Summary of Professional Accomplishments

1. Name: Jerzy Legut

2. Diplomas, degrees conferred in specific areas of science or arts, including the name of the institution which conferred the degree, year of degree conferment, title of the PhD dissertation

- Ph. D. in Mathematics, Wroclaw University of Science and Technology 1984 (June), Thesis topic: *Games of Fair Divisions*. Supervisor: prof. dr hab. Rastislav Telgarsky
- M. Sc. in Mathematics, speciality: applied mathematics, Wroclaw University of Science and Technology 1981 (July), Thesis topic: *Games of Fair Divisions and Lyapunov Theorem*. Supervisor: prof. dr hab. Rastislav Telgarsky

3. Information on employment in research institutes or faculties/departments or school of arts

- 2016-: Wroclaw University of Science and Technology, Faculty of Pure and Applied Mathematics, Lecturer
- 1984-1994 The Technical University of Wroclaw, Institute of Mathematics, Lecturer

Contents

4 Description of the achievements, set out in art. 219 para 1 point 2 of the Act

4.1 The list of articles constituting the scientific achievements

To present my scientific achievements, I chose the following publications:

[H1] Legut, J.: *Inequalities for α-optimal partitioning of a measurable space.* Proc. Amer. Math. Soc., 104, 1249-1251 (1988)

[H2] Legut, J. and Wilczyński, M.: *Optimal partitioning of a measurable space.* Proc. Amer. Math. Soc. 104, 262-264 (1988)

[H3] Legut J. and Wilczyński M.: *How to obtain a range of a nonatomic vector measure in* $\mathbb{R}^2$, J. Math. Anal. Appl. 394, 102-111 (2012)

[H4] Legut J.: *Connecting two points in the range of a vector measure*, Colloquium Mathematicum, vol. 153, No. 2, 163-167 (2018)

[H5] Legut J.: *How to obtain an equitable optimal fair division* Ann. Oper. Res. 284, 323-332 (2020)

[H6] Legut J.: *Simple fair division of a square*, J. Math. Econom., 86, 35-40, (2020)

All of the above articles have been published in journals on the Journal Citation Reports. According to the current scoring system (MEiN list of December 21, 2021), the journal in which the first two papers [H1] and [H2] were published is rated at 100 scors. The remaining papers were published in journals rated with 70 scors.

4.2 Introductory information

The present application concerns a second procedure regarding awarding the post-doctoral degree. I submitted the first application in April 2019 to Central Committee for Degrees and Titles. Unfortunately, the conducted procedure of Commission for Academic Degrees in the Mathematics Discipline of Wrocław University of Science and Technology ended by the decision to refuse awarding me the title of habilitated doctor (Resolution 3/2/2020 of March 3, 2020).

In the first application, as a scientific achievement, I presented 7 publications, of which three [H3], [H4] and [H5] I have left in this application. Following the suggestions of the reviewers from the first procedure, I included my older publication [H1] and my joint work [H2]. Paper [H6] was published after the commencement of the first habilitation procedure and therefore it was not included in it.

My scientific achievements consist of four parts:

- game theoretical approach to problems of fair division
- applications of fair division results to mathematical economics

- investigating properties of the range of nonatomic vectors measures and their applications to the theory of fair division
- methods of optimal partitioning of a measurable space and their applications to fair division and decision theory

The main results of the two first parts of my scientific activity were achieved in years 1984 - 1994. Most of these results were published in papers being in the base of JCR (Journal Citation Reports) journals. Some of them were presented on two international game theory conferences in USA (1988, 1991). I was also invited by various universities (in USA, Israel, Holland) to give lectures on my results obtained that time. I established cooperation with mathematicians from Holland as a result of which two joint articles were written and published ([39, 40]).

After twenty years break in my scientific activity I returned to work on the fair division problems. I have concentrated on the two last parts of my mathematical interests listed above.

4.3 Discussion of the most important results obtained in the publications that constitute the scientific achievement

4.3.1 Problem of fair division

Suppose we are given an object, (e.g. a cake) which is to be dvided among n persons (players) in such way that each person receives at least $1/n$ of the total worth of the cake according to his own value. We call such a division a fair one. A simple and well-known method of effecting such division between two persons is "for one to cut, the other to choose". At the beginning of this procedure, players decide, e.g. by drawing lots, which of them will cut the cake and which will choose one of the two remaining pieces. Each player may have a different opinion as to which pieces of the cake are most valuable to him. Therefore, a person who selects a piece of cake has a chance to get more than 1/2 of the value of the entire cake according to his rating. The problem of fair division was first formulated and described in 1949 by Steinhaus [57]. He posed the question of whether the "for one to cut, the other to choose" rule can be extended for $n > 2$. He found a solution for $n = 3$, and then Banach and Knaster (see [31, 56, 57, 58]) showed that the solution for $n = 2$ can be generalized to any number of players. Their result was later modified by Dubins and Spanier [23]. In turn, Fink [27] gave an algorithm in which the number of players need not be known. Brams and Taylor [7] discovered an interesting method of getting an envy-free partition in which neither player is interested in any piece of cake allocated to another player.

Dubins and Spanier [23] formulated the following mathematical model of the fair division problem. Let $\{\mu_i\}_{i=1}^n$, $(n > 1)$, be nonatomic probability measures defined on a measureable space $\{\mathcal{X}, \mathcal{B}\}$. This space represents the object (cake) which is to be divided among the players and measures $\{\mu_i\}_{i=1}^n$ describe the individual evaluation of sets belonging in $\mathcal{B}$.

By an *ordered partition* $P = \{A_i\}_{i=1}^n$ of the measurable space $\{\mathcal{X}, \mathcal{B}\}$ is meant a collection of measurable pairwise disjoint sets $A_1, \ldots, A_n$ summing to $\mathcal{X}$. Denote by $\mathscr{P}_n$ the set of all ordered measurable partitions and let $I = \{1, ..., n\}$ be the set of all players.

There exist several notions of fairness in the fair division theory literature.

Definition 4.1. Division $P = \{A_i\}_{i=1}^n \in \mathscr{P}_n$ is called:

- *proportional* if $\mu_i(A_i) \geq 1/n$ for all $i \in I$,
- *envy-free* if $\mu_i(A_i) \geq \mu_i(A_j)$ for all $i, j \in I$,
- *exact* if $\mu_i(A_j) = 1/n$ for all $i, j \in I$,
- *equitable* if $\mu_i(A_i) = \mu_j(A_j)$ for all $i, j \in I$.

The problem of fair division has been considered in many variants depending on the nature of goods to be divided and the fairness criteria. Different kinds of the players preferences and other criteria for evaluating the quality of the division has been analysed by various authors. The following two main directions are developed in the literature of fair division theory:

- proving the existence of a partition of $\mathcal{X}$ satisfying given criteria (e.g. Dubins i Spanier [23], Legut i Wilczyński [H2], [38], Sagara [48, 49], Weller [61]),
- providing procedures or algorithms for obtaining a fair divisions and applications of them to real-life situations (e.g. Brams i Taylor [7, 8], Brams, Taylor i Zwicker [9, 10], Woodall [62]),

In proving the existence of fair divisions, the famous theorem of Lyapunov [43] on the range properties of a nonatomic vector measure is often used:

Theorem 4.2. *If $\{\mu_i\}_{i=1}^n$ are nonatomic finite measures defined on the measurable space $\{\mathcal{X}, \mathcal{B}\}$ then the range $\vec{\mu}(\mathcal{B})$ of the mapping $\vec{\mu} : \mathcal{B} \to \mathbb{R}^n$ defined by*

$$\vec{\mu}(A) = (\mu_1(A), \ldots, \mu_n(A)), \; A \in \mathcal{B},$$

is convex and compact in $\mathbb{R}^n$.

The range $\vec{\mu}(\mathcal{B})$ will be called hereinafter as Lyapunov set. Using the above theorem it is easy to prove by induction the existence of exact fair divisions (cf. [37]).

Legut [32, 34] considered a problem of dividing a cake fairly among countably infinitely many players and proposed also a fair division model for continuum of players.
The results of the fair division theory can be applied in economics in the exchange and allocation of various commodities (cf. [33, 39, 44, 50]).

4.3.2 An α-optimal partition of a measurable space [H1,H2]

Let $\{\mu_i\}_{i=1}^n$ be nonatomic probability measures defined on a measurable space $\{\mathcal{X}, \mathcal{B}\}$. Dubins and Spanier [23] showed that if at least two measures are different, then there is a proportional partition, where each player can get more than $1/n$ of the value of the entire cake $\mathcal{X}$. The fair division theory analyzes partitions that maximize the individual measures of sets allocated to the players. Elton, Hill, and Kertz [26] defined an *optimal value* v^* of the fair division as follows

$$v^* := \sup_{P \in \mathscr{P}_n} \min_{i \in I} \mu_i(A_i), \tag{4.1}$$

and then they gave its estimation

$$(n + M - 1)^{-1} \leq v^* \leq M n^{-1}, \tag{4.2}$$

where

$$M := \sup_{P \in \mathscr{P}_n} \sum_{i=1}^{n} \mu_i(A_i).$$

Dubins and Spanier [23] extended the definition of proportional division for the case where players may have different shares in dividing the cake. Let

$$S_n = \{s = (s_1, \ldots, s_n) \in \mathbb{R}^n, s_i > 0,\ i \in I,\ \sum_{i=1}^{n} s_i = 1\},$$

be $(n-1)$-dimensional open simplex and let $\overline{S}_n$ denote the closure of this set in $\mathbb{R}^n$.

The coordinate α_i, $i \in I$, of the vector $\alpha = (\alpha_1, \ldots, \alpha_n) \in S_n$ represents the share of the i-th player in division of the cake. Dubins and Spanier [23] showed that for any $\alpha = (\alpha_1, \ldots, \alpha_n) \in S_n$ if at least two measures are different, there is a partition $P = \{A_i\}_{i=1}^n \in \mathscr{P}_n$ for which

$$\mu_i(A_i) > \alpha_i \quad \text{for all} \quad i \in I.$$

Inspired by the result of Dubins and Spanier [23] I introduced in paper [H1] to the theory of fair division the following definition.

