t)^{-\mu+1/2}} \frac{(1+t)^{-\mu+2}}{t^{3/2}} \exp\left(-\frac{(1-x)^2}{2t} - \frac{1}{2}\lambda_{-\mu,1}^2 t\right),$$

$$q_{0,1}^{(\mu)}(x,t,0) \overset{\mu}{\approx} \frac{x^{-2\mu}(1-x)}{1-x+t} \frac{(1+t)^{-\mu+2}}{t^{-\mu+1}} \exp\left(-\frac{x^2}{2t} - \frac{1}{2}\lambda_{-\mu,1}^2 t\right),$$

where $x\in(0,1), t>0$.

Let $B(t)$ be the n-dimensional Brownian motion starting from $x\in\mathbb{R}^n$. By $q^n(t,x)$ we denote the density function of the first exit time of Brownian motion from the unit ball $B(0,1)$ centered at the origin. Since $|B(t)|$ is a Bessel process with index $n/2-1$ (with reflecting condition at 0 for $n=1$), we have $q^n(t,x) = q_1^{(n/2-1)}(t,|x|)$ and, consequently,

Corollary 4. *For $x\in B(0,1)$ we have*

$$q^n(t,x) \overset{n}{\approx} \frac{(1-|x|)(1+t)^{n/2+1}}{(|x|+t)^{(n-1)/2}t^{3/2}} \exp\left(-\frac{(1-|x|)^2}{2t} - \frac{1}{2}\lambda_{n/2-1,1}^2 t\right).$$

Another consequence of Theorem 2 are the following formulae for $q_{0,1}^{(\mu)}(t,x,y)$. Indeed, employing additionally the representation (11) and properties of the Bessel function $J_\nu(x)$, we get

Corollary 5 (Corollary 3.2 in H2).
For $\mu<0$ and $x\in(0,1)$ we have

$$q_{0,1}^{(\mu)}(t,x,1) = x^{-\mu}\sum_{n=1}^{\infty} j_{-\mu,n}\frac{J_{-\mu}(j_{-\mu,n}x)}{J_{-\mu+1}(j_{-\mu,n})} e^{-\lambda_{-\mu,n}^2 t/2},$$

$$q_{0,1}^{(\mu)}(t,x,0) = \frac{2x^{-\mu}}{\Gamma(-\mu)}\sum_{n=1}^{\infty}(j_{-\mu,n})^{-\mu}\frac{J_{-\mu}(j_{-\mu,n}x)}{|J_{-\mu+1}(j_{-\mu,n})|^2} e^{-\lambda_{-\mu,n}^2 t/2}.$$

Finally, we present uniform asymptotics of the first exit time densities, that involve situations when x/t or $(1-x)/t$ (depending on the killing point) tend to zero or infinity. We start with introducing O_μ notation. For two functions $f,g:\mathbb{R}^2\to\mathbb{R}$ we denote $f(x)=O_\mu(g(x))$ whenever $|f(x)|\overset{\mu}{\lesssim} g(x)$ for x in an indicated range.

Theorem 6 (Theorem 4.2 in [H2]).
There exists $t_0>0$ such that the following asymptotics hold for all $t<t_0$: for $\mu>-1$ we have

$$q_1^{(\mu)}(t,x)=\frac{1-x}{\sqrt{2\pi t^3}}\frac{e^{-(1-x)^2/2t}}{x^{\mu+1/2}}\left(1+O_\mu\left(\frac{t}{x}\right)\right),\qquad x\in[0,1), \tag{23}$$

$$q_1^{(\mu)}(t,x)=\frac{1-x}{t^{\mu+1}}\frac{e^{-(1-x)^2/2t}}{2^\mu\Gamma(\mu+1)}\left(1+O_\mu\left(\left(\frac{x}{t}\right)^2+t\right)\right),\qquad x<1/2. \tag{24}$$

If $\mu<0$, then

$$q_{0,1}^{(\mu)}(t,x,0)=\frac{2x^{-2\mu}e^{-x^2/2t}}{(2t)^{-\mu+1}\Gamma(-\mu)}\left(1+O_\mu\left(e^{-2(1-x)/t}\right)\right),\qquad x\in[0,1), \tag{25}$$

$$q_{0,1}^{(\mu)}(t,x,0)=\frac{8(1-x)x^{-2\mu}}{(2t)^{-\mu+2}\Gamma(-\mu)}e^{-x^2/2t}\left(1+O_\mu\left(\frac{1-x}{t}+t\right)\right),\qquad x\in(0,1). \tag{26}$$

Remark. The asymptotic behaviour of $q_{0,1}^{(\mu)}(t,x,1)$ follows from the above-given theorem and the relationship (17).

The proof of Theorem 6 fills up the whole Section 4 in the article [H2]. First, in theorem 4.1 in [H2] we derive asymptotics in terms of the functions $p^{(\mu)}(t,x,y)$ and $q_{x/4,1}^{(-1/2)}(t,x,y)$. This is achieved by several tools: strong Markov property, inequalities for the modified Bessel function of the first kind $I_\nu(x)$, absolute continuity property (8) and some calculations reflecting intuitions worked out while studying the problem. The proof of Theorem 4.2 in [H2] focuses mainly on proper usage of the series representation of $q_{x/4,1}^{(-1/2)}(t,x,y)$ (related to (30)), which is equivalent to the hitting time density of the point 1 for the Brownian motion in the interval $(x/4,1)$.

4.4 The Brownian motion

4.4.1 Context and known results

We define the Brownian motion as the process generated by the half of the Laplacian $\frac{1}{2}\Delta$. The corresponding n-dimensional (global) heat kernel is given by $p(t,x,y)=(2\pi)^{-n/2}e^{-|x-y|^2/2t}$. Concerning the Dirichlet heat kernels, we recall the following general bounds provided by E. B. Davis in [23] (upper bound) and Q. S. Zhang in [66] (lower bound). For a $C^{1,1}$ bounded domain D there are constants $c_1,c_2,c_3,c_4,T>0$ such that

$$c_1\left(\tfrac{(1-|x|)(1-|y|)}{t}\wedge 1\right)\frac{e^{-c_2|x-y|^2/t}}{t^{n/2}}\leqslant p_D(t,x,y)\leqslant c_3\left(\tfrac{(1-|x|)(1-|y|)}{t}\wedge 1\right)\frac{e^{-c_4|x-y|^2/t}}{t^{n/2}} \tag{27}$$

for every $x,y\in D$ and $t<T$. Unfortunately, the constants c_2 and c_4 are different and different than 1/2, which appears in the global heat kernel. Therefore, the lower and upper bounds are incomparable with each other and incomparable with the global heat kernel for

large $|x-y|^2/t$. There are some results with correct exponents, but they completely fail in describing the boundary behaviour (see e.g. [61, 62, 63]). For instance, the main result of [62] (combined with Theorem 7), for simplicity restricted to convex domains, states that

$$p_D(t,x,y) \geqslant c\left(1 \wedge \frac{(\delta_D(x)\wedge\delta_D(y))^2}{t}\right)\frac{e^{-\lambda t/(\delta_D(x)\wedge\delta_D(y))^2}}{\left(1+\frac{1}{t}\left(\delta_D(x)\wedge\delta_D(y)\right)^2\right)^{(n+2)/2}}\, p(t,x,y),$$

for some $c>0$, where λ stands for the first eigenvalue of $-\Delta$ in the unit ball. However, the article was focused on asymptotics of the heat kernels with fixed space arguments, where the boundary behaviour plays marginal role. Until recently, precise two-sided estimates for Dirichlet heat kernels have been known only in such basic cases as a half-line and an interval (and their multidimensional extensions) as they are given by simple explicit formulae. Even the case of such a classical set as a ball turned out to require a more subtle approach, which will be shown in the next section.

As a quantitative version of the *property of not feeling the boundary* (cf. (1)) Michiel van den Berg showed in [61] the following bounds:

$$p(t,x,y) \geqslant p_D(t,x,y) \geqslant p(t,x,y)\left(1-e^{-\rho^2/t}\sum_{k=1}^{n}\frac{2^k}{(k-1)!}\left(\frac{\rho^2}{t}\right)^{k-1}\right), \tag{28}$$

where ρ is the distance between the interval $\overline{xy}$ and the boundary ∂D of the domain D, i.e.

$$\rho = \inf_{\substack{w\in\overline{xy}\\ z\in\partial D}} |w-z|.$$

A simple observation is that for $\rho > c\sqrt{t}$ the above bounds induce sharp two-sided estimates of $p_D(t,x,y)$. Nevertheless, this is the simplest case and is helpful only when the boundary has no major influence, which is indicated in the name of the property anyway.

Note that the long-time behaviour (i.e. for $t \geqslant T$, where $T>0$ is fixed) of p_D for bounded domains D can be easily deduced from the general theory (see [23], [24]), i.e. there is a comparability between p_D and

$$\delta_D(x)\delta_D(y)e^{-\lambda_1 t},$$

for every $x,y\in D$ and $t\geqslant T$, where λ_1 stands for the first eigenvalue of $-\Delta$ on D. Note that this kind of result can be derived from the spectral series representation of the kernel p_D in terms of the eigenfunctions and eigenvalues of the Laplacian in D (see for example [33]), i.e. it can be shown that for large times t the first component of the series dominates the others. However, this representation is ineffective for small t, when we have to deal with the cancellations of highly oscillating series - similarly as in the case of the Bessel process in the interval $(0,1)$.

There are few examples of sets, where the heat kernel is expressed by means of elementary functions, which allows one to obtain precise estimates. The simplest example is a half-line (as a subset of $\mathbb{R}$). Let it be $(0,\infty)$, for simplicity. Due to the famous reflection principle we have for $x,y,t>0$

$$p_{(0,\infty)}(t,x,y) = p(t,x,y)-p(t,x,-y) = p(t,x,y)\left(1-e^{-2xy/t}\right) \approx \left(1\wedge\frac{xy}{t}\right)p(t,x,y).$$

Since the coordinates of a Brownian motion are independent and due to rotational and translational invariance of heat kernels, the above estimate may be extended to any half-space $H \subset \mathbb{R}^n$, i.e. we have

$$p_H(t,x,y) = p(t,x,y)\left(1 - e^{-2\delta_H(x)\delta_H(y)/t}\right) \approx \left(1 \wedge \frac{\delta_H(x)\delta_H(y)}{t}\right) p(t,x,y). \tag{29}$$

Another example is an interval (a,b), $-\infty < a < b < \infty$:

$$p_{(a,b)} = \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \left[\exp\left(-\frac{(x-y+2(b-a))^2}{2t}\right) - \exp\left(-\frac{(x+y+2k(b-a))^2}{2t}\right)\right]. \tag{30}$$

Making some effort, one can show that for $0 < t < 1$ and $x, y \in (a,b)$ it holds

$$p_{(a,b)} \approx \frac{1}{b-a} p(t,x,y) \left(1 \wedge \frac{(x-a)(y-a)}{t}\right)\left(1 \wedge \frac{(b-x)(b-y)}{t}\right). \tag{31}$$

4.4.2 Dirichlet heat kernel of a ball

One can observe that the non-exponential factor in (31) is not comparable with the one in (27). Namely for $a = -1$, $b = 1$ and $x < -1/2$, $y > 1/2$ those factors are comparable with $(1+x)(1-y)/t^2$ and $(1+x)(1-y)/t$, respectively. This mismatch happens already for an interval, which is one of the simplest sets. The question that arises is ”what are the estimates for more complex sets?”.

An interval in the one-dimensional space $\mathbb{R}$ might be understand as a one-dimensional ball. Thus, the natural generalization of (31) would be estimates of the Dirichlet heat kernel for a ball in higher dimensions. This problem turns out to be significantly more complex, due to much more complicated relations between x, y and the boundary of the ball. Nevertheless, in the next theorem, which is the main result of [H3], we provide desired estimates for the ball $B = B(0,1)$ in any dimension.

Theorem 7 (Theorem 1 in [H3])**.**
For every $n \geqslant 1$ and $T > 0$ we have

$$p_B(t,x,y) \stackrel{d,T}{\approx} h(t,x,y)p(t,x,y), \tag{32}$$

for every $|x|, |y| < 1$ and $t < T$, where

$$h(t,x,y) = \left(1 \wedge \frac{(1-|x|)(1-|y|)}{t}\right) + \left(1 \wedge \frac{(1-|x|)|x-y|^2}{t}\right)\left(1 \wedge \frac{(1-|y|)|x-y|^2}{t}\right). \tag{33}$$

Before passing to the proof, let us make some comments.

1. The first term in the definition of h is the same as the factor in (27). However, adding the other term turns out to be crucial.

2. Is not obvious, but in dimension one estimates (32) and (31) are comparable. To see this, let us observe that if x and y are close to each other, the first term in (33) is dominating (since $[(1-|x|) \vee (1-|y|)] \gtrsim |x-y|$), and if x and y are far away from each other, the other one.

3. Theorem 7 may be partly considered as an extension of Theorem 1 for $\mu = (n-2)/2$, since for such indices the Bessel process is equivalent with the norm of Brownian motion, and then $p_1^{((n-2)/2)}(t,x,y) = \int_{|z|=x} p_B(t,x,z)d\sigma(z)$, where σ is the spherical measure on ∂B. Thus estimating the integral, we obtain estimates of $p_1^{((n-2)/2)}(t,x,y)$.

4. he estimates in Theorem 7 have been complemented in [57] with uniform short-time asymptotics with convergence rates under the conditions when the ratio $\delta_B(\frac{x+y}{2})/\sqrt{t}$ tends to 0 or infinity.

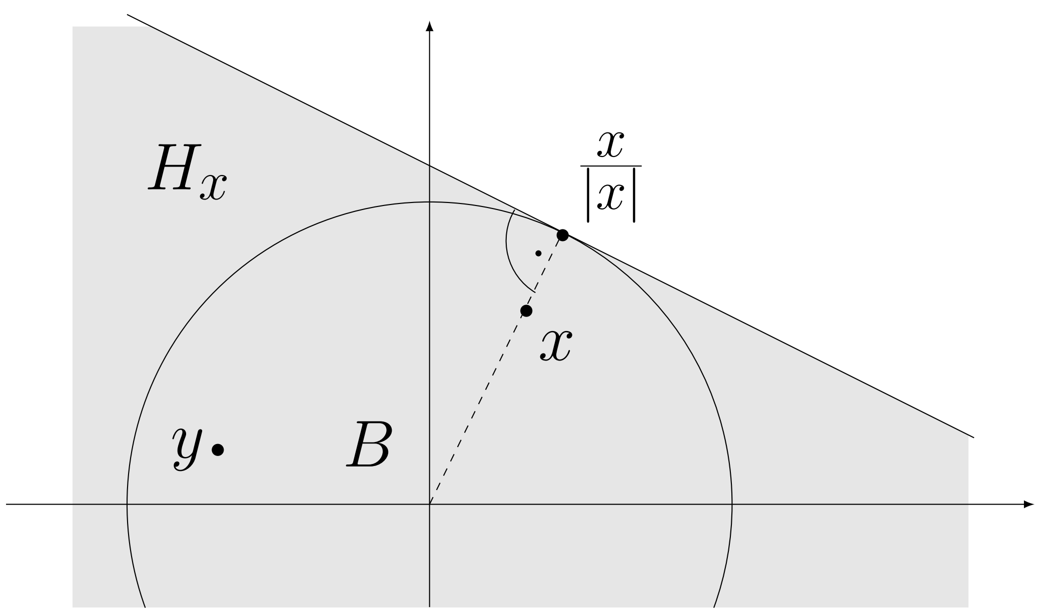
5. The density function $q_D(t,x,z)$ of the joint distribution of the first exit time and place is a normal inward derivative of $p_D(t,x,y)$ (see [33]) and consequently Theorem 7 immediately leads to its sharp two-sided estimates. This extends estimates of the first exit time density (without its dependence on exit place) derived in Corollary 4.