Definition 4.3. A partition $P^* = \{A_i^*\}_{i=1}^n \in \mathscr{P}_n$ is called an *α-optimal*, if the following equality holds

$$v^\alpha := \min_{i \in I} \left[\frac{\mu_i(A_i^*)}{\alpha_i} \right] = \sup_{P \in \mathscr{P}_n} \min_{i \in I} \left[\frac{\mu_i(A_i)}{\alpha_i} \right]. \tag{4.3}$$

The number v^α is called an *α-optimal value* of the problem of optimal partitioning of the measurable space. This is the largest possible value of the expression $\min_{i \in I} \left[\frac{\mu_i(A_i)}{\alpha_i} \right]$ which can be reached for a vector measure $\vec{\mu} = (\mu_1, \ldots, \mu_n)$ when dividing $\mathcal{X}$ for a fixed vector $\alpha = (\alpha_1, \ldots, \alpha_n) \in S_n$.

Definition 4.4. A partition $P = \{A_i\}_{i=1}^n \in \mathscr{P}_n$ is called an *equitable optimal* (or in short *optimal*) if it is α-optimal for $\alpha = (1/n, 1/n, \ldots, 1/n) \in S_n$.

It is easy to see that $v^* = v^\alpha/n$ for $\alpha = (1/n, 1/n, \ldots, 1/n) \in S_n$ where v^* is defined by (4.1).

The existence of α-optimal partitions follows from the theorem of Dvoretzky, Wald and Wolfowitz [25]:

Theorem 4.5. *If $\{\mu_i\}_{i=1}^n$ are nonatomic finite measures defined on the measurable space $\{\mathcal{X}, \mathcal{B}\}$ then the range $\vec{\mu}_P(\mathscr{P}_n)$ of the mapping $\vec{\mu}_P : \mathscr{P}_n \to \mathbb{R}^n$ defined by*

$$\vec{\mu}_P(P) = (\mu_1(A_1), \ldots, \mu_n(A_n)), P = \{A_i\}_{i=1}^n \in \mathscr{P}_n,$$

is convex and compact in $\mathbb{R}^n$.

It also follows from Theorem 4.5 that there exists a partition $P^0 = \{A_i^0\}_{i=1}^n \in \mathscr{P}_n$ satisfying the equality

$$M = \sup_{P \in \mathscr{P}_n} \sum_{i=1}^n \mu_i(A_i) = \sum_{i=1}^n \mu_i(A_i^0).$$

The number M can be interpreted as a "cooperative" value of the fair division problem (cf. [33]).

Let $r_i := \mu_i(A_i^0)$ and

$$m := \min\{r_i[r_i - \alpha_i(M-1)]^{-1} : [r_i - \alpha_i(M-1)]^{-1} > 0, \, i \in I\}.$$

In paper [H1] I presented elementary and short proof of the following theorem:

Theorem 4.6.

$$m \le v^\alpha \le M. \tag{4.4}$$

Inequalities (4.4) are not only a generalization of inequalities (4.2) for α-optimal partitions, but can also yield in some cases better estimates of the optimal value v^* defined by (4.1). An example of such estimation is provided in [H1]. The original proof of inequality (4.2) given by Elton, Hill and Kertz [26] is long and complicated. It uses advanced methods known from the measure theory. Due to the simplicity of the proof of inequality (4.4) provided in [H1] it is presented below.

Proof of Theorem 4.6

At first we show the inequality $v^\alpha \le M$. Suppose that $v^\alpha > M$. From the definition (4.3) of the number v^α we obtain $\alpha_i^{-1}\mu_i(A_i^*) > M$ for all $i \in I$. Hence we have $\sum_{i=1}^n \mu_i(A_i^*) > M$ which contradicts the definition of the number M.

To prove that $m \le v^\alpha$ we put $\mathbf{e}_i := (0, ..., 1, ..., 0) \in \mathbb{R}^n$ (1 is placed on the $i-$th coordinate). Clearly, $\mathbf{e}_i \in \vec{\mu}(\mathscr{P}_n)$ for all $i \in I$. Let W denote the convex hull of the set $\{\mathbf{r}, \{\mathbf{e}_i\}_{i=1}^n\}$ where $\mathbf{r} = (r_1, ..., r_n)$. It follows from Theorem 4.5 that $W \subset \vec{\mu}(\mathscr{P}_n)$. It is now sufficient to find a real number $t^* := \max\{t \in \mathbb{R} : t\alpha \in W\}$. Solving the following system of $n+1$ linear equations

$$\begin{cases} \beta_i + \beta_{n+1} r_i = \alpha_i, \quad i \in I, \\ \sum_{i=1}^{n+1} \beta_i = 1 \end{cases}$$

with respect to $\beta_i \ge 0$, $i = 1, 2, ..., n+1$, we obtain $t^* = m$. Hence we conclude that $m \le v^\alpha$ and the proof is complete.

□

The above method using the geometric properties of the $\vec{\mu}(\mathscr{P}_n)$ was later used in many publications concerning the estimation of optimal values related to various divisions of the measurable space (cf. [1, 2, 16, 17, 20, 21]).

An interesting issue is to describe the construction of $\alpha-$optimal partitions for nonatomic measures. Together with Maciej Wilczyński we presented in paper [H2] the general form of such divisions.

We may assume throughout that measures $\{\mu_i\}_{i=1}^n$ are absolutely continuous with respect to the same measure ϑ (e.g. $\vartheta = \sum_{i=1}^n \mu_i$). Denote by $f_i = d\mu_i/d\vartheta$ the Radon-Nikodym derivatives , i.e. functions f_i, satisfying the equalities:

$$\mu_i(A) = \int_A f_i \, d\vartheta, \; A \in \mathcal{B}, \; i \in I. \tag{4.5}$$

Define the following measurable sets

$$B_i(p) = \bigcap_{j=1, j\neq i}^{n} \left\{ x \in \mathcal{X} : p_i \alpha_i^{-1} f_i(x) > p_j \alpha_j^{-1} f_j(x) \right\},$$

$$C_i(p) = \bigcap_{j=1}^{n} \left\{ x \in \mathcal{X} : p_i \alpha_i^{-1} f_i(x) \geq p_j \alpha_j^{-1} f_j(x) \right\},$$

where $\alpha = (\alpha_1, \ldots, \alpha_n) \in S_n$, $p = (p_1, ..., p_n) \in \overline{S}_n$ and $i \in I$.

Legut i Wilczyński [H2] proved the following:

Theorem 4.7. *For all $\alpha \in S_n$ there exists a point $p^* \in \overline{S}_n$ and a corresponding partition $P^* = \{A_i^*\}_{i=1}^n$ which satisfies*
(i) $B_i(p^*) \subset A_i^* \subset C_i(p^*)$,
(ii) $v^\alpha = \dfrac{\mu_1(A_1^*)}{\alpha_1} = \dfrac{\mu_2(A_2^*)}{\alpha_2} = \ldots = \dfrac{\mu_n(A_n^*)}{\alpha_n}$.
and is α-optimal. Moreover, any partition satisfying (i) and (ii) is α-optimal.

The proof of this theorem is based on Sion's minimax theorem (see [4]).

Legut and Wilczyński [38] also investigated the existence and construction of $\alpha-$optimal partitions for countable and infinite number of players. Denote by $N := \{1, 2, ...\}$ the set of players and let $\{\mu_i\}_{i=1}^\infty$ be nonatomic probability measures defined on the measurable space $\{\mathcal{X}, \mathcal{B}\}$ describing individual assessments of measurable sets. Let $\alpha = \{\alpha_i\}_{i=1}^\infty$ be an infinite sequence of positive numbers that satisfy the condition $\displaystyle\sum_{i=1}^{\infty} \alpha_i = 1$. Denote by $\mathscr{P}_\infty$ the set of all measurable partitions $\{A_i\}_{i=1}^\infty$.

Legut [32] proved that for any sequence of positive numbers $\alpha = \{\alpha_i\}_{i=1}^\infty$ summing to 1 and any nonatomic probability measures $\{\mu_i\}_{i=1}^\infty$ there exists a partition $\{A_i\}_{i=1}^\infty \in \mathscr{P}_\infty$ such that

$$\mu_i(A_i) = \alpha_i, \quad i \in N.$$

Legut and Wilczyński [38] showed that the thesis of Theorem 4.7 is also true if it is formulated analogously for the case of an infinite countable number of players.

The problem of $\alpha-$optimal partitioning of a measurable space has been described and studied in many publications (cf. [16, 17, 20, 21, 48, 49]).

The construction of $\alpha-$optimal partitions for $\alpha = (1/n, 1/n, \ldots, 1/n) \in S_n$ shown in Theorem 4.7 can be used in the following classification problem (cf. [28, 29, 30]). Suppose we are given a continuous random variable X having one of the known distribution described by density functions $f_i : [0,1] \to \mathbb{R}_+$ $i \in I$. We don't know which is the true distribution of X. We consider a classification problem (cf. [28]) in which after one observation of $X(\omega)$ (realisation of the random variable X) we are to decide which is the true distribution of X.

Definition 4.8. A partition $P = \{A_i\}_{i=1}^n \in \mathscr{P}_n$ is called a *decision rule* if in case of $X(\omega) \in A_i$, we guess that X has density function f_i, $i \in I$.

Our objective is to minimize the largest probability of misclassification

$$\max_{i \in I} \mathbb{P}(X \notin A_i | \operatorname{dist} X = f_i),$$

over all measurable partitions $P = \{A_i\}_{i=1}^n \in \mathscr{P}_n$. Denote by

$$R = \inf \left\{ \max_{i \in I} \mathbb{P}(X \notin A_i | \operatorname{dist} X = f_i) : \{A_i\}_{i=1}^n \in \mathscr{P}_n \right\},$$

a minimal possible risk of misclassification. We obtain (cf. [29, 30])

$$R = \inf \left\{ \max_{i \in I} (1 - \mu_i(A_i)) : \{A_i\}_{i=1}^n \in \mathscr{P}_n \right\} = 1 - \sup \left\{ \min_{i \in I} \mu_i(A_i) : \{A_i\}_{i=1}^n \in \mathscr{P}_n \right\}.$$

Definition 4.9. A partition $P^* = \{A_i^*\}_{i=1}^n \in \mathscr{P}_n$ is said to be a minimax decision rule if

$$R = 1 - \min_{i \in I} \mu_i(A_i^*).$$

It is easy to see that the minimax decision rule $P^* = \{A_i^*\}_{i=1}^n \in \mathscr{P}_n$ defined above is at the same time the equitable optimal partition in the sense of Definition 4.4.

Jóźwiak and Legut [30] using Theorem 4.7 presented an example of decision rule construction in the case of two-dimensional random variables.

4.3.3 Construction of the Lyapunov set in $\mathbb{R}^2$ and its application in $\alpha-$optimal partitioning [H4]

In this section we present a method of finding the α-optimal value v^α and α-optimal partition of a measurable space $\{(0,1), \mathcal{B}\}$ for two players. Here $\mathcal{B}$ denote the family of measurable subsets of $(0,1)$. Legut and Wilczyński [H3] noticed that the Lyapunov set $\vec{\mu}(\mathcal{B})$ can be specified by a continuous and nondecreasing function $G : [0,1] \to [0,1]$ as folllows

$$\vec{\mu}(\mathcal{B}) = \left\{ (x,y) \in [0,1]^2 : 1 - G(1-x) \le y \le G(x) \right\}. \tag{4.6}$$

We show how to obtain this function. Denote by f_1 and f_2 density functions defined on interval $(0,1)$ corresponding to measures μ_1, μ_2 respectively. Define functions

$$F_i(x) = \int_0^x f_i(t)\,dt, \quad i = 1,2. \tag{4.7}$$

It follows from the compactness of the range $\vec{\mu}(\mathcal{B})$ that for any $t \in [0,1]$ there exists a set $D(t) \in \mathcal{B}$ such that

$$\mu_2(D(t)) = \max\{\mu_2(A) : \mu_1(A) = t,\ A \in \mathcal{B}\}. \tag{4.8}$$

Let $\mathbb{I}_A$ denote the indicator function of a set $A \in \mathcal{B}$. Let

$$r(x) := \left(\frac{f_2(x)}{f_1(x)}\right) \mathbb{I}_{\{f_1(x)>0\}}, \quad x \in (0,1). \tag{4.9}$$

Legut and Wilczyński [H4] used the Neyman-Pearson lemma (cf. [42]) to find the function G describing the boundary of set $\vec{\mu}(\mathcal{B})$. This methods depending on the property of the function r are described in the proposition below.

Proposition 4.10. *Assume that* $\{x : f_2(x) > 0\} \subset \{x : f_1(x) > 0\} = (0,1)$. *Then the following statements hold:*

1. *If the ratio* $r(x)$ *is decreasing on* $(0,1)$, *then* $D(x) = \left(0, F_1^{-1}(x)\right)$ *and hence* $G(x) = F_2(F_1^{-1}(x))$.
2. *If the ratio* $r(x)$ *is increasing on* $(0,1)$, *then* $D(x) = \left(F_1^{-1}(1-x), 1\right)$ *and hence* $G(x) = 1 - F_2(F_1^{-1}(1-x))$.
3. *If the ratio* $r(x)$ *is symmetric about* $x_0 = 1/2$ *and decreasing in* x *on* $(0,1/2)$, *then* $D(x) = \left(0, F_1^{-1}(x/2)\right) \cup \left(F_1^{-1}(1-x/2), 1\right)$ *and hence* $G(x) = F_2\left(F_1^{-1}(x/2)\right) + 1 - F_2\left(F_1^{-1}(1-x/2)\right)$.
4. *If the ratio* $r(x)$ *is symmetric about* $x_0 = 1/2$ *and increasing in* x *on* $(0,1/2)$, *then* $D(x) = \left(F_1^{-1}\left(\frac{1-x}{2}\right), F_1^{-1}\left(\frac{1+x}{2}\right)\right)$ *and hence* $G(x) = F_2\left(F_1^{-1}(\frac{1+x}{2})\right) - F_2\left(F_1^{-1}(\frac{1-x}{2})\right)$.

The following example presents a construction of function G using Proposition 4.10.

Example 4.11. Let f_1 be the uniform density on $(0,1)$ and f_2 the Cauchy density normalized and restricted to the interval $(0,1)$. These densities are given by

$$f_1(x) \equiv 1 \quad \text{and} \quad f_2(x) = \frac{\frac{\pi}{4}}{1 + \left[\frac{\pi}{2}(x - \frac{1}{2})\right]^2}, \quad x \in (0,1),$$

and the corresponding functions F_1 and F_2 defined by (4.7) satisfy

$$F_1(x) = x \quad \text{and} \quad F_2(x) = \frac{1}{2} \arctan\left(\frac{\pi}{2}\left(x - \frac{1}{2}\right)\right) + \frac{1}{2}, \quad x \in [0,1].$$

Since the ratio $r(x)$ is symmetric about $x_0 = 1/2$ and increases in x on $(0,1/2)$, it follows from Proposition 4.10 that the function $G(x)$ has the form:

$$G(x) = F_2\left(F_1^{-1}\left(\frac{1+x}{2}\right)\right) - F_2\left(F_1^{-1}\left(\frac{1-x}{2}\right)\right) = \arctan\left(\frac{\pi}{4}x\right), \quad x \in [0,1].$$

□

Legut and Wilczyński [H4] found a method of obtaining the function G for more general densities f_1, f_2. Define

$$\mathscr{R}(f_1, f_2) = \left\{ \left(\int_A f_1\, dt, \int_A f_2\, dt \right) \;:\; A \in \mathcal{B} \right\}.$$

It is easy to see that $\mathscr{R}(f_1, f_2) = \vec{\mu}(\mathcal{B})$. Define a function $\overline{H} : \mathbb{R} \to [0,1]$ by

$$\overline{H}(y) = \mu_1(\{x \;:\; f_2(x) > y f_1(x)\}) = \int_{\{x\,:\, f_2(x) > y f_1(x)\}} f_1(x)\, dx. \tag{4.10}$$

Define now function $f_2^*(x)$ by:

$$f_2^*(x) = \overline{H}^{\,-1}(x) \quad \text{for all} \quad x \in (0,1), \tag{4.11}$$

where

$$\overline{H}^{\,-1}(x) = \inf\{y \geq 0 \;:\; \overline{H}(y) \leq x\} \quad \text{for all} \quad 0 < x < 1.$$

Denote by

$$f_1^*(x) = \mathbb{I}_{(0,1)}(x) \tag{4.12}$$

the uniform density defined on $(0,1)$.

Legut and Wilczyński [H4] proved the following:

Theorem 4.12. *Let f_1, f_2 be probability density functions defined on $\{(0,1), \mathcal{B}\}$ and let f_1^* and f_2^* be the corresponding densities defined by (4.12) and (4.11) respectively. Then*

$$\mathscr{R}(f_1, f_2) = \mathscr{R}(f_1^*, f_2^*).$$

Moreover,

$$\mathscr{R}(f_1^*, f_2^*) = \left\{ (x,y) \in \mathbb{R}^2 \;:\; 0 \leq x \leq 1,\; 1 - G(1-x) \leq y \leq G(x) \right\},$$

where the function $G : [0,1] \to [0,1]$ has the form

$$G(x) = \int_{\{t: f_1(t) = 0\}} f_2(t)\, dt + \int_0^x f_2^*(t)\, dt \quad \text{for all} \quad x \in [0,1]. \tag{4.13}$$

The above theorem can be also used to obtain the set $\vec{\mu}_P(\mathscr{P}_2)$. It is easy to verify that $\vec{\mu}_P(\mathscr{P}_2)$ can be obtained by symmetric transformation of the set $\vec{\mu}(\mathcal{B})$ with respect to the line $x = \frac{1}{2}$, i.e.

$$\vec{\mu}_P(\mathscr{P}_2) = \left\{ (x,y) \in [0,1]^2 : (1-x, y) \in \vec{\mu}(\mathcal{B}) \right\}. \tag{4.14}$$

It follows from (4.6) and (4.14) that

$$\vec{\mu}_P(\mathscr{P}_2) = \left\{ (x,y) \in \mathbb{R}^2 \;:\; 0 \leq x \leq 1,\; 1 - G(x) \leq y \leq G(1-x) \right\}, \tag{4.15}$$

where function G is defined by (4.13).

Legut and Wilczyński [H4] used Theorem 4.12 to obtain α-optimal value and α-optimal partitions in two-dimensional case. They proved the following:

Theorem 4.13. *Let μ_1, μ_2 be probability measures defined on $\{(0,1), \mathcal{B}\}$ with corresponding densities f_1, f_2 and let $\alpha = (\alpha_1, \alpha_2) \in S_2$. Then*

$$v^\alpha = \frac{x_\alpha}{\alpha_1},$$

where x_α is the root of the equation

$$\frac{\alpha_2}{\alpha_1} x = G(1-x). \tag{4.16}$$

Moreover, the α-optimal partition has the form $\{\mathcal{X} \setminus A_2^\alpha, A_2^\alpha\}$, where A_2^α is any set satisfying $\mu_1(A_2^\alpha) = 1 - x_\alpha$ and

$$\{x : f_2(x) > y_\alpha f_1(x)\} \subset A_2^\alpha \subset \{x : f_2(x) \geq y_\alpha f_1(x)\}, \tag{4.17}$$

where $y_\alpha = \overline{H}^{-1}(1 - x_\alpha)$.

The following example illustrates an application of the above theorem.

Example 4.14. Let f_1 be the uniform density defined on $(0,1)$ and f_2 be density given by

$$f_2(x) = I_{(0,\frac{1}{2})}(x)(-8x(x-1)) + I_{[\frac{1}{2},1)}(x)8(x-1)^2, \quad 0 < x < 1.$$

Plots of the density functions f_1 and f_2 are shown in the figure below.

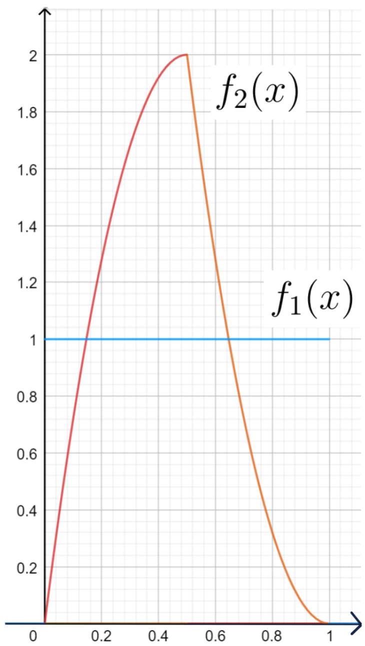

Figure 1. Density functions f_1 and f_2.