Corollary 8 (Corollary 2 in [H3]).
For any $T > 0$ we have

$$q_B(t,x,z) \overset{d,T}{\approx} \left(\frac{1-|x|}{t} + \frac{|x-z|^2}{t}\left(1 \wedge \frac{(1-|x|)|x-z|^2}{t}\right)\right) p(t,x,z) \tag{34}$$

whenever $|x| < 1$, $|z| = 1$ and $t < T$.

Sketch of the proof of Theorem 7.
First, let us introduce some notation. For $x \in B - \{0\}$ we denote by H_x the half-space that includes B and whose boundary is the hyperplane tangent to B at $\frac{x}{|x|}$.



Upper bound.

Combining (3), (6) and (29) and noting that $\delta_{H_x}(x) = 1 - |x|$, we get

$$\begin{aligned}
p_B(t,x,y) &= \int_{B(0,1)} p_B(t/2,x,z)p_B(t/2,z,y)dz \\
&\leqslant \int_{B(0,1)} p_{H_x}(t/2,x,z)p_{H_y}(t/2,z,y)dz \\
&\lesssim \int_{B(0,1)} \left(1 \wedge \frac{1-|x|}{t}\right) p(t/2,x,z) \left(1 \wedge \frac{1-|y|}{t}\right) p(t/2,z,y)dz \\
&\lesssim \left(1 \wedge \frac{1-|x|}{t}\right)\left(1 \wedge \frac{1-|y|}{t}\right) \int_{\mathbb{R}^n} p(t/2,x,z)p(t/2,z,y)dz \\
&= \left(1 \wedge \frac{1-|x|}{t}\right)\left(1 \wedge \frac{1-|y|}{t}\right) p(t,x,y),
\end{aligned}$$

which turns out to be equivalent to (32) whenever $1-|x| \geqslant \varepsilon$ or $|x-y| \geqslant \varepsilon$ for a fixed $\varepsilon > 0$. When $1-|x|, |x-y| < \varepsilon$ and additionally $x \in B(y/2|y|, 1/2)$, then it holds $\delta_{H_y}(x) < 2(1-|x|)$ and hence

$$\begin{aligned}
p_B(t,x,y) \leqslant p_{H_y}(t,x,y) &\leqslant \left(1 \wedge \frac{\delta_{H_y}(x)\delta_{H_y}(y)}{t}\right) p(t,x,y) \\
&\leqslant 2\left(1 \wedge \frac{(1-|x|)(1-|y|)}{t}\right) p(t,x,y) \leqslant 2h(t,x,y)p(t,x,y).
\end{aligned}$$

The remaining case, it is when $1-|x|, |x-y| < \varepsilon$ and additionally $x \in B - B(y/|y|, 1/2)$, is the most challenging one.

We start from the simple inequality, that follows from the monotonicity property (6)

$$p_B(t,x,y) \leqslant p_{H_x \cap H_y}(t,x,y).$$

Unfortunately, there exist no satisfactory estimates of $p_{H_x\cap H_y}(t,x,y)$ in the literature, although they are equivalent to estimates of the heat kernel in a two-dimensional cone (an interior of an angle). Let us denote by $\alpha_{x,y}$ the angle between vectors x and y. We apply very precisely the Strong Markov property and obtain for $\alpha_{x,y} < \pi/2$

$$p_{H_x\cap H_y}(t,x,y) \leqslant p_{H_y}(t,x,y) - p_{H_y}(t,\bar{x},y), \tag{35}$$

where

$$\bar{x} = \frac{2-|x|}{|x|}x$$

is a reflection of x with respect to the boundary ∂B of B. This somehow mimics the reflection principle. Note that the assumption $x \in B - B(y/|y|, 1/2)$ ensures $\bar{x} \in H_y$. Next, after some calculations, one can show that

$$p_{H_y}(t,x,y) - p_{H_y}(t,\bar{x},y) = (a(t,x,y) + b(t,x,y))p(t,x,y),$$

where

$$a(t,x,y) = \left(1 - \exp\left[-\tfrac{(1-|x|)(1-|y|\cos\alpha_{x,y})}{t}\right]\right)\left(1 - \exp\left[-\tfrac{(1-|y|)(1-|x|\cos\alpha_{x,y})}{t}\right]\right),$$

$$b(t,x,y) = \exp\left[-\tfrac{(1-\cos\alpha_{x,y})((1-|x|)+(1-|y|))}{t}\right]\left(1 - \exp\left[-\tfrac{2\cos\alpha_{x,y}(1-|x|)(1-|y|)}{t}\right]\right).$$

Bounding the first factor defining $b(t,x,y)$ by 1 and using the estimate $1-e^{-u} \approx 1 \wedge u$, $u>0$, we easily obtain $b(t,x,y) \lesssim 1 \wedge ((1-|x|)(1-|y|)/t)$, which gives the first term in (33). Furthermore, some geometrical arguments allow to us derive the bound

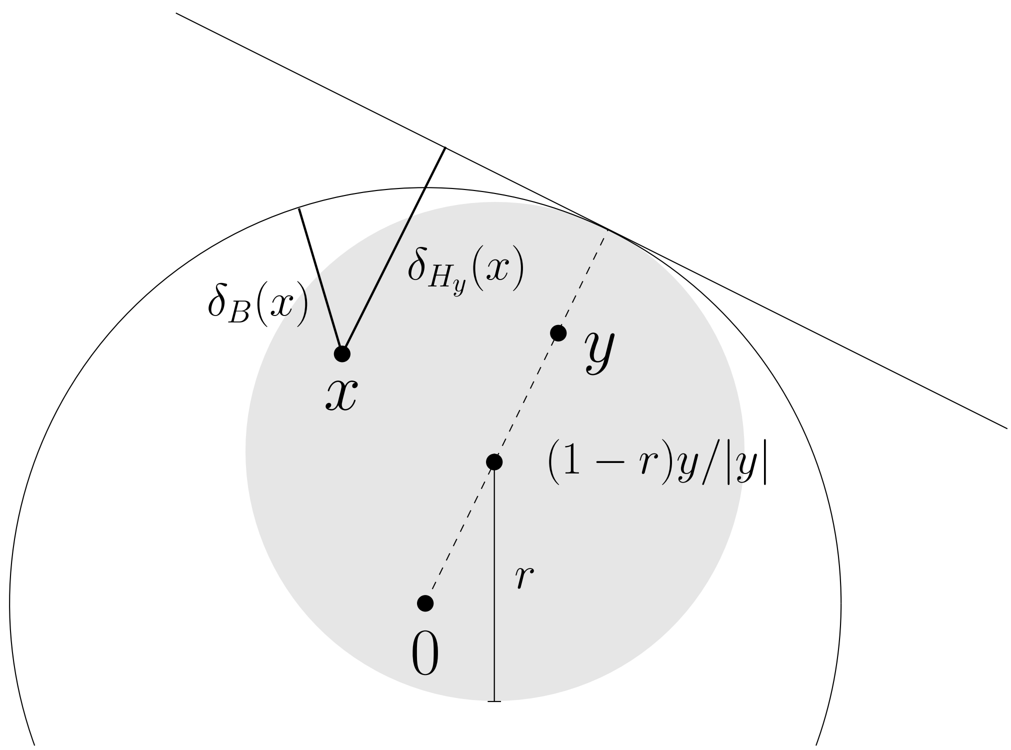
$$1-|x|\cos\alpha_{x,y} = (1-|x|) + |x|\frac{\sin^2\alpha_{x,y}}{1+\cos\alpha_{x,y}} \approx |x-y|^2,$$

and consequently

$$a(t,x,y) \leqslant \left(1 \wedge \frac{(1-|x|)|x-y|^2}{t}\right)\left(1 \wedge \frac{(1-|y|)|x-y|^2}{t}\right),$$

as required.

Lower bound.
Let us note that for $x \in B((1-r)y/|y|, r)$, for a fixed $r \in (0,1/2)$, we may expect that $p_B(t,x,y) \approx p_{H_y}(t,x,y)$ (the upper bound is obvious, by virtue of (6)). This is because the distances from x to ∂B and ∂H_y are comparable, i.e. $\delta_{H_y}(x) \overset{r}{\approx} \delta_B(x)$.



Using strong Markov property, we may write

$$p_B(t,x,y) = p_{H_y}(t,x,y) - R(t,x,y),$$

with

$$R(t,x,y) = \int_0^t \int_{\partial B} q_B(s,x,z) p_{H_y}(t-s,z,y)\, d\sigma(z) ds,$$

where σ denotes the spherical measure on ∂B. Since the upper bound for $p_B(t,x,y)$ is already proven, it implies the upper bound in Corollary 7. Thus, we get

$$R(t,x,y) = \int_0^t \int_{\partial B} \left(\frac{1-|x|}{t} + \frac{|x-z|^2}{t} \left(1 \wedge \frac{(1-|x|)|x-z|^2}{t} \right) \right) p(s,x,z) \\ \times \left(1 \wedge \frac{\delta_{H_y}(x)\delta_{H_y}(y)}{t-s} \right) p(t-s,z,y)\, d\sigma(z) ds.$$

It is shown in the proof of Proposition 3 in [H3], by some elaborate analysis, that for $x \in B(\frac{15}{16}\frac{y}{|y|}, \frac{1}{16})$ it holds

$$R(t,x,y) \overset{r}{\lesssim} p_{H_y}(t,x,y) \left(e^{-|x-y|^2/16t} + \frac{t}{\delta_B^2(x)} \right).$$

This gives us comparability of $p_B(t,x,y)$ and $p_{H_y}(t,x,y)$ under additional assumption of $|x-y|/\sqrt{t}$ and $\delta_B(x)/\sqrt{t}$ being large enough (however, the first one is easy to omit as small values of $|x-y|/\sqrt{t}$ are covered by (27)). Furthermore, this allows to obtain (Proposition 4 in [H3])

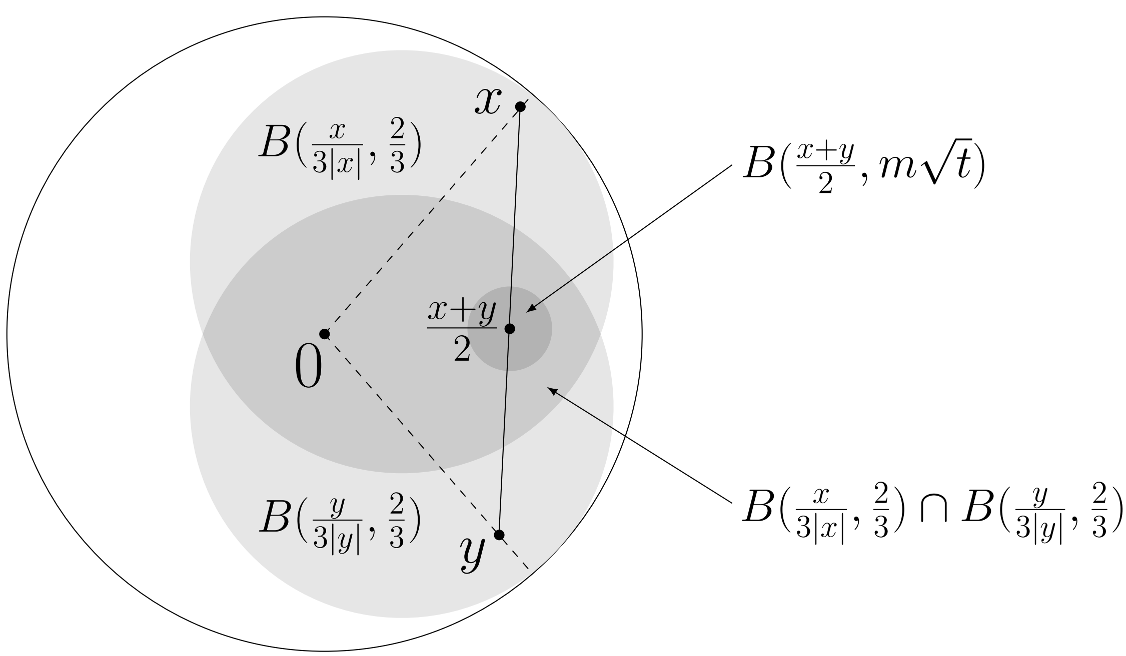
$$p_B(t,x,y) \gtrsim \left(1 \wedge \frac{(1-|x|)(1-|y|)}{t} \right) p(t,x,y), \qquad (36)$$

for any $x \in B(\frac{y}{3|y|}, \frac{2}{3})$, $\delta_B(x) \geqslant m\sqrt{t}$ and $t < t_0$, for some $m, t_0 > 0$. The final step takes advantage of properly applied Chapman-Kolmogorov identity. For any $|x-y|/\sqrt{t} > 4\sqrt{m}$ we have

$$B(\tfrac{x+y}{2}, m\sqrt{t}) \subset B(\tfrac{x}{3|x|}, \tfrac{2}{3}) \cap B(\tfrac{y}{3|y|}, \tfrac{2}{3})$$

and, by some geometrical arguments,

$$\delta_B(z) = 1 - |z| \gtrsim \sqrt{t} + |x-y|^2, \qquad z \in B(\tfrac{x+y}{2}, m\sqrt{t}).$$



Thus

$$
\begin{aligned}
p_B(t,x,y) &\geqslant \int_{B((x+y)/2,m\sqrt{t})} p_B(t/2,x,z)p_B(t/2,z,y)dz \\
&\geqslant \int_{B((x+y)/2,m\sqrt{t})} \left(1 \wedge \frac{\delta_B(x)\delta_B(z)}{t}\right)\left(1 \wedge \frac{\delta_B(y)\delta_B(z)}{t}\right) p(t/2,x,z)p(t/2,z,y)dz \\
&\overset{m,t_0}{\gtrsim} \left(1 \wedge \frac{\delta_B(x)(t+|x-y|^2)}{t}\right)\left(1 \wedge \frac{\delta_B(y)(t+|x-y|^2)}{t}\right) \\
&\qquad\qquad\qquad \times \int_{B((x+y)/2,m\sqrt{t})} p(t/2,x,z)p(t/2,z,y)dz \\
&\gtrsim h(t,x,y)p(t,x,y),
\end{aligned}
$$

where comparability of the last integral with $p(t,x,y)$ is separately proven in Lemma 1 in [H3]. For $|x-y|/\sqrt{t} \leqslant 4\sqrt{m}$ one may shift the center of the integration domain above by $m\sqrt{t}$ inward the ball B. This ends the proof.

□

4.4.3 Dirichlet heat kernel of $C^{1,1}$ convex domains

Once we know the estimate from Theorem 7 the next natural direction of research is to generalize them onto $C^{1,1}$ domains. This problem has been addressed in the article [H4]. It is well known that $C^{1,1}$ sets satisfy the inner and outer ball condition, which means that for any point z from the boundary ∂D of the set D there are two balls tangent to D at z such that one of them is completely included in D, and the other one in D^c. Furthermore, if D is bounded, then there exists a radius $r > 0$ such that for any $z \in \partial D$ the condition is satisfied with balls of radius at least r. We will denote the class of sets with such a property by $C^{1,1}_r(\mathbb{R}^n)$.