We find the α-optimal partition for measures μ_1, μ_2 with the above densities for $\alpha = (\frac{1}{3}, \frac{2}{3})$. First we need to construct function G describing the boundary of the range

$\vec{\mu}_P(\mathscr{P}_2)$ by (4.15). None of the assumptions of Proposition 4.11 for the ratio $r(x)$ defined by (4.9) hold. Hence, for determining the function G we will use method presented in Theorem 4.13. Note that function f_2 increases on $(0, 1/2)$ and decreases on $[1/2, 1)$. This implies that for $0 < y < 2$

$$\{x : f_2(x) > y f_1(x)\} = \{x : f_2(x) > y\} = (x_1, x_2),$$

where

$$x_1 = \frac{1}{2} - \sqrt{\frac{(2-y)}{8}} \quad \text{and} \quad x_2 = 1 - \sqrt{\frac{y}{8}}$$

solve the equations

$$y = -8x(x-1),\ x \in (0, 1/2) \quad \text{and} \quad y = 8(x-1)^2,\ x \in [1/2, 1)$$

respectively. Hence,

$$\overline{H}(y) = \int\limits_{\{x:\, f_2(x) > y f_1(x)\}} f_1(x)\, dx = \int\limits_{x_1}^{x_2} dx = \left(1 - \sqrt{\frac{y}{8}}\right) - \left(\frac{1}{2} - \sqrt{\frac{(2-y)}{8}}\right).$$

To find the function $\overline{H}^{-1}$, we note that the equation $\overline{H}(y) = x$ yields

$$(y-1)^2 = 1 - 16(1-x)^2 x^2,$$

which implies by a simple calculation that

$$\overline{H}^{-1}(x) = 1 - 2\left(x - \frac{1}{2}\right)\sqrt{-4x^2 + 4x + 1}, \quad x \in (0, 1).$$

Since $\int_{\{t: f_1(t) = 0\}} f_2(t)\, dt = 0$, it follows from (4.13) that $f_2^*(x) = \overline{H}^{-1}(x)$. Hence,

$$G(x) = \int_0^x f_2^*(t) dt = x + \frac{1}{6}\left(1 - 4(x-1)x\right)^{3/2} - \frac{1}{6}. \tag{4.18}$$

From the equation (cf. (4.16))

$$2x = G(1-x) = 1 - x + \frac{1}{6}(1 + 4x(1-x))^{\frac{3}{2}} - \frac{1}{6} \tag{4.19}$$

we obtain $x_\alpha \approx 0.433$ where G is given by (4.18).
The figure below shows the range $\vec{\mu}_P(\mathscr{P}_2)$ and the graphical solution of the equation (4.19).

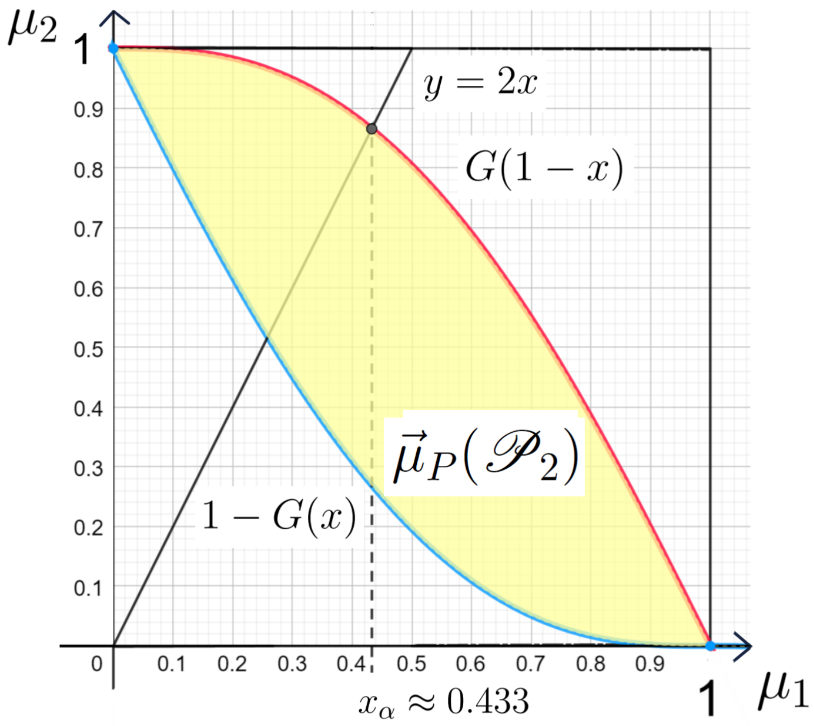

Figure 2. Graphical solution of the equation (4.19).

Next we calulate $y_\alpha = f_2^*(1-x_\alpha) \approx 0.811$, where $f_2^*(x) = 1-2\left(x - \frac{1}{2}\right)\sqrt{-4x^2+4x+1}$. It is easy to see that

$$D(1-x_\alpha) = [x_1, x_2],$$

where $D(t)$ is the set given by (4.8) and

$$x_1 = \frac{1}{2} - \sqrt{\frac{(2-y_\alpha)}{8}} \approx 0.114 \quad \text{and} \quad x_2 = 1 - \sqrt{\frac{y_\alpha}{8}} \approx 0.681.$$

Finally we obtain the explicit form of the α-optimal partition:

$$A_1^\alpha \approx (0, 0.114) \cup (0.681, 1), \quad A_2^\alpha \approx [0.114, 0.681].$$

□

An interesting issue in fair divisiom theory is estimating the minimum number of cuts needed to obtain a partition that meets certain criteria of fairness. Many specialists in this field have dealt with this issue (e.g. [5, 8, 47]). In the case of α-optimal partitioning of $(0,1)$ for two players, the minimum number of cuts can be easily estimated, because the form of A_2^α (cf. (4.17)) depends on the number of the sign changes of the function $f_2(x) - y_\alpha f_1(x)$. Legut and Wilczyński [H4] showed that for any natural number $k \in \mathbb{N}$ it is possible to construct such measures μ_1, μ_2 for which obtaining α-optimal partition requires the use of $2k$ cuts of $(0,1)$.

4.3.4 Some properties of subsets of the Lyapunov set [H4]

Paper [H4] deals with some properties of the range of nonatomic vector measure $\vec{\mu} = (\mu_1, \ldots, \mu_n)$ defined on measurable subsets $\mathcal{B}$ of unit interval $[0,1]$. Let $\mathcal{U}(k)$ denote a collection of all sets that are unions of at most k pairwise disjoint subintervals of $[0,1]$. Denote by $\langle \mathbf{a},\mathbf{b}\rangle$ $n-$dimensional closed line segment connecting $\mathbf{a} = (a_1, ..., a_n), \mathbf{b} = (b_1, ..., b_n) \in \mathbb{R}^n$. Stromquist and Woodall [60] proved that

$$\langle \mathbf{0},\mathbf{1}\rangle \subset \vec{\mu}(\mathcal{U}(n)), \tag{4.20}$$

where $\mathbf{0} := (0, ..., 0)$, $\mathbf{1} := (1, ..., 1) \in \mathbb{R}^n$. Legut [H4] showed that inclusion (4.20) can be formulated in more general form:

Theorem 4.15. *Let $A \in \mathcal{U}(k)$, $k \in \mathbb{N}$. Then*

$$\langle \boldsymbol{0}, \vec{\mu}(A)\rangle \subset \vec{\mu}\left(\mathcal{U}(n+k-1)\right).$$

It follows from Theorem 4.15 the following:

Proposition 4.16. *If the boundary of $\vec{\mu}\,(\mathcal{B})$ is a subset of $\vec{\mu}(\mathcal{U}(k))$ then*

$$\vec{\mu}\,(\mathcal{B}) = \vec{\mu}\left(\mathcal{U}(n+k-1)\right).$$

The main result of the paper [H4] is:

Theorem 4.17. *Let $A, B \in \mathcal{U}(k)$, $k \in \mathbb{N}$. Then*

$$\langle \vec{\mu}(A), \vec{\mu}(B)\rangle \subset \vec{\mu}\left(\mathcal{U}(2n+4k-3)\right).$$

Legut [H4] used Theorem 4.17 to present a simple proof of that part of the thesis of Lyapunov's Theorem (Theorem 4.2) which concerns the convexity of the range $\vec{\mu}(\mathcal{B})$. It can be concluded from Theorem 4.17 the following:

Proposition 4.18. *Assume that for some $k \in \mathbb{N}$ all extreme points of $\vec{\mu}(\mathcal{B})$ are in $\vec{\mu}(\mathcal{U}(k))$. Then $\vec{\mu}\,(\mathcal{B}) = \vec{\mu}(\mathcal{U}(2n+4k-3))$.*

Consider now case of two-dimensional vector measure $\vec{\mu} = (\mu_1, \mu_2)$ in which measures μ_1, μ_2 are defined on measurable subsets of $(0,1)$ respectively by density functions f_1 and f_2 satisfying

$$\{x : f_1(x) > 0\} = \{x : f_1(x) > 0\} = (0,1).$$

It follows from Proposition 4.18 and 4.10 the following:

Proposition 4.19. *Let $r(x)$ be the function defined by (4.9). The following implications hold*

- *If $r(x)$ is monotone on $(0,1)$, then $\vec{\mu}(\mathcal{B}) = \vec{\mu}(\mathcal{U}(2))$.*
- *If $r(x)$ is monotone on $(0,1/2)$ and symmetric about $x_0 = 1/2$, then $\vec{\mu}(\mathcal{B}) = \vec{\mu}(\mathcal{U}(3))$.*

4.3.5 A method of optimal partitiong of interval $[0,1)$ among n players [H5]

In this section we present an algorithm for obtaining an equitable optimal fair division of measurable space $\{[0,1), \mathcal{B}\}$ for nonatomic probability measures defined by some class of density functions $f_i : [0,1) \to \mathbb{R}_+$, $i \in I$, (cf. (4.5)). Here $\mathcal{B}$ denote family of measurable subsets of interval $[0,1)$.

Let $[a_j, a_{j+1})\}_{j=1}^m$ be a partition of interval $[0,1)$ such that

$$[0,1) = \bigcup_{j=1}^{m} [a_j, a_{j+1}), \;\; a_1 = 0, \, a_{m+1} = 1, \;\; a_{j+1} > a_j \; j \in J, \tag{4.21}$$

where $J = \{1, ..., m\}$.

Dall'Aglio, Legut and Wilczyński [22] found a method of $\alpha-$optimal partition of interval $[0,1]$ in case where density functions are simple, i.e.

$$f_i(x) = \sum_{j=1}^{m} h_{ij} \mathbb{I}_{[a_j, a_{j+1})}(x).$$

In their method they used linear programming algorithm. This method was generalized by Legut [35] who considered piecewise linear density functions:

$$f_i(x) = \sum_{j=1}^{m} (c_{ij}x + d_{ij}) \mathbb{I}_{[a_j, a_{j+1})}(x),$$

where $c_{ij}x + d_{ij} \geq 0$ for all $x \in [a_j, a_{j+1})$, $i \in I$, $j \in J$.

Now we define some class of density functions f_i, $i \in I$, using some properties of monotone likelihood ratio.

Definition 4.20. Density functions $f_i : [0,1) \to \mathbb{R}_+$, $i \in I$, satisfy on interval $[a,b) \subset [0,1)$ *strictly monotone likelihood ratio property* (in short SMLR) if for any $i, k \in I$, $i \neq k$, the ratios $\dfrac{f_i(x)}{f_k(x)}$ are strictly monotone on interval $[a,b) \subset [0,1)$.

Definition 4.21. Density functions $f_i : [0,1) \to \mathbb{R}_+$, $i \in I$, satisfy on interval $[0,1)$ *piecewise strictly monotone likelihood ratio property* (in short PSMLR) if there exists a partition $\{[a_j, a_{j+1})\}_{j=1}^m$ of interval $[0,1)$ satisfying (4.21) such that the density functions f_i have separately on each of the intervals strictly monotone likelihood ratio property (SMLR).

The following proposition could be helpful for checking whether given density functions f_i, $i \in I$, satisfy the PSMLR property.

Proposition 4.22. *Assume that density functions f_i, $i \in I$, are differentiable. Functions f_i, $i \in I$, satisfy PSMLR property iff the set*

$$Q := \{x \in (0,1) : f_i'(x) f_k(x) = f_i(x) f_k'(x), \; i, k \in I, \; i \neq k\} \tag{4.22}$$

is finite.

It follows from Proposition 4.22 that polynomials of positive degree being density functions defined on $[0,1)$ fulfill the PSMLR property.

Consider the problem of the equitable optimal fair division for two players with the following density functions

$$f_1(x) = \mathbb{I}_{[0,1)}(x) \quad \text{and} \quad f_2(x) = x \sin \frac{1}{x} + c,$$

where c is the constant satisfying $\int_0^1 f_1(x)dx = 1$. It is easy to verify, that in this case the set Q defined by (4.22) is infinite.

Define absolutely continuous and strictly increasing functions $F_i : [0,1] \to [0,1]$ by

$$F_i(t) = \int_{[0,t)} f_i(x)\, dx, \quad t \in [0,1], \quad i \in I. \tag{4.23}$$

For construction of the equitable optimal partition we need the following:

Proposition 4.23. *Suppose the densities f_i satisfy PSMLR property. Then for any numbers θ_1, θ_2 satisfying $a_j \leq \theta_1 < \theta_2 < a_{j+1}$, $j \in J$, and any $i, k \in I$, $i \neq k$ the one of the two following inequalities*

$$\frac{F_i(t) - F_i(\theta_1)}{F_i(\theta_2) - F_i(\theta_1)} < \frac{F_k(t) - F_k(\theta_1)}{F_k(\theta_2) - F_k(\theta_1)} \tag{4.24}$$

$$\frac{F_i(t) - F_i(\theta_1)}{F_i(\theta_2) - F_i(\theta_1)} > \frac{F_k(t) - F_k(\theta_1)}{F_k(\theta_2) - F_k(\theta_1)} \tag{4.25}$$

holds for each $t \in (\theta_1, \theta_2)$.

The inequalities (4.24) and (4.25) mean that there is a strict relative convexity relationship between the functions F_i and F_k, $i \neq k$, defined by (4.23). If the inequality (4.24) holds, then F_i is strictly convex with respect to F_k . This property is equivalent to the strict convexity of the composite function $F_i \circ F_k^{-1}$ on the interval $(F_k(a_j), F_k(a_{j+1}))$ (cf. [45]). It follows from a result of Shisha and Cargo [54] (Theorem 1) that $F_i \circ F_k^{-1}$ is strictly convex on $(F_k(a_j), F_k(a_{j+1}))$ if and only if the ratio $\dfrac{f_i(x)}{f_k(x)}$ is strictly increasing on (a_j, a_{j+1}). Hence the reverse implication in Proposition 4.20 is also true.

The relation of strict relative convexity induces on each interval (a_j, a_{j+1}) a strict partial ordering of the functions F_i (cf. [45]). Let $F_i \prec_j F_k$ denote that F_i is strictly convex with respect to F_k on (a_j, a_{j+1}). For each $j \in J$ define permutation $\sigma_j : I \to I$, such that

$$F_{\sigma_j(k+1)} \prec_j F_{\sigma_j(k)},$$

for $k = 1, ..., n-1$. Hence for all $t \in (a_j, a_{j+1})$ we have

$$\frac{F_{\sigma_j(k+1)}(t) - F_{\sigma_j(k+1)}(a_j)}{F_{\sigma_j(k+1)}(a_{j+1}) - F_{\sigma_j(k+1)}(a_j)} < \frac{F_{\sigma_j(k)}(t) - F_{\sigma_j(k)}(a_j)}{F_{\sigma_j(k)}(a_{j+1}) - F_{\sigma_j(k)}(a_j)}. \tag{4.26}$$

The following theorem proved by Legut [H5] presents an algorithm for obtaining an equitable optimal fair division for density functions having the PSMLR property.

Theorem 4.24. *Let a collection of numbers* z^*, $\{x_k^{*(j)}\}, k = 1, ..., n-1$, $j \in J$, *be a solution of the following nonlinear programming (NLP) problem*

$$\max z$$

subject to constraints

$$z = \sum_{j=1}^{m} \left[F_i(x^{(j)}_{\sigma_j(i)}) - F_i(x^{(j)}_{\sigma_j(i)-1}) \right] \quad i = 1, ..., n,$$

with respect to variables z, $\{x_k^{(j)}\}$, $k = 1, ..., n-1, j \in J$, *satisfying the following inequalities*

$$0 = a_1 \leq x_1^{(1)} \leq ... \leq x_{n-1}^{(1)} \leq a_2,$$
$$a_2 \leq x_1^{(2)} \leq ... \leq x_{n-1}^{(2)} \leq a_3,$$
$$...$$
$$a_m \leq x_1^{(m)} \leq ... \leq x_{n-1}^{(m)} \leq a_{m+1} = 1.$$

Then the partition $\{A_i^*\}_{i=1}^n \in \mathscr{P}_n$ *of the unit interval* $[0, 1)$ *defined by*

$$A_i^* = \bigcup_{j=1}^{m} \left[x^{*(j)}_{\sigma_j(i)-1}, x^{*(j)}_{\sigma_j(i)} \right), \; i \in I, \tag{4.27}$$

where $x_0^{*(j)} = a_j$, $x_n^{*(j)} = a_{j+1}$, $j \in J$, *is an equitable optimal fair division for the measures* μ_i, $i \in I$ *and* $v = z^*$ *is the optimal value.*

If for some $i \in I$ and $j \in J$, the equality $x^{*(j)}_{\sigma_j(i)-1} = x^{*(j)}_{\sigma_j(i)}$ holds we set $\left[x^{*(j)}_{\sigma_j(i)-1}, x^{*(j)}_{\sigma_j(i)} \right) = \emptyset$ in the union of intervals (4.27).

Theorem 4.24 can be generalized for the construction of $\alpha-$optimal partitions, but the formulation and proof of such theorem would be very complicated.

The following example presents the method described in the above theorem.

Example 4.25. Consider a problem of fair division for three players $I = \{1, 2, 3\}$ estimating measurable subsets of the unit interval $[0, 1)$ using measures μ_i, $i = 1, 2, 3$, defined respectively by the following density functions

$$f_1(x) := 12 \left(x - \frac{1}{2} \right)^2, f_2(x) := 2x, \; f_3(x) := \mathbb{I}_{[0,1)}(x), \quad x \in [0, 1).$$

We use the algorithm described in Theorem 4.24 to obtain an equitable optimal fair division. First we need to divide the interval $[0, 1)$ into some subintervals on which the densities f_i, $i = 1, 2, 3$, separably satisfy SMLR property. For this reason we find the set Q defined by (4.22). It is easy to check that $Q = \{\frac{1}{2}\}$ and hence by Proposition 4.22 the densities f_i, $i = 1, 2, 3$, satisfy the SMLR property on intervals $[0, \frac{1}{2})$ and $[\frac{1}{2}, 1)$. Denote cumulative strictly increasing distribution functions by $F_i(t) = \int_0^t f_i(x)\,dx$, $i = 1, 2, 3$. Then we have

$$F_1(t) = 4t^3 - 6t^2 + 3t,\ F_2(t) = t^2,\ F_3(t) = t, \quad t \in [0, 1).$$

Based on the inequalities (4.26) we establish the proper order of assignments of subintervals of the intervals $[0, \frac{1}{2})$ and $[\frac{1}{2}, 1)$ to each player as follows: we take midpoints $\{\frac{1}{4}\}$ and $\{\frac{3}{4}\}$ of the two intervals and verify that

$$\frac{F_1(1/4) - F_1(0)}{F_1(\frac{1}{2}) - F_1(0)} > \frac{F_3(1/4) - F_3(0)}{F_3(\frac{1}{2}) - F_3(0)} > \frac{F_2(1/4) - F_2(0)}{F_2(\frac{1}{2}) - F_2(0)},$$

and

$$\frac{F_3(3/4) - F_3(0)}{F_3(1) - F_3(\frac{1}{2})} > \frac{F_2(3/4) - F_2(0)}{F_2(1) - F_2(\frac{1}{2})} > \frac{F_1(3/4) - F_1(0)}{F_1(1) - F_1(\frac{1}{2})}.$$

Hence, we obtain permutations

$$\sigma_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \quad \text{and} \quad \sigma_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}.$$