We also make an additional assumption of convexity of the domains in order to obtain exponential behaviour of Dirichlet heat kernels of the same order as in the global heat kernel $p(t,x,y)$. Namely, S. R. S. Varadhan showed (Corollary 4.7 in [64]) that for a domain D

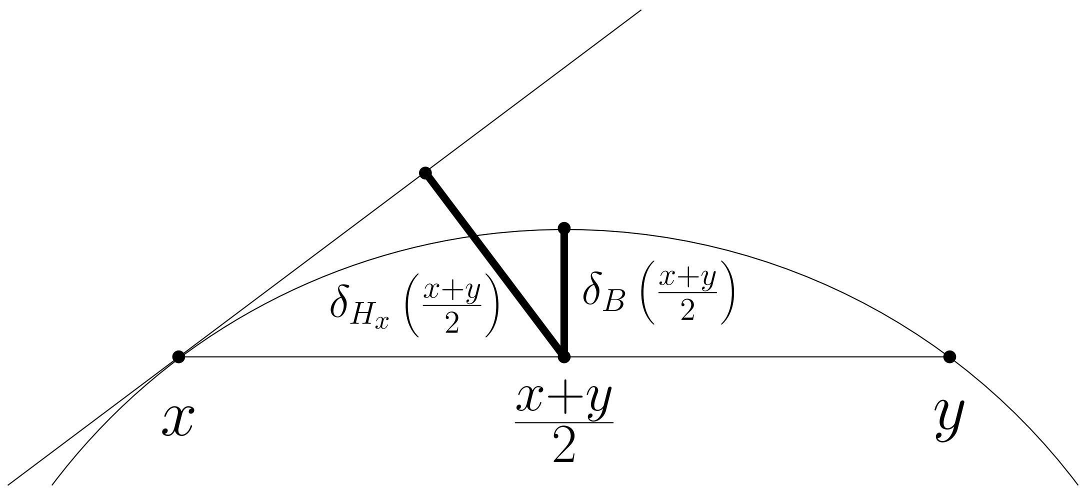
$$
\lim_{t\to 0} t \ln\left(p_D(t,x,y)\right) = \frac{1}{2} d_D^2(x,y),
$$

where $d_D(x,y)$ is the infimum of lengths of arcs included in D and connecting x and y. If D was concave, there would be $x, y \in D$ such that $d_D(x,y) > |x-y|$ and consequently $e^{-d_D^2(x,y)/4t} << e^{-|x-y|^2/2t}$ for t small enough. One could clearly try to obtain estimates with the term $-d_D^2(x,y)/2t$ in the exponent, but this seems to be a much more challenging task and rather a material for further research, as there is expected an additional exponential factor of the form $e^{c(x,y)/t^{1/3}}$ related to Buslaev conjecture [12, 35].

Another question that might be asked when analysing Theorem 7 is whether $|x-y|^2$ appearing in (33) is related to the exponent $e^{-|x-y|^2/2t}$ in the global heat kernel $p(t,x,y)$ or rather to the geometry of the ball. Indeed, for x, y on (for simplicity) the boundary ∂B of the unit ball the expression $|x-y|^2$ is comparable to the following distances (see the figure below)

$$d_1(x,y) := \delta_B\left(\frac{x+y}{2}\right) = 1-\sqrt{1-\tfrac{1}{4}|x-y|^2},$$

$$d_2(x,y) := \delta_{H_x}\left(\frac{x+y}{2}\right) = \delta_{H_y}\left(\frac{x+y}{2}\right) = \tfrac{1}{4}|x-y|^2.$$



The definition of H_x and the distances $d_1(x,y)$ and $d_2(x,y)$ might be easily extended for $C^{1,1}$ domains. Indeed, H_x stands for any half-space such that $D \subset H_x$ and $\delta_D(x) = \delta_{H_x}(x)$. Such a half-space might not be unique but the Lebesgue measure of points with such a ambiguity is zero.

The first presented result is the general upper bound for heat kernels, which involves the distance of type $d_2(x,y)$.

Theorem 9 (Theorems 3.2, 3.4 and Corollary 3.5 in [H4])**.**
Let $D \subset \mathbb{R}^n$ be a $C^{1,1}_r$, $r>0$, domain. There is a constant $C = C(T,n,r) > 0$ such that

$$\begin{aligned}
&p_D(t,x,y)\\
&\leqslant C\,p(t,x,y)\left[\left(1\wedge\frac{\delta(x)\delta(y)}{t}\right)+\left(1\wedge\frac{\delta_{H_x}(x)\delta_{H_x}(y)}{t}\right)\left(1\wedge\frac{\delta_{H_y}(y)\delta_{H_y}(x)}{t}\right)\right]\\
&\leqslant 4Cp(t,x,y)\left[\left(1\wedge\frac{\delta(x)\delta(y)}{t}\right)+\left(1\wedge\frac{\delta_{H_x}(x)\delta_{H_x}\left(\frac{x+y}{2}\right)}{t}\right)\left(1\wedge\frac{\delta_{H_y}(y)\delta_{H_y}\left(\frac{x+y}{2}\right)}{t}\right)\right],
\end{aligned}$$

where $x,y \in D$, $t<T$. Furthermore, if $\angle(H_x,H_y) \geqslant \frac{1}{2}\pi$, then the constant C is absolute.

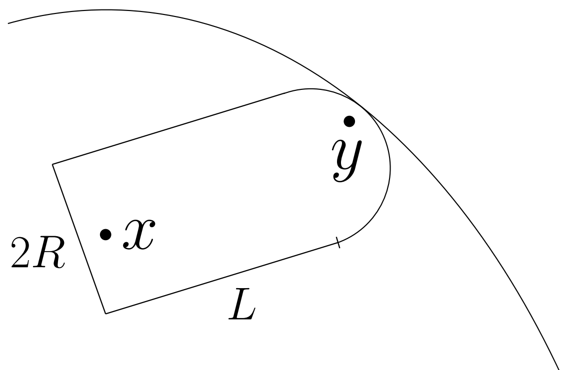
The general idea of the proof was similar as in the case of a ball, but when comes to the details there were more challenging problems to solve.

The lower bound is of similar form, but $d_2(x,y)$ is replaced with $d_1(x,y)$.

Theorem 10 (Theorem 4.3 in [H4]).
For any convex set $D \in C^{1,1}_r$ and $T > 0$ there is $C = C(n, r, T)$ such that

$$\begin{aligned} &p_D(t,x,y) \\ &\geqslant Cp(t,x,y)\left[\left(1 \wedge \frac{\delta_D(x)\delta_D(y)}{t}\right) + \left(1 \wedge \frac{\delta_D(x)\delta_D\left(\frac{x+y}{2}\right)}{t}\right)\left(1 \wedge \frac{\delta_D(y)\delta_D\left(\frac{x+y}{2}\right)}{t}\right)\right] \quad (37)\\ &\approx Cp(t,x,y)\left(1 \wedge \frac{\delta_D(x)\left(\delta_D\left(\frac{x+y}{2}\right)+\sqrt{t}\right)}{t}\right)\left(1 \wedge \frac{\delta_D(y)\left(\delta_D\left(\frac{x+y}{2}\right)+\sqrt{t}\right)}{t}\right). \end{aligned}$$

The first step in the proof was to show the bound $p_D(t,x,y) \geqslant C\left(1 \wedge \frac{\delta_D(x)\delta_D(y)}{t}\right)p(t,x,y)$ in the case of the set $D = J_{R,L} := B_n(0,R) \cup ((0,L) \times B_{n-1}(0,R))$ (a half of a "pill") and $x = (L-\sqrt{t}, 0, 0, ..., 0)$, $y_1 \leqslant 0$, where C does not depend on R and L. Next, it was generalized for any $D \in C^{1,1}_r$ and $\delta_D(x) \geqslant \sqrt{t}$ by inscribing $J_{R,L}$ in D such that $\delta_{J_{R,L}}(x) = \delta_D(x)$. Eventually, using suitably Chapman-Kolmogorov we obtain (37).