Now we are ready to formulate an NLP problem as in Theorem 4.24:

$$\max z$$

subject to constraints

$$z = F_1(x_1^{(1)}) - F_1(0) + F_1(1) - F_1(x_2^{(2)}),$$

$$z = F_2(\tfrac{1}{2}) - F_2(x_1^{(2)}) + F_2(x_2^{(2)}) - F_2(x_2^{(1)}),$$

$$z = F_3(x_1^{(2)}) - F_3(x_1^{(1)}) + F_3(x_2^{(1)}) - F_3(\tfrac{1}{2}),$$

with respect to the variables $z, \{x_k^{(j)}\}$ $k = 1, 2, j = 1, 2$, satisfying the following inequalities

$$0 \le x_1^{(1)} \le x_1^{(2)} \le \tfrac{1}{2} \le x_2^{(1)} \le x_2^{(2)} \le 1.$$

Solving the above NLP problem we obtain

$$z^* \approx 0.4843,\ x_1^{*(1)} \approx 0.1426,\ x_1^{*(2)} = a_2 = 0.5,\ x_2^{*(1)} \approx 0.6269, x_2^{*(2)} \approx 0.9367.$$

Hence, we get the equitable optimal fair division $\{A_i^*\}_{i=1}^3 \in \mathscr{P}$ of the unit interval $[0, 1)$, where

$$A_1^* = [0, x_1^{*(1)}) \cup [x_2^{*(2)}, 1), \quad A_2^* = [x_2^{*(1)}, x_2^{*(2)}) \quad \text{and} \quad A_3^* = [x_1^{*(1)}, x_2^{*(1)}).$$

The optimal value $v = z^* \approx 0.4843$. In the figure below the equitable optimal fair division is presented.

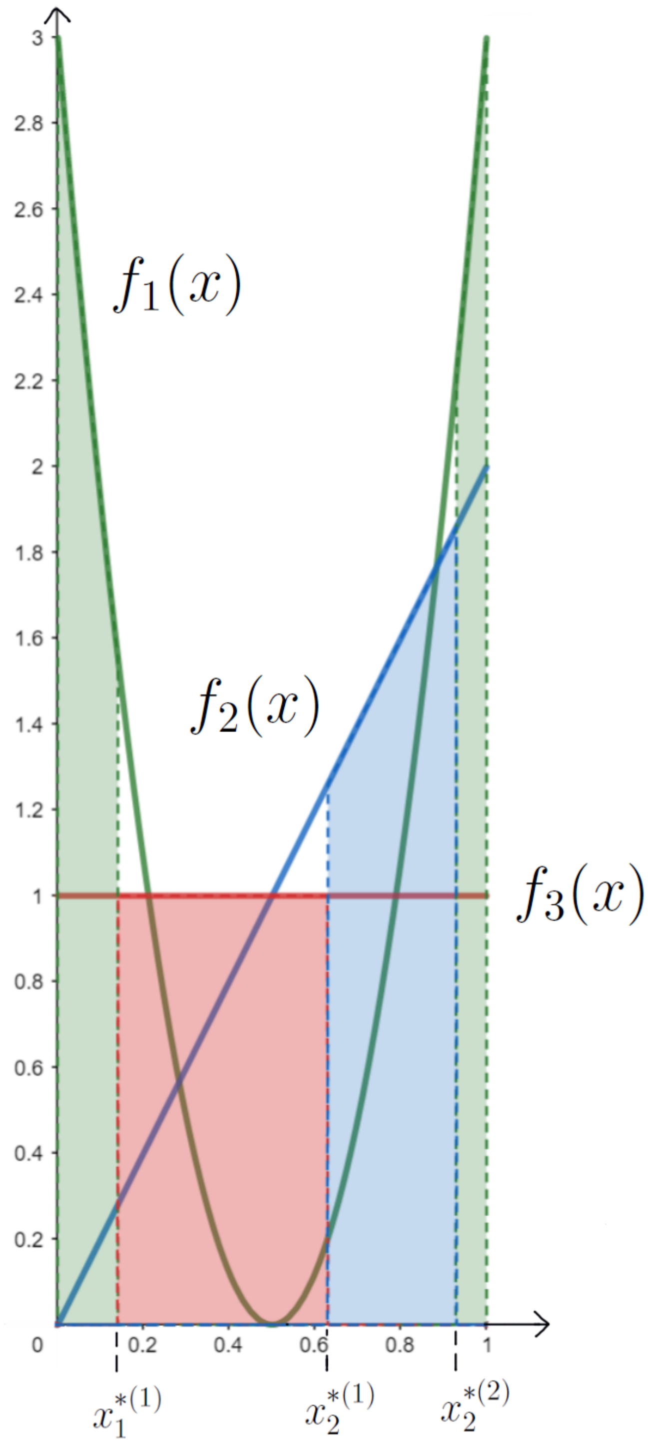

Figure 3. Illustration of the equitable optimal fair division

Figure 3 shows areas marked with green, blue and red. The fields of these areas correspond respectively to the values $\{\mu_i(A_i^*)\}_{i=1}^3$ of the sets $\{A_i^*\}_{i=1}^3$ forming the optimal equitable division.

□

Legut [H5] used the method presented in Theorem 4.24 to show how to obtain also equitable ε-optimal divisions in case where the set Q defined by (4.22) is countably infinite. The definition of an equitable ε-optimal division is given below.

Definition 4.26. A partition $P^\varepsilon = \{A_i^\varepsilon\}_{i=1}^n \in \mathscr{P}_n$ is said to be an equitable ε-optimal fair division if for all $i \in I$

$$\mu_i(A_i^\varepsilon) > v - \varepsilon,$$

where v is the optimal value.

4.3.6 Simple fair division of square $(0,1)^2$ [H6]

The most interesting results in fair division (cake-cutting) problems were obtained for cutting the unit interval $\mathcal{X} = (0,1)$ into n subintervals $\{(x_{k-1}, x_k)\}_{k=1}^n$, where

$$0 = x_0 < x_1 < ... < x_{n-1} < x_n = 1,$$

in such a way that each player receives his piece as a contiguous interval. Such partition is called *simple division.* Stromquist [59] and independently Woodall [62] proved under some weak assumptions the existence of envy-free simple divisions of a cake represented by the unit interval $(0,1)$. The existence of simple equitable divisions was also proved by Cechlárová et al. [11] and Aumann and Dombb [3] . An interesting and very short proof of the existence of an equitable and simple division of the unit interval $(0,1)$ relied on the classical Borsuk-Ulam theorem and was given by Chéze [12].

I published in paper [H6] some results concerning the existence of simple fair division of unit square $(0,1)^2$. In this section $\mathcal{B}$ denote family of measurable subsets of $(0,1)^2$.

Suppose that the i-th player evaluates the Borel subsets of $(0,1)^2$ using nonatomic probability measure ν_i which are absolutely continuous with respect to the Lebesgue measure λ defined on Borel subsets of $(0,1)^2$. We assume that there exist density functions $u_i(x,y) : (0,1)^2 \to \mathbb{R}_+$ such that

$$u_i(x,y) > 0 \quad \text{for all} \quad (x,y) \in (0,1)^2 \quad \text{and} \tag{4.28}$$

$$\nu_i(A) = \iint_A u_i(x,y)dxdy, \quad \text{for all} \quad A \in \mathcal{B}. \tag{4.29}$$

The partition of the unit square can be interpreted as the division of a land. Such divisions are regarded by economists as one of the most important applications of the fair division theory in practice. In the literature on this subject are presented various procedures and algorithms for the division of two-dimensional objects that meet various criteria (see [6, 19, 39, 40, 44, 50]). Woodall [62] noticed that the problem of dividing the unit square fairly can be reduced to the problem of dividing a one-dimensional segment by simply projecting the square $(0,1)^2$ onto $(0,1)$. Unfortunately, if the number of players is large, the narrow rectangles obtained in this way are useless in practical applications.

Denote by $\mathbb{C}_n$ the following set of finite sequences

$$\mathbb{C}_n := \{\mathbf{c} = (c_1, ..., c_r) : \, c_j \in \mathbb{N}, \, j = 1, ..., r, \, \sum_{j=1}^r c_j = n, \, 2 \le r \le n\}.$$

For a given $\mathbf{c} = (c_1, ..., c_r) \in \mathbb{C}_n$ consider the following procedure of fair division:

1. Divide the square $(0,1)^2$ using horizontal cuts starting at heights $\{h_1, ..., h_{r-1}\}$.
2. Next, each of the resulting r rectangles $(0,1) \times (h_{j-1}, h_j)$, $j = 1, ..., r$, where $h_r = 1$ and $h_0 = 0$, we leave unpartitioned (if $c_j = 1$) or we divide it by cuts parallel to the Oy axis into $c_j > 1$ parcels.

Using this procedure we obtain n rectangles to be assigned to the n players. We introduce a two-dimensional version of the one-dimensional definition of simple divisions.

Definition 4.27. A partition $P_{\mathbf{c}} = \{A_i\}_{i=1}^n$ is called a *simple division* of $(0,1)^2$ for $\mathbf{c} = (c_1, ..., c_r) \in \mathbb{C}_n$, if there exist numbers $\{h_j\}_{j=0}^r$ and $\{x_k^{(j)}\}_{k=0}^{c_j}$ satisfying

$$0 = h_0 < h_1 < ... < h_{r-1} < h_r = 1, \tag{4.30}$$

and

$$0 = x_0^{(j)} < x_1^{(j)} < ... < x_{c_j}^{(j)} = 1, \tag{4.31}$$

such that each set A_i is given by:

$$A_i := (x_{k-1}^{(j)}, x_k^{(j)}) \times (h_{j-1}, h_j) \Leftrightarrow i = \sum_{m=0}^{j-1} c_m + k, \tag{4.32}$$

where $c_0 = 0$.

It follows from the assumption $\nu_i \ll \lambda$, $i \in I$, that the boundaries of sets A_i have measure 0 according to ν_i, $i \in I$. Hence

$$\nu_i \left(\cup_{l=1}^n A_l\right) = 1, \quad i \in I,$$

For this reason, to simplify the notation, we consider open sets A_i defined by (4.32). The division scheme described in Definition 4.27 is illustrated by an example presented in Figure 4.