If, for a convex set $D \in C^{1,1}_r$, there is a constant C such that $1 < \delta_{H_x}\left(\frac{x+y}{2}\right)/\delta_D\left(\frac{x+y}{2}\right) < C$ holds for any $x, y \in D$, then the bounds in Theorems 9 and 10 are equivalent (up to a multiplicative constant) and they provide two-sided sharp estimates. Unfortunately, such a constant may not exist. In order to describe sets for which we can establish two-sided estimates, let us introduce the following two characteristics of a a strictly convex $C^{1,1}$ domain D:

$$Q_D := \inf_{w,z\in\partial D, w\neq z} \frac{\delta_D\left(\frac{w+z}{2}\right)}{\delta_{H_w}\left(\frac{w+z}{2}\right)},$$

$$R_D := \min\left\{ \inf_{\substack{w,z\in\partial D, w\neq z\\ \delta_D\left(\frac{w+z}{2}\right)\leqslant 1}} \frac{\delta_D\left(\frac{w+z}{2}\right)}{\delta_{H_w}\left(\frac{w+z}{2}\right)}, \inf_{\substack{w,z\in\partial D, w\neq z\\ \delta_D\left(\frac{w+z}{2}\right)>1}} \sup_{\substack{m\in\overline{wz}\\ \delta_D(m)>1}} \frac{\delta_D(m)}{\delta_{H_w}(m)} \right\},$$

as well as the corresponding families of sets:

$$\mathcal{S}_Q := \left\{D \in C^{1,1}(\mathbb{R}^n) : D \text{ is strictly convex}, Q_D > 0\right\},$$
$$\mathcal{S}_R := \left\{D \in C^{1,1}_r(\mathbb{R}^n) \text{ for some } r > 0 : D \text{ is strictly convex}, R_D > 0\right\}.$$

The condition $Q_D > 0$ means that for any $w, z \in \partial D$ the distance from the midpoint $\frac{w+z}{2}$ to the boundary ∂D is comparable with the distances to P_w and P_z. In case $R_D > 0$ the condition is weaker whenever $\delta_D\left(\frac{w+z}{2}\right) > 1$, as we only require existence of a point at the interval $\overline{wz}$ whose distance to ∂D is greater than 1 and comparable to distance to P_w. Some additional comments:

1. $0 \leqslant Q_D \leqslant R_D \leqslant 1$.
2. In the definition of Q_D and R_D the points w and z are taken from the boundary of D only, and not from its interior, which makes it easier to estimate these characteristics.
3. If $D \in S_Q$, then D is bounded (Lemma 5.2 in [H4]). In fact, R_D is introduced in order to deal with unbounded domains.
4. We assume strict convexity since else the complexity of the problem appears to be unexpectedly high and general two-sided bounds seems extremely difficult to derive. We refer the reader to Example 3, where a behaviour of the heat kernel of a very simple non-strictly convex set is examined.
5. Both of the classes $\mathcal{S}_Q$ and $\mathcal{S}_R$ contain nontrivial and important examples (see Examples 1 and 2). It seems also not easy to construct a strictly convex $\mathcal{C}_r^{1,1}$ set which does not belong to $\mathcal{S}_R$.

Theorem 11. *If $D \in \mathcal{S}_Q$ then*

$$\begin{aligned} &p_D(t,x,y)\\ &\overset{r,Q_D,T}{\approx} p(t,x,y)\left[\left(1\wedge\frac{\delta_D(x)\delta_D(y)}{t}\right)+\left(1\wedge\frac{\delta_D(x)\delta_D\left(\frac{x+y}{2}\right)}{t}\right)\left(1\wedge\frac{\delta_D(y)\delta_D\left(\frac{x+y}{2}\right)}{t}\right)\right]\\ &\overset{Q_D}{\approx} \quad p(t,x,y)\left[\left(1\wedge\frac{\delta_D(x)\delta_D(y)}{t}\right)+\left(1\wedge\frac{\delta_{H_x}(x)\delta_{H_x}(y)}{t}\right)\left(1\wedge\frac{\delta_{H_y}(y)\delta_{H_y}(x)}{t}\right)\right] \end{aligned}$$

holds for $x, y \in D$, $0 < t < T$.

Moreover, it turns out that $\mathcal{S}_Q$ is the exact subclass of $C^{1,1}$ domains for which the lower bound from Theorem 10 is equivalent (up to a multiplicative constant) to the upper bound.

Theorem 12. *Let D be a strictly convex $C^{1,1}$ set. Then $D \in \mathcal{S}_Q$ if and only if*

$$\begin{aligned} &p_D(t,x,y) \\ &\overset{D,T}{\approx} p(t,x,y)\left[\left(1\wedge\frac{\delta_D(x)\delta_D(y)}{t}\right)+\left(1\wedge\frac{\delta_D(x)\delta_D\left(\frac{x+y}{2}\right)}{t}\right)\left(1\wedge\frac{\delta_D(y)\delta_D\left(\frac{x+y}{2}\right)}{t}\right)\right]. \end{aligned} \tag{38}$$

holds for $x, y \in D$, $0 < t < T$.

After relaxing the condition $Q_D > 0$ into $R_D > 0$, the heat kernel $p_D(t, x, y)$ keeps admitting two-sided estimates of the form of the upper bound from Theorem 9.

Theorem 13. *If $D \in \mathcal{S}_R$, then*

$$p_D(t,x,y) \overset{r,T,R_D}{\approx} p(t,x,y)\left[\left(1\wedge\frac{\delta(x)\delta(y)}{t}\right)+\left(1\wedge\frac{\delta_{H_x}(x)\delta_{H_x}(y)}{t}\right)\left(1\wedge\frac{\delta_{H_y}(y)\delta_{H_y}(x)}{t}\right)\right]$$

holds for $x, y \in D$, $0 < t < T$, where H_x, H_y are any half-spaces such that $D \subset H_x, H_y$ and $\delta_D(x) = \delta_{H_x}(x)$, $\delta_D(y) = \delta_{H_y}(y)$.

Remark 14. Similarly as in (22) and Corollary 8, the density function $q_D(t,x,y)$ of the joint distribution of the first exit time and place is a normal inward derivative of $p_D(t,x,y)$, and therefore one can easily transform every Dirichlet heat kernel estimates, denoted generally by $f(t,x,y)$, to the estimates of $q_D(t,x,y)$ by calculating the limit

$$\lim_{\delta(y)\to 0} \frac{f(t,x,y)}{\delta(y)}.$$

For example, by virtue of Theorem 13, we deduce that if $D \in \mathcal{S}_R$, then $q_D(t,x,y)$ admits two-sided estimates of the form

$$\begin{aligned}&\lim_{\delta(y)\to 0} \frac{p(t,x,y)}{\delta(y)}\left[\left(1\wedge\frac{\delta(x)\delta(y)}{t}\right)+\left(1\wedge\frac{\delta_{H_x}(x)\delta_{H_x}(y)}{t}\right)\left(1\wedge\frac{\delta_{H_y}(y)\delta_{H_y}(x)}{t}\right)\right]\\&= p(t,x,y)\frac{1}{t}\left[\delta(x)+\left(1\wedge\frac{\delta_{H_x}(x)\delta_{H_x}(y)}{t}\right)\delta_{H_y}(x)\right],\end{aligned}$$

where we used $\delta(y) = \delta_{H_y}(y)$.

Examples

The first example concerns domains that are interiors of a paraboloid and its generalizations. Such sets are usually difficult to study, since neither they are bounded nor their complements are bounded; see [2, 41] for some result concerning the first exit time of such sets and [30] for quantitatively sharp heat kernel estimates.