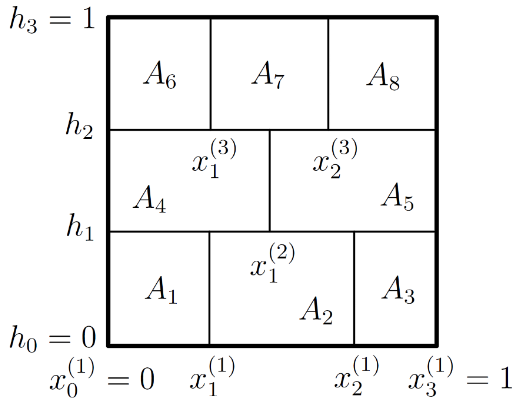

Figure 4. An example of the division scheme of square $(0,1)^2$ for $n = 8$ and $\mathbf{c} = (3,2,3)$.

I proved in paper [H6] the following theorem.

Theorem 4.28. *For any probability nonatomic measures* $\nu_i \ll \lambda$, $i \in I$, *defined by (4.29), satisfying (4.28) and for any* $\mathbf{c} = (c_1, ..., c_r) \in \mathbb{C}_n$ *there exist numbers* $\{h_j\}_{j=0}^r$ *and* $\{x_k^{(j)}\}_{k=0}^{c_j}$ *fulfilling (4.30), (4.31) such that the following equalities hold*

$$\nu_i(A_i) = \nu_m(A_m) \quad \textit{for all} \quad i, m \in I,$$

where $P_{\mathbf{c}} = \{A_i\}_{i=1}^n$ *is a simple division defined by (4.32). Moreover, the division* $P_{\mathbf{c}} = \{A_i\}_{i=1}^n$ *is unique.*

The equitable partition $P_{\mathbf{c}} = \{A_i\}_{i=1}^n$ mentioned in the thesis of the above theorem does not have to be a proportional division. The proof of Theorem 4.28 is not constructive and is based on the Borsuk-Ulam theorem. The method of constructing a simple proportional division of square $(0,1)^2$ was presented in the proof of the following theorem:

Theorem 4.29. *For any probability nonatomic measures* $\nu_i \ll \lambda$, $i \in I$, *defined by (4.29), satisfying (4.28) and for any* $\mathbf{c} = (c_1, ..., c_r) \in \mathbb{C}_n$ *there exist numbers* $\{h_j\}_{j=0}^r$ *and* $\{x_k^{(j)}\}_{k=0}^{c_j}$ *fulfilling (4.30), (4.31) and a permutation* $\sigma : I \to I$ *such that:*

$$\nu_{\sigma(i)}(A_i) \geq \frac{1}{n} \quad \textit{for all} \quad i \in I, \tag{4.33}$$

where $P_{\mathbf{c}} = \{A_i\}_{i=1}^n$ *is a simple division defined by (4.32).*

The following example illustrates the constructive method of obtainig proportional division of the unit square among three players.

Example 4.30. Suppose three players $I = \{1, 2, 3\}$ estimate measurable subsets of square $(0,1)^2$ using probability measures $\{\nu_i\}_{i=1}^3$ with corresponding density functions u_i, $i \in I$, given by

$$u_1(x,y) = x + y, \quad u_2(x,y) = 4xy, \quad u_3(x,y) = 4x(1-y), \quad \text{for} \quad (x,y) \in (0,1)^2.$$

We show a method of obtaining a proportional simple division of the unit square for $\mathbf{c} = (1,2) \in \mathbb{C}_3$. The division scheme is presented in figure below.

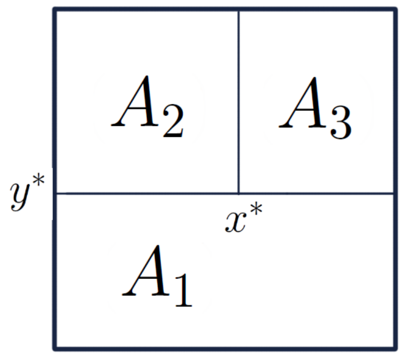

Figure 5. Division scheme for $\mathbf{c} = (1,2) \in \mathbb{C}_3$.

We need to find only two numbers $y^* = h_1$ and $x^* = x_1^{(2)}$ (cf. Figure 4 and 5) and a permutation $\sigma : I \to I$ such that

$$\nu_{\sigma(1)}(A_1) = \nu_{\sigma(1)}((0,1) \times (0,y^*)) \geq \frac{1}{3},$$

$$\nu_{\sigma(2)}(A_2) = \nu_{\sigma(2)}((0,x^*) \times (y^*,1)) \geq \frac{1}{3},$$

$$\nu_{\sigma(3)}(A_3) = \nu_{\sigma(3)}((x^*,1) \times (y^*,1)) \geq \frac{1}{3}.$$

Presented method below is based on a procedure of fair division of a cake found by Banach and Knaster and reported by Steinhaus [57] in 1949.

Define continuous functions $w_i : [0,1] \to [0,1]$ by

$$w_i(t) := \int_0^t dy \int_0^1 u_i(x,y)dx, \quad i \in I.$$

Performing simple calculations, we obtain

$$w_1(t) = \frac{1}{2}(t + t^2), \quad w_2(t) = t^2, \quad w_3(t) = 2t - t^2, \quad t \in [0,1].$$

Let

$$y^* = \min\left\{ t : \max_{i \in I} w_i(t) \geq \frac{1}{3} \right\}.$$

The way of determining the number y^* is illustrated in the figure below.

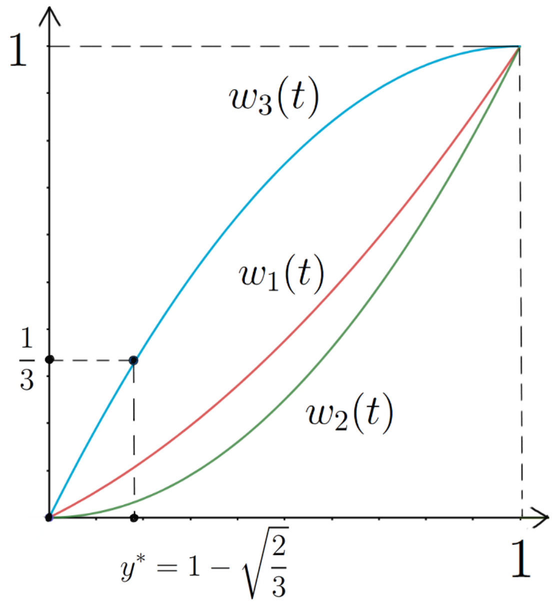

Figure 6. Graphic illustration of determining the number y^*.

Simple calculations show that

$$y^* = 1 - \sqrt{\frac{2}{3}} \approx 0.184, \quad \text{and} \quad w_3(y^*) = \frac{1}{3}.$$

Hence

$$\nu_3(A_1) = \nu_{\sigma(1)}((0,1)\times(0,y^*)) = \frac{1}{3},$$

and the set $A_1 = (0,1)\times(0,y^*)$ is assigned to the third player. Then we set $\sigma(1) = 3$. It is easy to see that

$$\nu_i((0,1)\times(y^*,1)) \geq \frac{2}{3}, \quad \text{for} \quad i = 1,2.$$

Rectangle $(0,1)\times(y^*,1)$ can be easily divided fairly among the remaining two players by a cutting it at a point x^*. For this reason we can apply the procedure "for one to cut, the other to choose". Assume that the second player is to cut the rectangle into two parts and the first player is to choose one of the resulting piece. The second player must cut vertically the rectangle at point x^* satisfying

$$\nu_2((0,x^*)\times(y^*,1)) = \nu_2((x^*,1)\times(y^*,1)) \tag{4.34}$$

to ensure that he receives at least half of the rectangle $(0,1)\times(y^*,1)$ according to his own measure. Solving equation (4.34) with respect to x^* we obtain $x^* = \frac{\sqrt{2}}{2}$. Now the first player estimates the resulting rectangles according to his own measure:

$$\nu_1((0,x^*)\times(y^*,1)) = \frac{1}{12}(-2\sqrt{2} + 4\sqrt{3} + \sqrt{6}) \approx 0,546,$$

$$\nu_1((x^*,1)\times(y^*,1)) = \frac{1}{12}(-4 + 2\sqrt{2} - 4\sqrt{3} + 5\sqrt{6}) \approx 0,346.$$

Hence the first player obviously chooses the rectangle $(0,x^*)\times(y^*,1)$ and the remaining piece $(x^*,1)\times(y^*,1)$ is assigned to the second player with

$$\nu_2((x^*,1)\times(y^*,1)) = \frac{1}{3}(\sqrt{6} - 1) \approx 0,483.$$

We set $\sigma(2) = 1$ and $\sigma(3) = 2$. Finally we get the following permutation $\sigma : I \to I$

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \tag{4.35}$$

establishing the assignmets of sets A_i, $i \in I$, to the players and obtained partition is proportional.

□

The main result of the paper [H6] is proving the existence of an equitable simple and proportional division of square $(0,1)^2$. The proof of this theorem is based on Theorem 4.28 and 4.29.

Theorem 4.31. *For any probability nonatomic measures $\nu_i \ll \lambda$, $i \in I$, defined by (4.29), satisfying (4.28) and for any $\mathbf{c} = (c_1, ..., c_r) \in \mathbb{C}_n$, there exists a permutation $\sigma : I \to I$ and an equitable simple and proportional division $\{A_i\}_{i=1}^n$ defined by (4.32) such that:*

$$\nu_{\sigma(i)}(A_i) \geq \frac{1}{n} \quad \text{for all} \quad i \in I. \tag{4.36}$$

The construction of proportional fair division presented in Example 4.30 determines the assigment of sets A_i, $i \in I$, to the players according to the permutation (4.35). It follows from the proof of Theorem 4.28 that this permutation remains the same for an partition which is at the same time simple, equitable and proportional.

It turns out that the thesis of Theorem 4.31 can be extended to more complex partitions. Suppose $n = 8$. We start the division scheme with one horizontal cut, then each of the two resulting rectangles we divide by two vertical cuts. We finish with four horizontal cuts of the remaining four rectangles obtaining in the end 8 parcels.

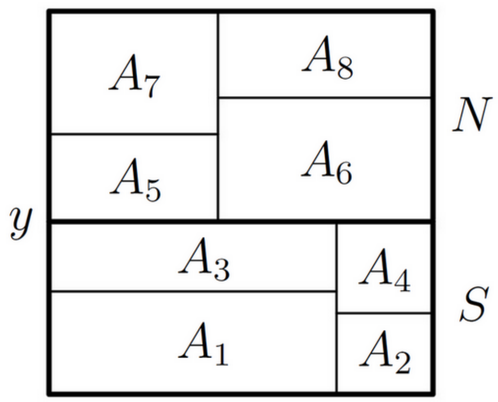

Figure 7. An example of the division scheme obtained by mixed horizontal and vertical cuts.

In my further research, I plan to generalize the result of Theorem 4.31 to the case of simple division of set $(0, 1)^k$ for $k > 2$. Also, I plan to investigate existence a simple division of the $(0, 1)^2$ square that is also envy-free.

4.4 Contribution description of the postdoctoral researcher to the scientific achievement

Papers [H1, H3, H5, H6] were written by myself.

Paper [H2]

- The idea to investigate whether there are methods to construct an $\alpha-$optimal partition is the postdoctoral student's own contribution.
- The hypothesis and the concept of the paper were developed together with the co-author.

- The formulation of Theorem 3 and its proof were developed together with dr hab. Maciej Wilczyński.

I estimate my participation in writing the paper [H2] at 50%.

Paper [H4]

- The formulation of the problem concerning the construction of a range of two-dimensional nonatomic vector measure is the postdoctoral's own contribution.
- The idea for writing the paper was developed together with dr hab. Maciej Wilczyński.
- Putting a hypothesis on the method of determining the range of a vector measure is the habilitation candidate's own contribution.
- I estimate my participation in proving Theorem 2 at 30 %
- The development of three examples illustrating the obtained results constitutes my own contribution.
- The entire Chapter 4 on application of obtained results to fair division problems was written by myself.

I estimate my participation in writing the paper [H4] at 50%.

4.5 Discussion of selected publications of the scientific achievements

My other scientific achievements are presented in the following articles:

[D1] Legut J.: *Market Games with a Continuum of Indivisible Commodities*, International Journal of Game Theory, 15, 1-7 (1985)

[D2] Legut J.: *The Problem of Fair Division for Countably Many Participants*, J. Math. Anal. Appl., 109, 83-89 (1985)

[D3] Legut J. : *A Game of Fair Division with a Continuum of Players.* Colloquium Mathematicum, vol LIII, 323-331 (1987)

[D4] Legut J.: *A Game of Fair Division in Normal Form*, Colloquium Mathematicum, vol LVI, 179-184 (1988)

[D5] Legut J. and Wilczyński M.: *Optimal partitioning of a Measurable Space into Countably Many Sets,* Probability Theory and Related Fields, 86, 551-558 (1990)

[D6] Legut J.: "On Totally Balanced Games Arising from Cooperation in Fair Division", *Games and Economic Behavior*, 2, 47-60 (1990)

[D7] Legut J., Potters J.A.M. and Tijs S.H.: *Economies with Land - A Game Theoretical Approach,* Games Econom. Behav. vol. 6, Issue 3, 416-430 (1994)

[D8] Legut J., Potters J.A.M. and Tijs S.H. (1995): *A transfer Property of Equilibrium Payoffs in Economies with Land* , Games Econom. Behav. vol. 10, Issue 2, 355-375 (1995)

[D9] Dall'Aglio M., Legut J., Wilczyński M.: *On Finding Optimal Partitions of a Measurable Space*, Mathematica Applicanda, vol. 43(2), 193-206 (2015)

[D10] Legut J.: *Optimal Fair Division for Measures with Piecewise Linear density Functions,* International Game Theory Review, vol. 19, No. 2, 175009, (2017)

[D11] Legut J.: *On a method of obtaining an approximate solution of an exact fair division problem,* Mathematica Applicanda, vol. 46 (2), 245-256 (2018)

[D12] Legut J. and Wilczyński M.: *How to obtain maximal and minimal subranges of two-dimensional vector measures.* Tatra Mt. Math. Publ. 74 85–90 (2019)

The first three papers [D1, D2, D3] concern the use of the results of game theory in the problem of fair division and constituted the basis of my doctoral dissertation.

In paper [D1] I introduced a model of a new class of market games in which players exchange goods described by means of measurable subsets of a certain space, and utility functions are represented by nonatomic probability measures. Then I showed that the new class contains totally balanced games (see ([52])) and overlaps with the class of market games previously defined by Shapley and Shubik [55].

The results of papers [D2] and [D5] were mentioned in Section 4.3.2 in commentary to Theorem 4.7.

In [D3] I proposed a fair division model in which an infinite number of players are represented by the unit interval [0.1]. I showed that in this model there exist partitions which are $\epsilon-$ fair.

Paper [D4] concerns the representation of fair division games in a strategic form. The main result of this work is proving the existence of a Nash equilibrium point of these games in pure strategies. This equilibrium point corresponds to the optimal division of a measurable space.

The publications [D1, D2, D3, D4, D5] were cited in book entitled "Fair division-from cake-cutting to dispute resolution" written by well-known specialists in the fair division theory - Brams and Taylor [8].

In paper [D6] I proposed a method of analysing a secondary division of an object $\mathcal{X}$ using the theory of cooperative games. In this method players form coalitions to improve the initial partition and then a cooperative game is defined. It turned out that these games are totally balanced and then have nonempty core. A method of obtaining an imputation from this core is found. A characterization and some properties of such class of games are presented. This results were considered in the literature of fair division and cooperative games theory (cf. [13], [14], [18], [17]).

In paper [D7] a cooperative game v_E associated with an economy with land E (an economy of Debreu-type in which land is the unique commodity) is defined. The set of all TU-games of type v_E is investigated and the set of equilibrium payoffs (in the TU-sense) of the economy E is described as a subset of the core of v_E. The authors proved that equilibrium payoffs can be extended to population monotonic allocation schemes in the sense of Sprumont. The results of this article were mentioned in other papers (cf. [20, 21, 46, 50, 51])

Paper [D8] deals with an exchange economy of Debreu type with only one commodity (land). The authors investigate NTU-games connected to these kinds of economies. The main result of this paper is that equilibrium payoffs is the NTU-model are connected to equilibrium payoff in the TU-model considered in [D6] by b-transfer - a concept introduced by Shapley [53].

Papers [D9] and [D10] were mentioned in the introductory part of Section 4.3.5. They concern the methods of constructing the optimal division of a measurable space according to the measures defined with the help of the density functions described by simple ([D9]) and piecewise linear functions ([D10]).

In paper [D11] I proposed an algorithm of obtainig the approximate solution to the problem of exact fair division of the unit interval [0.1]. Resulting partition is at the same time proportional, exact and equitable (see Definition 4.1). In addition, I have provided an example to illustrate this algorithm for three players.

Paper [D12] deals with certain properties of a two-dimensional range of a nonatomic vector measure. Let $(\mathcal{X}, \mathcal{B})$ be a measureable space with a nonatomic vector measure $\vec{\mu} = (\mu_1, \mu_2)$. Denote by $R(Y) = \{\vec{\mu}(Z) : Z \in \mathcal{B}, Z \subseteq Y\}$. For a given $p \in \vec{\mu}(\mathcal{B})$ consider a family of measurable subsets $\mathcal{B}_p = \{Z \in \mathcal{F} : \vec{\mu}(Z) = p\}$. Dai and Feinberg [15] proved the existence of a maximal subset $Z^* \in \mathcal{B}_p$ having the maximal subrange $R(Z^*)$. They showed also existence of a minimal subrange $M^* \in \mathcal{B}_p$ having the minimal subrange $R(M^*)$. In paper [D12] we present a method of obtaining the maximal and the minimal subsets. Hence, we get simple proofs of the results of Dai i Feinberg [15].

5 Presentation of significant scientific or artistic activity carried out at more than one university, scientific or cultural institution, especially at foreign institutions

At the beginning of my research activity in 1988 (in July) I visited the Faculty of Mathematics of the Georgia Institute of Technology in Atlanta (USA) and I gave a lecture on the fair division theory with the presentation of my own scientific results.

In the early nineties, I started a scientific cooperation with the Faculty of Mathematics of the Catholic University in Nijmegen (the Netherlands), which I have visited many times. The result of this collaboration was the publication of two scientific articles on applications of the fair division theory in economics ([39, 40]). I presented the results of our cooperation in 1991 at the International Conference on Game Theory and Economics at Stony Brook University in New York.

In 1992 (in July), I gave a lecture at the Economics Department of Tel Aviv University on some applications of game theory in problems of fair division.

In 2013 (in October), I established cooperation with Professor Marco Dall'aglio from the Luiss Universita Guido Carli in Rome. The results of this cooperation regarding some methods of fair division were published in paper [22].

In 2022 from April 1 to June 30 I completed an internship at the Mathematics Institute of the University of Silesia, whose supervisor was prof. dr hab. Szymon Plewik.

The purpose of this internship was to establish scientific cooperation in research on the properties of a range of a nonatomic vector measure.

6 Presentation of teaching and organizational achievements as well as achievements in popularization of science or art

During my professional career at the Wrocław University of Science and Technology, I gave lectures on many mathematical subjects, such as: linear algebra, mathematical analysis, statistics, probability theory, differential equations, and survey data analysis.

I have received the following awards for my teaching activities:

- Award of the Dean of the Faculty of Fundamental Problems of Technology for didactic achievements - 1986
- Award of the Director of the Institute of Mathematics for didactic achievements - 1989
- Rector's Award in recognition of outstanding contribution to the university's activities - 2017

In the years 2016-2022 I was the promoter of 20 bachelor's theses and 9 master's theses in mathematics. Some of these works concerned the subject of my research. These were the following theses:
Bachelor's theses:

- Michał Krząstek - Application of the theory of cooperative games in the assessment of the bargaining power of political parties
- Bartosz Lewandowski - Application of game theory in the fair division of a holding company
- Taras Kostiuk - Analysis of the competitive strategies of companies entering the market based on the models of non-cooperative games
- Krystian Kasprzyk - Optimal fair division of two-dimensional objects
- Patrycja Niewęgłowska - The method of fair division of two-dimensional objects with imposed restrictions and its application in practice
- Piotr Trzeciak - Determining the image of a two-dimensional vector measure and its applications

Masters theses:

- Krystian Kasprzyk - Minimax decision rules for identifying an unknown distribution of a random variable
- Sandra Mróz - Applications of fair division theory in cooperative games

When it comes to activities that popularize mathematics, in 2018 I took part in the commission checking works of the mathematics olympiad organized for secondary schools.

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