Example 1 (Proposition 5.6. in [H4])**.**
Consider a domain $U = \{x \in \mathbb{R}^n : x_n > a|(x_1,...,x_{n-1})|^p\}$, where $p \geqslant 2$, $n \geqslant 2$ and $a > 0$. Then $U \in \mathcal{S}_R$. As a consequence, the heat kernel $p_U(t,x,y)$ admits estimates from Theorem 13 with constants depending on n, T, a, p.

Next, we consider bounded domains in $\mathbb{R}^2$ with analytical boundary.

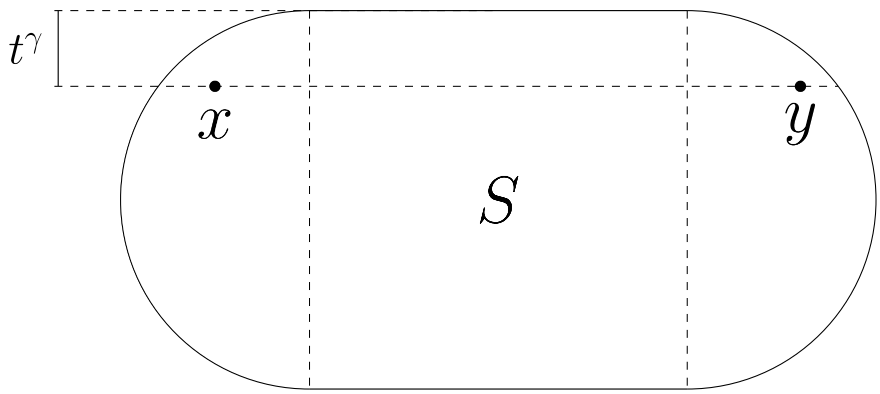
Example 2 (Proposition 5.7. in [H4])**.**
For $n = 2$ the class $\mathcal{S}_Q$ contains strictly convex bounded domains with analytical boundary. As a consequence, the heat kernels of such sets admit estimates from both: Theorem 13 and Theorem 12.

Finally, let us define the following set

$$S = \Big(B_2((-1,0),1)\Big) \cup \Big((-1,1)\times(-1,1)\Big) \cup \Big(B_2((1,0),1)\Big) \subset \mathbb{R}^2,$$

which is a square $(-1,1)\times(-1,1)$ with two semicircles added to its left and right sides. It is known as a *stadium*. The next example shows that for some range of arguments the heat kernel $p_S(t,x,y)$ is comparable neither to the bound from Theorem 9 nor to the one from Theorem 10. Note that the space arguments realizing the indicated behaviour of $p_S(t,x,y)$ are located at opposite ends of the 'flat' part of the boundary, which suggests that non-strict convexity is indeed the property that impacts on the incomparability of the bounds.

Example 3 (Example 5.1. in [H4]).
Let $x, y \in S$ be such that $x_1 < -1$, $y_1 > 1$, $x_2, y_2 = 1 - t^\gamma$, $\gamma > 0$, and $\delta_S(x), \delta_S(y) < t^{1+\gamma}$ with $t < 1$ (see the picture below).



For $0 < \gamma \leqslant \frac{1}{2}$ we have

$$p_S(t,x,y) \approx p(t,x,y)\left[\left(1 \wedge \frac{\delta(x)\delta(y)}{t}\right) + \left(1 \wedge \frac{\delta_{H_x}(x)\delta_{H_x}(y)}{t}\right)\left(1 \wedge \frac{\delta_{H_y}(y)\delta_{H_y}(x)}{t}\right)\right],$$

and for $\gamma \geqslant \frac{2}{3}$ it holds

$$p_S(t,x,y) \approx p(t,x,y)\left[\left(1 \wedge \frac{\delta_S(x)\delta_S(y)}{t}\right) + \left(1 \wedge \frac{\delta_S(x)\delta_S\left(\frac{x+y}{2}\right)}{t}\right)\left(1 \wedge \frac{\delta_S(y)\delta_S\left(\frac{x+y}{2}\right)}{t}\right)\right].$$

However, for $\frac{1}{2} < \gamma < \frac{2}{3}$ we have

$$p_S(t,x,y) \approx \frac{\delta_S(x)\delta_S(y)}{t^{3(1-\gamma)}} p(t,x,y), \tag{39}$$

while

$$\left(1 \wedge \frac{\delta_S(x)\delta_S(y)}{t}\right) + \left(1 \wedge \frac{\delta_S(x)\delta_S\left(\frac{x+y}{2}\right)}{t}\right)\left(1 \wedge \frac{\delta_S(y)\delta_S\left(\frac{x+y}{2}\right)}{t}\right) \approx \frac{\delta_S(x)\delta_S(y)}{t}$$

and

$$\left(1 \wedge \frac{\delta(x)\delta(y)}{t}\right) + \left(1 \wedge \frac{\delta_{H_x}(x)\delta_{H_x}(y)}{t}\right)\left(1 \wedge \frac{\delta_{H_y}(y)\delta_{H_y}(x)}{t}\right) \approx \frac{\delta_S(x)\delta_S(y)}{t^{2-\gamma}} = \frac{\delta_S(x)\delta_S(y)}{t^{3(1-\gamma)+2\left(\gamma-\frac{1}{2}\right)}}.$$

4.5 Description of contribution

Below, I specify my contribution to each of the articles in the achievement.

[H1] Concept and execution of the proof of upper bounds of $p_1^{(\mu)}(t,x,y)$ (Proposition 3.2) and of crucial steps in the proof of lower bound (Proposition 3.1), i.e. in the case when $x \leqslant 1/4$ and $\mu > -1/2$. The co-authors have earlier obtained bounds covering some of these results, but the methods they proposed were either standard or have not been used in the article.

[H2] Full contribution.

[H3] The upper bound has been proven mainly by me, except the inequality (35) ((3.2) in [H3]), which I proposed anyway. In the case of the lower bounds I authored the ideas of the proofs and performed initial calculations. While working on upper bounds I have proposed the correct hypothesis.

[H4] Full contribution.

4.6 Summary and significance

My achievement might be generally described as derivation of sharp estimates of Dirichlet heat kernels and first exit time and place distribution densities for the Brownian motion and the Bessel process, with special care of exponential behaviour. These two processes are closely related to each other, since the norm of Brownian motion is distributed as the Bessel process with a suitable index. The studied functions are fundamental in their fields and they are extremely important from the point of view of not only stochastic processes, but mathematical analysis and physics as well, and they have been intensively studied for many years. Nonetheless, the presented results are first of their kind. Such precise estimates give better insight to the behaviour of the studied functions, allow one to compare the Dirichlet heat kernels with the global ones and enable further research on e.g. asymptotics or resolvent kernels. Estimates of the obtained form are already known for Green functions or Poisson kernels for the said processes (see [29, 68]), and also for the heat kernels of a big class of Lévy and Markov processes (see e.g. [6, 7, 16, 31] and references therein), where however the exponential behaviour does not exist. It was therefore a natural direction of research to tackle analogous problems in the case of Dirichlet heat kernels for the Brownian motion and the Bessel process. I have shown that they might be approached successfully, which encourages one to explore the topic deeper.

The success of the research was possible thanks to the combination of the probabilistic and analytical methods. Many geometrical arguments played also an important role. This mixture seems to be crucial to obtain optimal results in this field. Such an approach should be also adaptable to other settings, which includes e.g. other diffusions or sets with less smooth boundary. In fact, some ideas from [H3] have been already used in [54] to obtain uniform estimates for the Green function of the hyperbolic ball.

The topic of the achievement, despite its long history, still belongs to the mainstream of research in mathematics. The obtained results have been not only mentioned, but also applied or extended by other mathematicians. In particular, the estimates of the Fourier-Bessel heat kernel from [H1] have been used to estimate the probability density of a maximum of the Bessel bridge [36], to prove boundedness of a γ-Littlewood-Paley-Stein operator [5] and a maximal operator associated with Fourier-Bessel expansion [40], and to show uniform L^1 boundedness for the semigroup acting on some atoms related to atomic Hardy spaces [11]. Furthermore, the estimates of the first exit time density for the Bessel process from [H2] helped the author of [59] with finding the asymptotic population growth rate and the authors of [3] with bounding a distance between càdlàg Lévy process X on a separable Banach space and the family of processes adapted to the natural filtration of X. Additionally, in [4] new bounds, independent of the index, for tails and moments of the first exit times of Bessel process have been found, while the one that could be concluded from [H2] depend on the index. Finally, the article [H3] was one of the motivations to conduct the research that led to the article [48]. The last item from the achievement, the paper [H4], has not been cited yet, but it is very recent. Nevertheless, it seems to be the most valuable one, as it deals with a large class of sets and, for example, it includes the main result of [H3] as a special case.

Summing up, my achievement contributes to the classical potential theory by estimating Dirichlet heat kernels and first exit time and place distribution densities for the Brownian and the Bessel process, comparing them to the global heat kernels. Such a problem relates to the *property of not feeling the boundary* introduced by Marc Kac in 1950 and might be considered as a continuation of research in this direction. The innovative approach was based on combing probabilistic and analytical methods and may be adapted to other settings. The obtained results have already gained some recognition and have been successfully applied by other mathematicians. Finally, my contribution to each of the articles from the achievement is full or dominating, which confirms my abilities to conduct research.

5 Significant scientific activity carried out at more than one university or scientific institution, especially at foreign institutions

In the periods of time 13/05/2015–30/09/2015, 2/10/2017–18/02/2018 and 18/02/2019–25/08/2019 I completed 3 research internships at Nanyang Technological University (NTU), Singapore, where I was hired as a Research Fellow. My supervisor there was prof. Nicolas Privault. I got for the first internship through an application, while two other were by invitation. In particular, I helped with preparing the grant application from which my third stay was funded. Relatively short periods of duration are due to private reasons only: a long distance from Poland and my wife's inability to work in her profession in Singapore - the country is famous of its very strict regulations.

During the internships I researched problems related to the Stein method, approximation of the normal distribution and their application to random graphs. Singapore seems to be a perfect place to explore these topics, as a student of Charles Stein and, at the same time, the author of the Stein-Chen method (a variation of the Stein method) Louis Chen is Singaporean and he works at National University of Singapore. Despite the fact that the studied topics were new to me, the cooperation with prof. Privault quickly developed and we finished in 2015 first joint work [50], finally published in 2018. Since then we have published three more articles [51, 52, 53]. One of the most important results of our collaboration is solving a 30 years long standing problem of the convergence rate to the normal distribution of the normalized number of copies of a fixed graph in the Erdős-Rényi random graph model $G(n, p)$ [51].

Although about 3 years have passed since the last stay in Singapore, we are with prof. Privault in constant contact and continue our cooperation. In particular, I visited him for two weeks in March 2022 and we are conducting some new research that are planned in my grant.

6 Teaching and organizational achievements as well as achievements in popularization of science.

Teaching achievements:

- During my employment, that is from 2010 (I completed my PhD as an assistant) I have conducted **121 courses** - both: tutorials (111) and lectures (10). Most of them (106), which is rather typical at the Wrocław University of Science and Technology, were conducted for non-mathematical studies (science studies) at various university faculties. Four of them were held in English. Furthermore, I have taught 12 courses for mathematics students, that includes Mathematical Analysis, Introduction to Probability, Probability Theory, Discrete Mathematics and Introduction to Logic and Set Theory.

  In addition, as a substitute, I conducted courses such as Statistics or Introduction to Stochastic Processes. In the detailed list below, I only included the courses of which I was the main lecturer.

Courses for non-mathematical studies:

Course	Tut.	Lect.
Mathematical Analysis (part 1 or 2)[1]	47	4
Algebra[1]	26	1
Discrete Mathematics	22	1
Probability Theory		1
Mathematics (in English)	1	1
Algebra[1] (in English)	2	

Courses for mathematical studies:

Course	Tut.	Lect.
Introduction to Probability	4	
Probability Theory	4	
Functional Analysis	1	
Discrete Mathematics	1	2
Introduction to Logic and Set Theory	1	
Mathematical Analysis M2	1	
Mathematical Analysis M3	1	

- I am a **supervisor of two groups of courses** at the Faculty of Pure and Applied Mathematics - Discrete Mathematics and (co-supervisor) of Graph Theory. I am responsible for preparing and developing the program and defining learning outcomes of these courses.

- I was the **auxiliary supervisor** in Kamil Bogus' PhD thesis.

- Three students were given the **bachelor's degree** title under my supervision:
  Jana Wilczyńska (2019)
  Klaudia Pytel (2020)
  Barbara Maziarz (2021).

Organizational achievements:

- In 2019, together with Prof. K. Bogdan, Dr. Sc. K. Kaleta and Dr. Sc. Sztonyk we founded and continue to run a **scientific seminar** *Random graphs and discrete structures*. The aim of the seminar is to expand scientific interests at the university towards discrete mathematics, and in particular random graphs, which play an important role in modern mathematics. One of the motivations for starting the seminar was my research on random graphs together with prof. Nicolas Privaut during my research stays in Singapore.

- In 2015 and 2019, I was a member of the **organizing committee** of the international **conference** *Probability and Analysis* held in Będlewo, Poland. In this year's edition I belong additionally to the **scientific committee**.

- During the years 2011–2014 I was a member of the Council of the Faculty of Fundamental Problems of Technology at Wrocław University of Science and Technology. Since 2021 I am a member of the Council of the Faculty of Pure and Applied Mathematics.

[1]Under this name various similar courses are gathered.

- In 2011-2014 I helped with the **organization of national finals** of qualifications for the International Championship of Mathematical and Logical Games held at the Wrocław University of Technology, whose winners take part in the international finals in Paris.

- In 2009, I was a co-founder of the students ***Mathematics Science Club***, which is still active today.

Popularization of science:

- In the winter semester of the academic years 2020/21 and 2021/22 I conducted an original course within **Studium Talent** program. Courses of this type are intended for high school students - their goal is to interest students in a given subject (mathematics, in this case) and, through the final exam, enable them to gain additional points in recruitment to the Wrocław University of Science and Technology.

- On 9^{th} of October 2021 I gave a popular science **lecture** titled ”Grapho-mania” as a part of a countrywide event *Night of innovations*.

- On 25^{th} and 28^{th} of April 2022 I gave **two lectures** for high school students as a part of a voivodeship event *Week of Mathematics* organized by Wrocław University of Science and Technology.

- In the summer semester of the school year 2021/22 I was running a **math club** in a high school.

- In the academic year 2021/2022 I was a **tutor** of a talented high school student and was developing his interest in Mathematics.

7 Other important information about professional career

- In 2022 my project *Central limit theorem for general functionals with applications to random graphs* received **funding** from the National Science Centre (the “NCN”) under the SONATA 17 call.

- In 2021 I was elected for two years to **Academia Iuvenum** - a new body at the Wrocław University of Science and Technology associating (currently) 24 outstanding young (up to 35 years old) scientists from the university. The aim of Academia Iuvenum is to support young scientists and to motivate them to interdisciplinary intellectual exchange.

- In 2014, 2017 and 2021 I received Wrocław University of Science and Technology **Rector’s prize** for outstanding contribution to the development of the university

- In the winter semester 2018-19 I received **targeted subsidy** from the Ministry of Science and Higher Education for young scientists.

- In 2010 I won the 3^{rd} prize in the XLIV national ***Competition*** *for the best student thesis in the theory of probability and applications of mathematics* organized by the Polish Mathematical Society.

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